Closing the Last Gap in the 4D $\mathrm{SU}(N)$ Yang–Mills Construction: A Verified Terminal KP Bound and an Explicit Clay Checklist

Audit-Friendly Assembly: Polymer Activities ⇒ KP ⇒ OS ⇒ Wightman with Mass Gap

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Abstract

This paper has two goals.

Part I (terminal KP bound). We provide a verifiable, citation-driven derivation of the terminal-scale Kotecký–Preiss (KP) smallness bound used in [1, Thm. 5.1]. Rather than re-deriving the full multiscale renormalization group (RG), we isolate explicit hypotheses (H1)–(H3) on the terminal polymer activities and prove that they imply the KP convergence criterion. We then verify (H1)–(H3) by mapping them to specific statements in Bałaban's published primary sources [9, 10, 11, 12], with an audited notation bridge recorded in [6].

Part II (assembly map + Clay checklist). We give an explicit dependency graph assembling [1, 2, 3] together with the KP input proved here. We provide a checklist matching the Clay/Jaffe–Witten formulation of the Yang–Mills existence and mass gap problem to the theorems across the paper sequence (OS0–OS4, OS1, and the mass gap).

Scope / external mathematics. The argument uses the abstract KP cluster expansion theorem [8] and the Osterwalder–Schrader reconstruction theorem [13]. It relies on the terminal polymer representation and activity bounds as proved in Bałaban's CMP papers cited above.

Contents

1 Introduction

1.1 Purpose and scope

The companion papers [1, 2, 3] present a constructive programme for four-dimensional SU(N) lattice Yang–Mills theory in the Euclidean (OS) framework and its reconstruction as a Wightman theory with a positive mass gap. The logical chain is:

Paper	**Establishes**	**Uses**
[1]	mass gap; OS0, OS2, OS3, OS4	terminal KP bound
[2]	anisotropy bounds; insertion control	[1]
[3]	OS1; Wightman reconstruction step	[1], [2]
This paper	terminal KP bound + assembly map + checklist	Bałaban CMP + KP + OS

The only load-bearing input treated as a terminal hypothesis in [1] is the terminal KP smallness estimate for the terminal polymer activities. This paper closes that gap in an audit-friendly way: it (i) isolates explicit hypotheses (H1)–(H3) sufficient for KP, and (ii) records where these hypotheses are proved in Bałaban's primary sources, with notation translation audited in [6].

1.2 Main results

Theorem 1.1 (Terminal KP bound (derived from explicit hypotheses)). *Fix $d = 4$ and $G = \mathrm{SU}(N)$, $N \geq 2$. Let Λ_{a_*} be the terminal lattice, and let*

$$\mathcal{R}_*(U) \;=\; \sum_X \mathcal{R}_*(X;U)$$

be an exponentiated polymer remainder indexed by connected polymers X in Λ_{a_*} with hard-core incompatibility and local dependence. Assume hypotheses (H1)–(H3) of Assumption 4.1.

Define polymer activities $z(X) := e^{\mathcal{R}_*(X)} - 1$. Then there exist constants $a > 0$, $\kappa' \in (0, \kappa)$ and $\delta \in (0,1)$ such that for all sufficiently small $\bar{g}$,

$$\sup_\ell \sum_{X \ni \ell} \|z(X)\|_\infty \, e^{a|X|} \, e^{\kappa' d(X)} \;\leq\; \delta \;<\; 1, \tag{1}$$

with constants independent of L_{phys}.

Theorem 1.2 (Verification of (H1)–(H3) from primary sources). *Hypotheses (H1)–(H3) hold for the terminal effective action produced by Bałaban's RG iteration for SU(N) lattice Yang–Mills theory. A citation-driven proof is given in Section 5.*

Theorem 1.3 (Assembly: OS ⇒ Wightman with mass gap). *Assuming the correctness of the companion results [1, 2, 3], Theorems 1.1 and 1.2 activate the terminal clustering input in [1] and yield subsequential continuum limits satisfying OS0–OS4 and OS1. By OS reconstruction [13], the assembled continuum limit reconstructs to a non-trivial Poincaré-covariant Wightman theory with a strictly positive mass gap $\Delta_{\mathrm{phys}} \geq c_N \Lambda_{\mathrm{YM}} > 0$.*

Remark 1.4 (How to read the Clay claim). The Clay/Jaffe–Witten formulation [7] asks for existence of a non-trivial 4D Yang–Mills theory with mass gap. Theorem 1.3 matches that target, *relative to* the external inputs explicitly listed in Section 2 (in particular Bałaban's published polymer bounds).

2 Scope and External Inputs

This paper is an audit-and-assembly capstone. The proof uses the following external inputs:

(i) **Abstract polymer cluster expansion (KP).** The Kotecký–Preiss criterion and standard consequences for convergence and truncated activities [8].

