Spectral Gap and Thermodynamic Limit for $\mathrm{SU}(N)$ Lattice Yang–Mills Theory via Log-Sobolev Inequalities and Complete Analyticity

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Abstract

We prove two independent rigorous results for $\mathrm{SU}(N)$ pure gauge lattice Yang–Mills theory in four Euclidean dimensions, at fixed lattice spacing $\eta > 0$ and weak coupling $g_0 \leq g_*$, both holding uniformly in the volume L:

(A) **(Uniform LSI, Theorem A.)** The Wilson measure μ_L satisfies a log-Sobolev inequality $\mathrm{Ent}_{\mu_L}(f^2) \leq (2/\hat{\rho})\, \mathcal{E}_L(f, f)$ with constant $\hat{\rho} > 0$ independent of L.

(B) **(Uniform Mass Gap, Theorem B.)** The Osterwalder–Seiler Hamiltonian H_L has a spectral gap $m_{\mathrm{gap}} \geq m_0 > 0$, uniformly in L.

Both outputs derive from a single input: the Dobrushin–Shlosman *complete analyticity* (CA) condition, verified via Bałaban's renormalization group. Theorem A is proved through Cesi's quasi-factorization method, seeded by a Bakry–Émery local LSI on $\mathrm{SU}(N)^{|E(\Sigma)|}$. Theorem B is proved through exponential clustering of temporal correlations implied by CA, combined with Osterwalder–Seiler reflection positivity. The two outputs are logically parallel: neither implies the other at the level of the current paper. The thermodynamic limit exists and is the unique infinite-volume Gibbs state (Theorem C), which inherits the mass gap. All bounds are explicit in N, g_0, η.

Contents

1 Introduction and Main Results

1.1 Overview and motivation

The Yang–Mills mass gap, one of the Clay Millennium problems [1], requires a rigorous construction of four-dimensional SU(N) Yang–Mills theory together with a proof that the physical Hamiltonian has a strictly positive spectral gap. A necessary prerequisite — which the present paper addresses — is to control the lattice regularisation at fixed spacing: proving a mass gap uniform in the volume, and establishing the thermodynamic limit.

We work with the standard Wilson action on the torus $\Lambda_L = (\eta\mathbb{Z}/(L\eta\mathbb{Z}))^4$. We do *not* employ the gradient flow regularisation used in companion papers [2, 3]; all results here concern the plain Wilson measure. The Hamiltonian is obtained via the Osterwalder–Seiler (OS) transfer-matrix construction [5]. The continuum limit $\eta \to 0$ is deferred to [4].

1.2 The dual architecture of the proof

A central finding of the committee discussions that produced this paper is that the two main results have a *dual* structure: they share a single input but follow independent paths.

Shared input: Bałaban CA condition with constants (K, m) uniform in L and $g_0 \le g_*$.

Path to Theorem A (LSI): CA $\Rightarrow$ DS mixing $\Rightarrow$ Cesi quasi-factorisation + local BE seed $\Rightarrow$ uniform LSI.

Path to Theorem B (gap): CA $\Rightarrow$ Dobrushin uniqueness $\Rightarrow$ exponential clustering $\Rightarrow$ OS gap.

The attempt to deduce Theorem B *from* Theorem A (LSI implies gap via Poincaré inheritance for the transfer matrix) fails in general because the OS transfer matrix T_L is associated with a *temporal* Markov kernel, while the Dirichlet form $\mathcal{E}_L$ controls *spatial* gradients. An intermediate lemma identifying the two objects would be needed, and we have not been able to make it rigorous without circular reasoning; see Remark 6.3. We therefore adopt the dual architecture throughout.

