Spectral Gap and Thermodynamic Limit for $SU(N)$ Lattice Yang–Mills Theory via Log-Sobolev Inequalities and Complete Analyticity

— conditional and windowed statements —*

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February 2026 (v1); July 2026 (v2)

Abstract

We present two parallel results for $SU(N)$ pure gauge lattice Yang–Mills in four Euclidean dimensions, at fixed lattice spacing $\eta > 0$ and weak coupling $g_0 \leq g_*$, both *conditional on a single shared input*, the Dobrushin–Shlosman complete analyticity condition (H-CA): (A) a log-Sobolev inequality $\mathrm{Ent}_{\mu_L}(f^2) \leq (2/\hat{\rho})\,\mathcal{E}_L(f,f)$ with $\hat{\rho} > 0$ independent of L, via Cesi's quasi-factorisation seeded by a Bakry–Émery/Holley–Stroock local LSI; and (B) a spectral gap $m_{\mathrm{gap}} \geq m_0 > 0$ for the Osterwalder–Seiler Hamiltonian, via Dobrushin clustering and reflection positivity. The two outputs are logically parallel; neither implies the other here (the bridge lemma remains open, Remark 6.3). *Version 2 corrects the status of the shared input:* v1 declared (H-CA) verified from Balaban's infrastructure; per the audited chain (2602.0053/0054 v2) that route is conditional on (H-DOB-blk)+(H-P0) and valid only in the volume window $L \leq e^{C/g_0^2}$. Accordingly all results are stated in two regimes: windowed (audit-backed conditional) and all-volume (under the strictly stronger bare hypothesis (H-CA)$_\infty$, required for the thermodynamic limit, Theorem C). What is unconditional and machine-verified: the curvature computation $\mathrm{Ric}_{SU(N)} = \frac{N}{4}$, the Holley–Stroock block-seed arithmetic, the variance/entropy decompositions and the failure of pointwise inheritance, the Dobrushin contraction machinery and its window arithmetic, the clustering⇒transfer-gap lemma on explicit operators, and the convergence of Cesi's geometric factor. All bounds remain explicit in N, g_0, η.

****Version 2.** Audited replacement of v1 within the programme-wide verification pass (2602.0041 v3, 2602.0051–0057 v2, 2602.0063 v3, 2602.0069–0077 v2, 2602.0087 v3, 2602.0088 v3). Principal changes: (i) **F-CA**: v1's "Theorem 4.2" ("Assumption 4.1 holds") declared the complete-analyticity input verified on a one-paragraph justification; per the audited chain (2602.0053/0054 v2) the Balaban-derived mixing is conditional on (H-DOB-blk)+(H-P0) *and windowed* — Theorem 4.2 is retagged as a conditional import, Assumption 4.1 becomes Hypothesis (H-CA). (ii) **F-WIN**: the audited route provides mixing only within the volume window $L \leq e^{C/g_0^2}$ ((H-XOVER)); v1's "uniformly in *all L*" and the $L \to \infty$ limit of Theorem C exceed it strictly — v2 distinguishes (H-CA)$_{\mathrm{win}}$ (audit-backed, windowed) from (H-CA)$_\infty$ (bare hypothesis, needed for Theorem C). (iii) **F-P0**: the profile exponent in Proposition 5.5 is tagged as the ledger's unpinned (H-P0) datum. (iv) **F-REF**: companions [2,3] flagged unaudited; v1's "[4] in preparation" is now 2602.0063 v3; the overlap with the audited chain is cross-cited and this paper is placed as the fixed-η consolidation of the LSI route (route 1 of 2602.0088 v3's two-route ledger). v1's honest self-limitations (Remarks 1.2, 5.2, 5.4, 6.3) are preserved verbatim. All exact content is machine-verified in the companion suite (`verification/2602-0089/`, 6/6 PASS).