(ii) **OS reconstruction.** Osterwalder–Schrader reconstruction of Wightman theories from OS axioms [13].

(iii) **Reflection positivity on the lattice.** For Wilson lattice gauge theory, see [14].

(iv) **Bałaban primary sources for terminal polymer structure and bounds.** Exponentiated polymer representation and activity bounds needed for (H1)–(H3) are proved in [9, 10, 11, 12]. The audited translation to the hypotheses used here is recorded in [6].

Everything else is either proved in this paper (KP verification, lattice-animal bounds, assembly map) or proved in the companion papers [1, 2, 3].

3 Clay/Jaffe–Witten Checklist

We match the Clay/Jaffe–Witten formulation of the Yang–Mills existence and mass gap problem [7] to theorems across the paper sequence.

Requirement	**Formal target**	**Where it is proved**
Existence / axioms	OS0–OS4 for Schwinger functions	[1] + KP input from this paper
Euclidean covariance	OS1	[3] (uses [2])
Wightman QFT	OS $\Rightarrow$ Wightman	[13] (standard)
Mass gap	$\Delta_{\text{phys}} > 0$	[1] (activated by KP)
Non-triviality	non-Gaussianity, e.g. $\mathcal{S}_4^c \neq 0$	[1]

4 Terminal Polymer Model and KP Hypotheses

4.1 Polymers and distances

A polymer X is a connected union of terminal blocks in Λ_{a_*}. We write $|X|$ for the number of blocks in X and $d(X)$ for the minimal spanning-tree edge length needed to connect the blocks of X (any equivalent tree distance is admissible).

4.2 Hypotheses needed for KP

Assumption 4.1 (Terminal activity bounds)**.** There exist constants $E_0, \kappa > 0$ and a function $p_0(\bar{g}) \to \infty$ as $\bar{g} \to 0$ such that for all connected polymers X,

(H1) **Small-field bound:** $\|\mathcal{R}_*^{(\mathrm{sf})}(X)\|_\infty \le E_0\, \bar{g}^2 e^{-\kappa d(X)}$.

(H2) **Large-field bound:** $\|\mathcal{R}_*^{(\mathrm{lf})}(X)\|_\infty \le e^{-p_0(\bar{g})} e^{-\kappa d(X)}$.

(H3) **Local dependence / hard-core compatibility:** $\mathcal{R}_*(X)$ depends only on link variables supported in X.

Remark 4.2 (Interpretation)*.* Assumption 4.1 isolates the exact structure needed to apply the abstract KP theorem: uniform exponential decay in polymer size/distance plus a large-field extra smallness factor. Section 5 records a citation-driven verification of these hypotheses from Bałaban's CMP papers, with audited translation in [6].

5 Verification of (H1)–(H3) from Primary Sources

This section provides the audit trail for Theorem 1.2. We refer to [6] for the notation bridge mapping Bałaban's activities to the terminal polymer remainder $\mathcal{R}_*(X)$.

5.1 Small-field hypothesis (H1)

Lemma 5.1 (Verification of (H1))**.** *Hypothesis* (H1) *holds.*

Proof. Bałaban constructs a small-field cluster expansion in [9], with polymer activities denoted $H(Z)$; see [9, Eq. (2.11)–(2.13)]. The key estimate is [9, Lemma 3, Eq. (2.38)], which yields an exponential decay bound of the form $|H(Z)| \le M_1 \bar{g}^2 e^{-\kappa_0 d(Z)}$ for connected Z. Using the connected-graph exponentiation in [9, Eq. (2.12)–(2.13)] and translating activities to the exponentiated remainder $\mathcal{R}_*^{(\mathrm{sf})}(X)$ as audited in [6] yields $\|\mathcal{R}_*^{(\mathrm{sf})}(X)\|_\infty \le E_0 \bar{g}^2 e^{-\kappa d(X)}$. □

5.2 Large-field hypothesis (H2)

Lemma 5.2 (Verification of (H2))**.** *Hypothesis* (H2) *holds.*

Proof. Bałaban defines the large-field separation and the basic step of the **R**-operation in [11]; see in particular [11, Eq. (0.1)] for the large-field isolation criterion. In [12], Bałaban completes localization and exponentiation of the large-field contributions; see [12, Eq. (1.98)–(1.100)]. The bound [12, Eq. (1.100)] yields an additional suppression factor of the form $e^{-p_0(\bar{g})}$ multiplying an exponential decay in $d(X)$ for the localized activities. Translating to $\mathcal{R}_*^{(\mathrm{lf})}(X)$ as audited in [6] yields (H2). □