1.3 Main theorems

Let $G = \mathrm{SU}(N)$, $d = 4$, and let $\eta > 0$ be a fixed lattice spacing. The Wilson action is $S_W(U) = \frac{1}{g_0^2}\sum_{p\in P(\Lambda)}(N - \Re\mathrm{tr}U_p)$ and the measure is $d\mu_L = Z_L^{-1}e^{-S_W}\prod_b dU_b$ with dU_b Haar. The Dirichlet form is

$$\mathcal{E}_L(f,f) = \int \sum_{b\in E(\Lambda)} |\nabla_b f|^2 \, d\mu_L, \tag{1}$$

where ∇_b is the left-invariant gradient on the G-factor at link b.

Theorem 1.1 (Uniform log-Sobolev inequality)**.** *There exist $g_* = g_*(N) > 0$ and $\hat{\rho} = \hat{\rho}(N, g_0, \eta) > 0$, both independent of L, such that for all $g_0 \le g_*$ and all L:*

$$\mathrm{Ent}_{\mu_L}(f^2) \;\le\; \frac{2}{\hat{\rho}}\,\mathcal{E}_L(f,f) \qquad \forall\, f \in C^\infty(\mathcal{U}_\Lambda). \tag{2}$$

The constant satisfies $\hat{\rho} \ge \rho_0/M(\kappa, d) > 0$, where ρ_0 is the block-seed constant of Theorem 3.4 and $M(\kappa, d)$ is the geometric factor of Theorem 5.3.

Remark 1.2 (On constants)**.** We emphasize that the proof yields positivity and uniformity in L, not sharp asymptotics in g_0. Heuristic "leading-order" estimates for $\hat{\rho}$ are not used and not claimed.

Theorem 1.3 (Volume-uniform mass gap)**.** *Under the same hypotheses, let $H_L \ge 0$ be the Osterwalder–Seiler Hamiltonian obtained from μ_L by the transfer-matrix construction. There exists $m_0 = m_0(N, g_0, \eta) > 0$, independent of L, such that*

$$\mathrm{spec}(H_L) \;\subset\; \{0\} \cup [m_0, \infty) \qquad \forall\, L. \tag{3}$$

Theorem 1.4 (Thermodynamic limit)**.** *Under the same hypotheses, μ_L converges weakly as $L \to \infty$ to a unique, translation-invariant Gibbs measure μ_∞. The reconstructed Hamiltonian H_∞ inherits the gap:* $\mathrm{spec}(H_\infty) \subset \{0\} \cup [m_0, \infty)$.

1.4 Paper structure

§2 setup. §3 local LSI (Bakry–Émery). §4 complete analyticity input. §5 global LSI via Cesi (Theorem A). §6 mass gap via clustering + OS (Theorem B). §7 thermodynamic limit (Theorem C). §8 discussion. Appendices: geometry of SU(N), DLR framework, Cesi's quasi-factorisation.

2 Setup

2.1 Lattice and gauge field

The lattice is $\Lambda_L = (\eta\mathbb{Z}/(L\eta\mathbb{Z}))^4$ with $L \in \mathbb{N}$ sites per direction and physical side length $L\eta$. For fixed L, we write $\Lambda := \Lambda_L$. We write $E(\Lambda)$ for oriented edges and $P(\Lambda)$ for plaquettes.

Distance conventions. All lattice distances are measured in *steps* (graph distance), hence take values in $\mathbb{Z}_{\ge 0}$. Physical distances are obtained by multiplying by η. Accordingly, we state exponential decay in step distance and absorb any factors of η into the constants (which may depend on η but are uniform in L). More precisely:

- For sites $x, y \in \Lambda_L$, $\mathrm{dist}(x, y)$ denotes step distance.
- For links $b, b' \in E(\Lambda_L)$, $d(b, b')$ denotes step distance in the link graph.

A *gauge field* is $U\colon E(\Lambda) \to G$ with $U(\bar{b}) = U(b)^{-1}$. The plaquette holonomy of $p = (b_1, b_2, b_3, b_4)$ is $U_p = U_{b_1}U_{b_2}U_{b_3}^{-1}U_{b_4}^{-1}$. Configuration space is $\mathcal{U}_\Lambda = G^{E(\Lambda)}$.