1 Introduction and Main Results

1.1 Overview

This paper is the fixed-η consolidation of the LSI route of the programme (route 1 in the two-route ledger of 2602.0088 v3): it packages uniform LSI, uniform gap, and the thermodynamic limit as consequences of one mixing input. The continuum limit is 2602.0063 v3 (v1 cited it as "in preparation"). Overlapping territory — the LSI chain 2602.0041 v3/0051–0057 v2 and the gap/DLR pair 2602.0053/0054 v2 — is now cross-cited with audited version numbers; the ledger names used below are theirs.

1.2 The dual architecture

Shared input: (H-CA). Path A: (H-CA)⇒DS mixing⇒Cesi+local seed⇒LSI. Path B: (H-CA)⇒Dobrushin contraction⇒clustering⇒OS gap. The failure of the direct bridge A⇒B is Remark 6.3 (retained from v1, which was honest here).

1.3 Main theorems

Setup as in v1 §2 (Wilson action, Dirichlet form (1), OS transfer matrix (4)–(7); distances in lattice steps).

Hypothesis (H-CA) (v1's Assumption 4.1). There exist $g_* > 0$, $K > 0$, $m > 0$ depending only on N, d, η such that for all $g_0 \le g_*$, all finite V, all $x \in V$ and boundary conditions τ, σ differing only at distance $\ge r$ from x: $\left\| \frac{d\mu_V^\tau}{d\mu_V^\sigma}(\cdot \text{ at } x) - 1 \right\|_\infty \le K e^{-mr}$. *Two regimes:* $\textbf{(H-CA)}_{\text{win}}$: constants controlled for volumes $L \le L_*(g_0) = e^{C/g_0^2}$ — this is what the audited route (2602.0053/0054 v2) delivers, itself conditional on (H-DOB-blk)+(H-P0). $\textbf{(H-CA)}_\infty$: constants uniform in *all* L — a strictly stronger bare hypothesis, not backed by the audited chain; required for Theorem C.

Theorem 1.1 (Uniform LSI — conditional)**.** *Under (H-CA)*$_{\text{win}}$ *(resp. (H-CA)*$_\infty$*) there are* $g_*(N) > 0$ *and* $\hat\rho = \hat\rho(N, g_0, \eta) > 0$ *independent of* L *such that for* $g_0 \le g_*$ *and all* $L \le L_*(g_0)$ *(resp. all* L*):* $\mathrm{Ent}_{\mu_L}(f^2) \le \frac{2}{\hat\rho}\mathcal{E}_L(f, f)$*, with* $\hat\rho \ge \rho_0/((1 + C_{\text{DS}})M(\kappa, d))$*.*

Remark 1.2 (On constants; v1's Remark 1.2, retained). The proof yields positivity and L-uniformity (within the regime), not sharp asymptotics in g_0. The seed ρ_0 costs $e^{-C_1\ell_0/g_0^2}$ (Theorem 3.4) — the Holley–Stroock price already recorded in the audited LSI chain.

Theorem 1.3 (Volume-uniform mass gap — conditional)**.** *Under the same regimes, the OS Hamiltonian* H_L *satisfies* $\mathrm{spec}(H_L) \subset \{0\} \cup [m_0, \infty)$ *with* $m_0 = \kappa_t/\eta > 0$ *independent of* L *(within the regime).*

Theorem 1.4 (Thermodynamic limit — conditional on (H-CA)$_\infty$ only)**.** *Under (H-CA)*$_\infty$*,* $\mu_L \to \mu_\infty$ *weakly, the unique translation-invariant Gibbs state;* H_∞ *inherits the gap.* This theorem cannot be run on (H-CA)$_{\text{win}}$: the $L \to \infty$ limit exhausts every finite window *[suite check 6 prints the window sizes:* $L_* = e^{C/g_0^2}$ *is huge but finite at fixed* g_0*].*

2 Setup

As v1 (§2.1–2.3): Wilson measure, Dirichlet form with left-invariant gradients, OS transfer matrix $T_L = B^*B \geq 0$, $\hat{T}_L = T_L/\lambda_0$, $H_L = -\eta^{-1}\log \hat{T}_L$, connected temporal correlator $\langle F_0; G_n\rangle_{\mu_L} = (\hat{F}_0\Omega, \hat{T}_L^n P_{\Omega^\perp}\hat{G}_0\Omega)$.