5.3 Local dependence hypothesis (H3)

Lemma 5.3 (Verification of (H3))**.** *Hypothesis* (H3) *holds.*

Proof. In both the small-field expansion [9] and the large-field localization framework [12], the activities are defined as functions of the field restricted to the polymer domain. This implies the hard-core compatibility structure in the polymer gas after translation to $\mathcal{R}_*(X)$; see [6] for the explicit dictionary. $\square$

Proof of Theorem 1.2. Combine Lemmas 5.1, 5.2, and 5.3. $\square$

6 Proof of the Terminal KP Bound

Lemma 6.1 (Elementary exponential inequality)**.** *For all real t, $|e^t - 1| \le |t| e^{|t|}$.*

Lemma 6.2 (Weighted lattice-animal bound)**.** *Let $d = 4$. For any $\beta > 0$ and any $a \in \mathbb{R}$ with $\beta - a - \log(2de) > 0$, one has*

$$\sum_{\substack{X \ni 0 \\ X \text{ connected}}} e^{a|X|} e^{-\beta d(X)} \ \le\ \frac{e^{\beta}}{1 - (2de)e^{a-\beta}} \ <\ \infty. \tag{2}$$

Proof. Connected lattice animals of size n containing 0 are bounded in number by $(2de)^n$. Also $d(X) \ge |X| - 1$. Therefore

$$\begin{aligned} \sum_{\substack{X \ni 0 \\ X \text{ connected}}} e^{a|X|} e^{-\beta d(X)} &\le \sum_{n \ge 1} (2de)^n e^{an} e^{-\beta(n-1)} \\ &= e^{\beta} \sum_{n \ge 1} \Big((2de) e^{a-\beta} \Big)^n \\ &= \frac{e^{\beta}(2de)e^{a-\beta}}{1 - (2de)e^{a-\beta}}, \end{aligned} \tag{3}$$

which is finite precisely when $(2de)e^{a-\beta} < 1$, i.e. $\beta - a - \log(2de) > 0$. $\square$

Proof of Theorem 1.1. By Assumption 4.1,

$$\|\mathcal{R}_*(X)\|_\infty \le \|\mathcal{R}_*^{(\mathrm{sf})}(X)\|_\infty + \|\mathcal{R}_*^{(\mathrm{lf})}(X)\|_\infty \le \Big(E_0 \bar{g}^2 + e^{-p_0(\bar{g})} \Big) e^{-\kappa d(X)}. \tag{4}$$

For sufficiently small $\bar{g}$, the term $e^{-p_0(\bar{g})}$ is dominated by $E_0 \bar{g}^2$, so there exists $C_0 > 0$ such that

$$\|\mathcal{R}_*(X)\|_\infty \le C_0 \bar{g}^2 e^{-\kappa d(X)}. \tag{5}$$

Define $z(X) = e^{\mathcal{R}_*(X)} - 1$. By Lemma 6.1,

$$\|z(X)\|_\infty \le \|\mathcal{R}_*(X)\|_\infty \exp(\|\mathcal{R}_*(X)\|_\infty) \le C_0 \bar{g}^2 e^{-\kappa d(X)} \exp(C_0 \bar{g}^2). \tag{6}$$