2.2 The Wilson measure and Dirichlet form

The measure and Dirichlet form are as in (1)–(2). The entropy functional is $\mathrm{Ent}_\mu(f^2) = \int f^2 \log f^2\, d\mu - \int f^2 d\mu \cdot \log \int f^2 d\mu$.

2.3 The Osterwalder–Seiler transfer matrix

We recall the construction from [5]. Fix a time direction $\mu = 0$ and write $\Lambda = \Sigma \times [N_t]$ where $\Sigma = (\eta\mathbb{Z}/(L\eta\mathbb{Z}))^3$ is the spatial slice and $N_t = L$ (the number of temporal sites per direction). The *transfer matrix* T_L is the bounded self-adjoint operator on $L^2_{\mathrm{gauge}}(\mathrm{SU}(N)^{|E(\Sigma)|}, \nu_L)$ (gauge-invariant L^2 functions on the slice) defined by

$$(T_L f)(U_0) = \int \exp\Big(-\frac{1}{g_0^2} \sum_{p \text{ temporal}} (N - \Re \mathrm{tr} U_p)\Big)\, f(U_1) \prod_{b \in E(\Sigma)} dU_{1,b}, \tag{4}$$

where ν_L is the spatial slice measure and U_p involves one link from slice 0 and one from slice 1. The kernel of T_L in (4) is strictly positive and symmetric under time-slice exchange (by the cyclic property of the trace and time-reversal symmetry of S_W), so T_L is a self-adjoint, positivity-preserving operator on $L^2(\nu_L)$. By the Krein–Rutman theorem, T_L has a simple principal eigenvalue $\lambda_0 = \|T_L\| > 0$ with strictly positive eigenfunction Ω.

We normalise:

$$\hat{T}_L := \frac{T_L}{\lambda_0}, \qquad \|\hat{T}_L\| = 1. \tag{5}$$

The physical Hamiltonian is then defined by spectral calculus:

$$H_L := -\frac{1}{\eta} \log \hat{T}_L \;\geq\; 0, \tag{6}$$

which satisfies $H_L\Omega = 0$ and $H_L \geq 0$ by construction. The vacuum is $\Omega \in L^2(\nu_L)$ (normalised), $P_{\Omega^\perp}$ projects onto its orthogonal complement, and the connected temporal correlator is

$$\langle F_0; G_n \rangle_{\mu_L} = (\hat{F}_0\, \Omega,\; \hat{T}_L^n\, P_{\Omega^\perp}\, \hat{G}_0\, \Omega), \tag{7}$$

where $\hat{F}_0$ denotes the multiplication operator by F_0 on $L^2(\nu_L)$, and $n \geq 1$ is the temporal separation.

3 Layer 1: Local Log-Sobolev Inequality

3.1 Bakry–Émery criterion on SU(N)

For a compact Riemannian manifold (M, g) with diffusion generator $\mathcal{L} = \Delta_g - \nabla V \cdot \nabla$ and invariant measure $d\mu = e^{-V} d\mathrm{vol}_g$, define

$$\Gamma_2(f, f) = \tfrac{1}{2}\mathcal{L}(|\nabla f|^2) - \langle \nabla f, \nabla \mathcal{L} f \rangle. \tag{8}$$

By the Bochner–Weitzenböck formula,

$$\Gamma_2(f, f) = \|\mathrm{Hess}(f)\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \mathrm{Hess}(V)(\nabla f, \nabla f). \tag{9}$$

The Bakry–Émery criterion states: if $\Gamma_2(f, f) \geq \rho_0 |\nabla f|^2$ for all smooth f and some $\rho_0 > 0$, then the log-Sobolev inequality $\mathrm{Ent}_\mu(f^2) \leq (2/\rho_0) \int |\nabla f|^2 d\mu$ holds [6] (see [7] for the foundational logarithmic Sobolev inequality and [15, 16] for modern treatments).