3 Layer 1: Local Log-Sobolev Inequality

Proposition 3.1 (Curvature of $SU(N)^m$; v1 Prop. 3.1 — verified)**.** *With* $\langle X, Y\rangle = -2\mathrm{tr}(XY)$*:* $\mathrm{Ric}_{SU(N)}(X,X) = \frac{N}{4}\|X\|^2$*, fibrewise on products.*

The audit flagged the normalisation cascade in v1's Appendix A ("factor of $\frac{1}{2}$ relative to the Killing form") as fragile-looking; the suite settles it: exact algebra gives $\mathrm{Ric} = \frac{1}{2}$ for $SU(2)$ and numerically 0.7500 for $SU(3)$, i.e. $N/4$ in both cases — v1's constant is *correct* [check 1].

Lemma 3.2 (Haar LSI via Bakry–Émery, $V = 0$), Remark 3.3 (gauge directions; why Holley–Stroock instead of geodesic convexity) as in v1.

Theorem 3.2 (Local LSI on blocks; v1 Thm. 3.4)**.** *For blocks* $|\Delta| \leq \ell_0$ *and any boundary condition* τ*,* μ_Δ^τ *satisfies LSI(*ρ_0*) with* $\rho_0 = \rho_{\mathrm{Haar},\ell_0}\exp(-2C_1\ell_0/g_0^2) > 0$*,* $C_1 = 2N\,n_p$ *(*$n_p = 6$ *in* $d = 4$*), uniformly in* τ *and* L*.*

The oscillation count (13) and the Holley–Stroock degradation $e^{-\mathrm{osc}}$ are verified exactly (finite-chain spectral version, plus the arithmetic of (13)) [check 2]. The e^{-C/g_0^2} price is the same object as the audited chain's seed cost — no sharpness is claimed (Remark 1.2).

4 Complete Analyticity: The Shared Input

v1's Theorem 4.2 is retagged. v1 stated "Assumption 4.1 holds" as a theorem with a one-paragraph justification citing [13, 14, 17, 18]. Per the audited chain this is a *conditional import*:

Theorem 4.1 (Conditional import; v1 Thm. 4.2 retagged)**.** *Within the volume window* $L \leq e^{C/g_0^2}$ *and conditionally on (H-DOB-blk)+(H-P0) (2602.0053/0054 v2), the Balaban infrastructure (propagator decay [13], averaging [14], convergent expansions [17, 18]) yields (H-CA)*$_{\mathrm{win}}$*. No derivation of (H-CA)*$_\infty$ *from primary sources is on record.*

Proposition 4.3 (CA⇒DS mixing via Dobrushin contraction $\|C\|_{\ell^\infty\to\ell^\infty} \leq K\sum_r c_d r^3 e^{-mr} < 1$) survives as a clean implication; its threshold arithmetic, the exponential decay of $[(I-C)^{-1}C]$, and a window exhibit ($\delta(L)$ crossing 1 under audited-style constant degradation) are verified on explicit influence matrices [check 4].

5 Layer 2: Global LSI via Cesi (Theorem A)

Lemma 5.1 (law of total variance and its entropy analogue, integrated form) is exact and verified [check 3]; Remark 5.2 (no pointwise inheritance) is not just honest but *true in the strong sense*: the suite exhibits a conditioning value y^* with $\mathrm{Var}_{\mu(\cdot|y^*)}(f) > \mathrm{Var}_\mu(f)$ [check 3]. Theorem 5.3 (Cesi) and Remark 5.4 (correct citation: Prop. 2.1 + §4 of [11]; v1 already fixed a phantom "Theorem 1.3") as in v1; the geometric factor $M(\kappa, d) = \sum_k (1 + C_{\mathrm{DS}})^k e^{-\kappa 2^k} < \infty$ (double-exponential domination) is verified numerically for small κ and large C_{DS} [check 6]. The proof of Theorem 1.1 is v1's three steps, now reading (H-CA) in the appropriate regime.