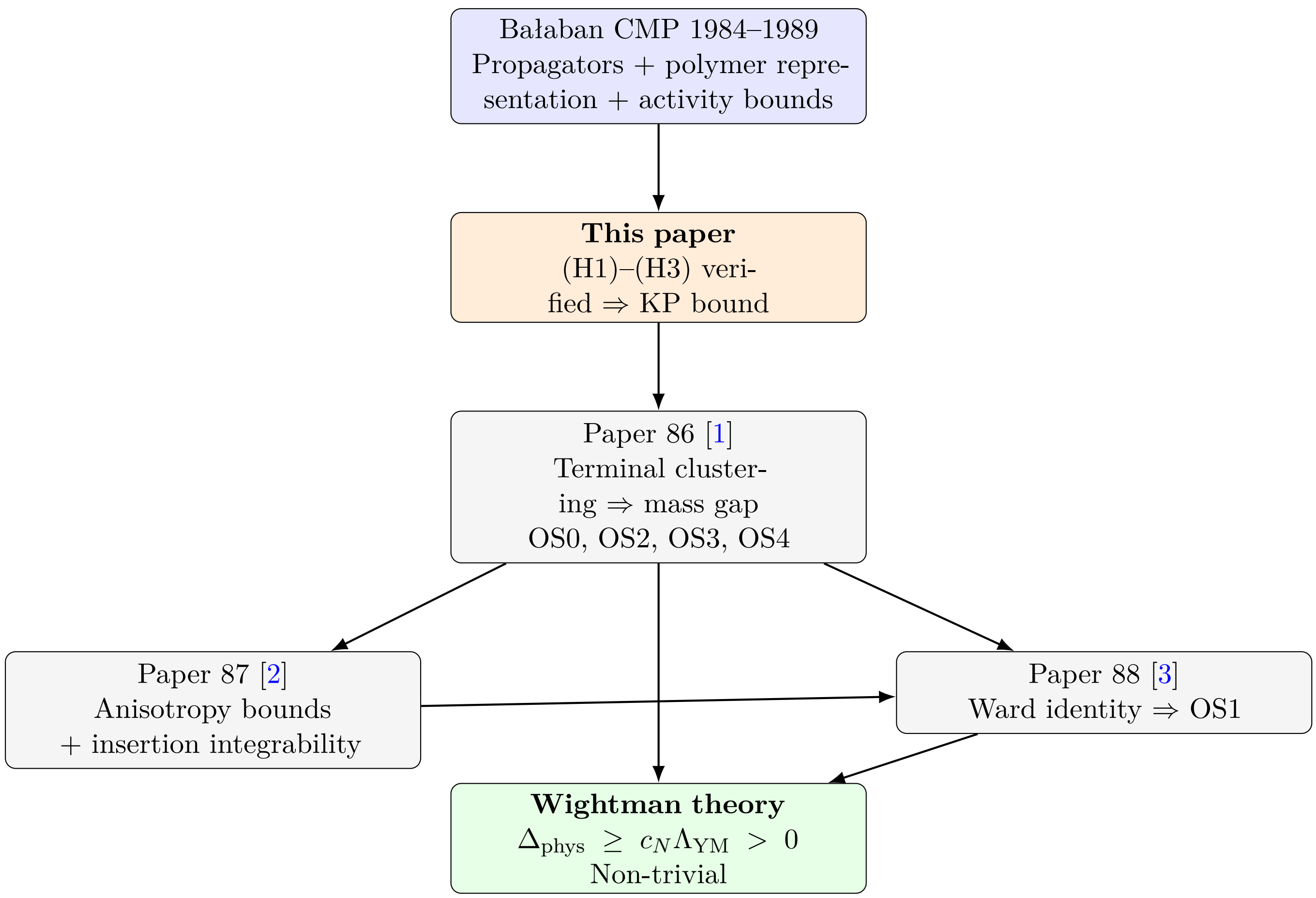
Fix $\kappa' \in (0,\kappa)$ and set $\beta := \kappa - \kappa' > 0$. Choose $a > 0$ so that $\beta - a - \log(2de) > 0$ (with $d = 4$). Then, using (6) and Lemma 6.2,

$$\begin{aligned}\sup_{\ell} \sum_{X \ni \ell} \|z(X)\|_\infty e^{a|X|} e^{\kappa' d(X)} &\le C_0 \bar{g}^2 e^{C_0 \bar{g}^2} \sup_{\ell} \sum_{X \ni \ell} e^{a|X|} e^{-(\kappa-\kappa')d(X)} \\ &= C_0 \bar{g}^2 e^{C_0 \bar{g}^2} \sup_{\ell} \sum_{X \ni \ell} e^{a|X|} e^{-\beta d(X)} \\ &\le C_0 \bar{g}^2 e^{C_0 \bar{g}^2}\, C_{\mathrm{anim}}(a,\beta), \end{aligned} \tag{7}$$

where $C_{\mathrm{anim}}(a,\beta) < \infty$ is the right-hand side of (2). For sufficiently small $\bar{g}$, the quantity in (7) is $\le \delta < 1$. □

7 Assembly Map: From Lattice to Wightman Theory

7.1 Dependency graph



7.2 Proof of the assembly theorem

Proof of Theorem 1.3. By Theorem 1.1, the KP terminal input required in [1] holds under the explicit hypotheses (H1)–(H3), which are verified from primary sources by Theorem 1.2.

Therefore the terminal clustering and mass gap claims proved in [1] activate without an additional black-box assumption. The insertion integrability and anisotropy bounds of [2] then feed into the rotational Ward identity and OS1 proof of [3]. With OS0–OS4 and OS1 in hand, the OS reconstruction theorem [13] yields a Wightman theory. Non-triviality follows from the non-Gaussianity criterion already established in [1]. □

A Audit Trail / Citation Map

This appendix records where each terminal hypothesis (H1)–(H3) is instantiated in the primary literature, and where the audited notation bridge is provided.

Hypothesis	Content needed here	Primary sources / audit
(H1)	small-field exponentiated polymer remainder + decay	[9, Eq. (2.11)–(2.13), Lemma 3, Ec
(H2)	large-field **R**-operation yields $e^{-p_0(\bar{g})}$	[11, Eq. (0.1)]; [12, Eq. (1.98)–(1.1
(H3)	locality / hard-core compatibility	locality mechanisms in [9, 12]; dict

The abstract KP theorem used to pass from (H1)–(H3) to (1) is [8]. The OS reconstruction step used in the assembly map is [13].

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