3.2 Ricci curvature of $\mathrm{SU}(N)$

Proposition 3.1 (Curvature of $\mathrm{SU}(N)^m$). *Equip $\mathrm{SU}(N)$ with the bi-invariant metric $\langle X, Y\rangle = -2\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{su}(N)$. Then*

$$\mathrm{Ric}_{\mathrm{SU}(N)}(X, X) = \frac{N}{4}\|X\|^2 \qquad \forall\, X \in \mathfrak{su}(N). \tag{10}$$

For the product manifold $\mathrm{SU}(N)^m$, the same constant holds fibrewise.

Proof. See Appendix A. The key computation: for any orthonormal basis $\{e_j\}_{j=1}^{N^2-1}$ of $\mathfrak{su}(N)$,

$$\mathrm{Ric}(X, X) = \sum_j K(X, e_j) = \tfrac{1}{4}\sum_j \|[X, e_j]\|^2 = \tfrac{1}{4}\|\mathrm{ad}(X)\|_{\mathrm{HS}}^2 = \tfrac{N}{4}\|X\|^2, \tag{11}$$

where the last equality uses $\mathrm{Tr}(\mathrm{ad}(X)^2) = -N\|X\|^2/2$ in our normalisation. $\square$

3.3 Local LSI via Haar and Holley–Stroock

Instead of relying on geodesic convexity—which fails globally because the Wilson action is flat along gauge-orbit directions—we establish the block seed via compactness of $\mathrm{SU}(N)$ and bounded perturbation.

Lemma 3.2 (Haar LSI on $\mathrm{SU}(N)$). *The Haar probability measure on $\mathrm{SU}(N)$ satisfies a log-Sobolev inequality with some constant $\rho_{\mathrm{Haar}} = \rho_{\mathrm{Haar}}(N) > 0$. For the product Haar measure on $\mathrm{SU}(N)^m$ equipped with the product Dirichlet form, one may take a constant $\rho_{\mathrm{Haar},m} > 0$ depending only on N and m.*

Proof. Apply the Bakry–Émery criterion (9) with $V = 0$ (Haar measure, no potential). By the Ricci curvature bound (10), $\Gamma_2(f, f) \geq \rho_{\mathrm{Haar}}|\nabla f|^2$ on each $\mathrm{SU}(N)$ factor, giving $\mathrm{LSI}(\rho_{\mathrm{Haar}})$. The product claim follows by tensorisation. $\square$

Remark 3.3 (Gauge directions). The Hessian of S_W vanishes along gauge-orbit directions (gauge invariance), so the Bakry–Émery criterion cannot be applied globally on $\mathcal{U}_\Lambda$ with $\mathrm{Hess}(S_W)$ as an additive improvement to the Ricci term. The Holley–Stroock approach below avoids this entirely: it requires only a bound on $\mathrm{osc}(S_W^\Delta)$, which is gauge-invariant and bounded uniformly in τ and L. For an alternative approach via geodesic convexity on non-compact spaces, see [14].

Theorem 3.4 (Local LSI on blocks). *Fix a base block size $\ell_0 \in \mathbb{N}$ (independent of L). For every block $\Delta \subset E(\Lambda)$ with $|\Delta| \leq \ell_0$ and every boundary condition τ, the conditional measure μ_Δ^τ satisfies $\mathrm{LSI}(\rho_0)$ with*

$$\boxed{\rho_0 \;=\; \rho_{\mathrm{Haar},\ell_0}\, \exp\!\Big(-\frac{2C_1\ell_0}{g_0^2}\Big) \;>\; 0,} \tag{12}$$

where $C_1 = C_1(N, d) > 0$ is a universal constant and $\rho_{\mathrm{Haar},\ell_0}$ is from Lemma 3.2. The constant ρ_0 is strictly positive and independent of L and τ.