Proposition 5.1 (Large-field suppression; v1 Prop. 5.5 — profile tagged). $\mu_L(b \in \Omega_{\mathrm{lf}}) \le C_0 e^{-cp_0(g_0)^2}$ *with* $p_0(g_0) = (-\log g_0)^{p_0}$. The exponent $p_0 > 0$ is the ledger's unpinned (H-P0) profile datum (cf. F-P0 of 2602.0088 v3); "universal" in v1 overstated its provenance.

6 Layer 3: Mass Gap via Clustering and OS (Theorem B)

Theorem 6.1 (temporal clustering from (H-CA) via the Dobrushin argument; $m \sim |\log g_0|$ is part of the *import*, not free) and Lemma 6.2 (clustering$\Rightarrow \|\hat{T}_L P_{\Omega^\perp}\| \le e^{-\kappa_t}$, using $T_L = B^*B \ge 0$, density of local vectors, and the spectral-measure argument) survive; the lemma's mechanics are verified exactly on explicit B^*B operators, including the n-th-root saturation and $m_0 = \kappa_t/\eta$ [check 5]. Remark 6.3 (no bridge from Theorem A to Theorem B; the $1/N_t$ obstruction) retained verbatim — the audit confirms this negative remark as correct and important.

7 Thermodynamic Limit (Theorem C)

Proposition 7.1 (tightness, Tychonoff) unconditional. Theorem 7.2 (existence, uniqueness, DLR, clustering of μ_∞) is proved exactly as in v1 *but requires (H-CA)$_\infty$*: the uniqueness argument (32) sends $\mathrm{diam}(\Lambda) \to \infty$ and the gap persistence takes $L \to \infty$, both of which leave every finite window. v2 makes this the only place where the stronger hypothesis enters — and Theorem C is therefore strictly more conditional than Theorems A/B.

8 Discussion

Architecture. The dual (parallel-outputs) structure and the single choke point survive the audit unchanged — indeed the audit *used* it: one retag (Theorem 4.1) propagates cleanly. *Placement.* Theorems A/B at fixed η within the window are the consolidation of 2602.0041 v3/0051–0057 v2 and 0053/0054 v2; nothing here strengthens those audited results — it packages them behind one named input. The bridge lemma (§8.2 of v1) remains open; the gradient-flow companions [2, 3] remain unaudited; the continuum step is 2602.0063 v3 with its own Assumption ledger.

Note added in v2

Findings F-CA (assumption declared theorem), F-WIN (all-L claims exceed the audited window; (H-CA)$_{\mathrm{win}}$ vs (H-CA)$_\infty$ split, Theorem C now visibly the most conditional result), F-P0 (profile tag on Prop. 5.5), F-REF (unaudited companions flagged; cross-citation of the audited chain). Confirmed correct under audit: Prop. 3.1 ($N/4$, suite-adjudicated), Rmk 5.2's integrated- only inheritance, Rmk 5.4's citation repair, Rmk 6.3's negative result. Numbering of v1 preserved. Suite: `verify_2602_0089.py`, 6 deterministic checks, exit 0 iff 6/6 (7/7-style conventions of the series).

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[21] L. Eriksson, ai.viXra:2602.0041 **v3** and 2602.0051–0057 **v2** (audited LSI chain).

[22] L. Eriksson, ai.viXra:2602.0053/0054 **v2** (audited gap/DLR pair; source of (H-DOB-blk), (H-P0), window).

[23] L. Eriksson, ai.viXra:2602.0088 **v3** (two-route ledger; this paper consolidates route 1).