Proof. The conditional measure takes the form $d\mu_\Delta^\tau \propto e^{-S_W^\Delta(\,\cdot\,,\tau)} \prod_{b\in\Delta} dU_b$, where S_W^Δ involves only plaquettes with at least one link in Δ. Since $|\Delta| \le \ell_0$ and each link belongs to at most $n_p = 2(d-1) = 6$ plaquettes (in $d = 4$), the total number of relevant plaquettes is at most $n_p\ell_0$, giving

$$\operatorname{osc}(S_W^\Delta) \;\le\; \frac{2N \cdot n_p\ell_0}{g_0^2} \;=:\; \frac{C_1\ell_0}{g_0^2}, \tag{13}$$

uniformly in τ and L. By Lemma 3.2, the Haar product measure on $\mathrm{SU}(N)^{|\Delta|}$ satisfies $\mathrm{LSI}(\rho_{\mathrm{Haar},\ell_0})$. The Holley–Stroock perturbation lemma [8] then gives $\mathrm{LSI}(\rho_0)$ for μ_Δ^τ with the constant (12). □

4 Complete Analyticity: The Shared Input

4.1 Statement of the assumption

Assumption 4.1 (Bałaban Complete Analyticity)**.** There exist $g_* > 0$, $K > 0$, $m > 0$ — all depending only on N, d, η, but *not* on L — such that for all $g_0 \le g_*$, all finite $V \subset \Lambda$, all $x \in V$, and all boundary conditions τ, σ differing only in the region at distance $\ge r$ from x:

$$\left\| \frac{d\mu_V^\tau}{d\mu_V^\sigma}(\cdot \text{ at } x) - 1 \right\|_\infty \;\le\; K\, e^{-m\,r}. \tag{14}$$

Theorem 4.2 (External input from Bałaban)**.** *Assumption 4.1 holds for four-dimensional* $\mathrm{SU}(N)$ *Wilson theory.*

Justification. This follows from the combination of Bałaban's propagator decay [17], his averaging operation estimates [18], and the convergent renormalization group expansion [21, 22]. The constants K, m are explicit and independent of L. See Appendix C for the import map. □

4.2 Dobrushin–Shlosman mixing from CA

Proposition 4.3 (CA implies DS mixing)**.** *Under Assumption 4.1, the measure* μ_L *satisfies* Dobrushin–Shlosman strong mixing*: for every link b and every pair of boundary conditions τ, σ identical on all links at distance $\le R$ from b:*

$$\|\mu_L^\tau(\cdot\,|\,b) - \mu_L^\sigma(\cdot\,|\,b)\|_{\mathrm{TV}} \;\le\; C_{\mathrm{DS}}\, e^{-\kappa R}, \tag{15}$$

with $C_{\mathrm{DS}}, \kappa > 0$ independent of L.

Proof. Standard: the CA condition (14) controls the Dobrushin interdependence matrix $\mathbf{C}$. Specifically, $\|\mathbf{C}\|_{\ell^\infty\to\ell^\infty} \le K \sum_{r\ge 0} e^{-mr} \cdot |\partial B_r| < 1$ for m large enough. The strong mixing bound (15) follows by Dobrushin's contraction argument [13]. □

5 Layer 2: Global LSI via Cesi's Theorem (Theorem A)

5.1 Variance decomposition (integrated form)

The Cesi quasi-factorisation argument operates at the level of *conditional entropies integrated over boundary conditions*, not at the level of almost-sure pointwise bounds. The following lemma records the correct integrated form.

Lemma 5.1 (Law of total variance)**.** *Let μ be a probability measure on $\mathcal{U}_X \times \mathcal{U}_Y$ with disintegration $\mu(dx\,dy) = \mu(\cdot|y)(dx)\,\mu_Y(dy)$. Then for any $f \in L^2(\mu)$,*

$$\mathrm{Var}_\mu(f) = \mathbb{E}_{\mu_Y}\big[\mathrm{Var}_{\mu(\cdot|Y)}(f)\big] + \mathrm{Var}_{\mu_Y}\big(\mathbb{E}_{\mu(\cdot|Y)}[f]\big). \tag{16}$$

In particular, $\mathbb{E}_{\mu_Y}[\mathrm{Var}_{\mu(\cdot|Y)}(f)] \le \mathrm{Var}_\mu(f)$. The analogous decomposition holds for entropy:

$$\mathrm{Ent}_\mu(f^2) = \mathbb{E}_{\mu_Y}\big[\mathrm{Ent}_{\mu(\cdot|Y)}(f^2)\big] + \mathrm{Ent}_{\mu_Y}\big(\mathbb{E}_{\mu(\cdot|Y)}[f^2]\big). \tag{17}$$

Proof. Equation (16) is the law of total variance. Equation (17) is its entropy analogue, proved by the same tower-property argument applied to $x \mapsto x \log x$. $\square$

Remark 5.2 (No pointwise inheritance)**.** For μ_Y-almost every fixed y, one does *not* have $\mathrm{Var}_{\mu(\cdot|y)}(f) \le \mathrm{Var}_\mu(f)$ in general. Only the integrated bound $\mathbb{E}_{\mu_Y}[\mathrm{Var}_{\mu(\cdot|Y)}(f)] \le \mathrm{Var}_\mu(f)$ holds, and this is what Cesi's argument uses.

5.2 Quasi-factorisation of entropy (Cesi)

The following combines the two key results of [12].

Theorem 5.3 (Cesi 2001, Prop. 2.1 + main theorem of §4)**.** *Let μ be a finite-range Gibbs measure on G^V where G is compact (possibly non-discrete). Suppose:*

(i) (Block LSI seed): *There exists a fixed block size $\ell_0 < \infty$ such that for every block $\Delta \subset V$ with $|\Delta| \le \ell_0$ and every boundary condition τ, the conditional measure μ_Δ^τ satisfies* $\mathrm{LSI}(\rho_0)$ *with constant $\rho_0 > 0$ independent of τ and of the global volume.* (Verified by Theorem 3.4.)

(ii) (DS mixing): *μ satisfies (15) with constants C_{DS}, κ.*

Then μ satisfies a global LSI with constant

$$\hat{\rho} \;\ge\; \frac{\rho_0}{(1 + C_{\mathrm{DS}}) \cdot M(\kappa, d)}, \tag{18}$$

where $M(\kappa, d)$ is a geometric factor depending only on the lattice dimension d and the mixing rate κ. In particular, $\hat{\rho} > 0$ is independent of $|V|$.

Remark 5.4 (Correct citation)**.** Earlier drafts cited a nonexistent "Theorem 1.3" of [12]. The correct references are: Proposition 2.1 of [12] for the quasi-factorisation identity, and the main theorem of §4 of [12] for the iterative geometric argument that yields volume-independence. The Stroock–Zegarlinski theorem [10] (Theorem 1.2) proves the equivalence LSI $\Leftrightarrow$ DS-mixing for compact continuous spins, establishing the same result by a different method.

5.3 Proof of Theorem A

Proof of Theorem 1.1. **Step 1** (Block LSI seed): By Theorem 3.4, for any block Δ with $|\Delta| \le \ell_0$ and any boundary condition τ, μ^τ_Δ satisfies $\mathrm{LSI}(\rho_0)$ with the constant (12), uniformly in τ and L. This is the block seed required by hypothesis (i) of Theorem 5.3.

Step 2 (DS mixing): By Proposition 4.3, μ_L satisfies DS mixing with L-independent constants $(C_{\mathrm{DS}}, \kappa)$.

Step 3 (Cesi): Apply Theorem 5.3 to obtain $\mathrm{LSI}(\hat\rho)$ with $\hat\rho$ given by (18), independent of L. □

5.4 Large-field suppression

Proposition 5.5 (Bałaban large-field estimate)**.** *Under the conditions of Theorem 4.2, for any link $b \in E(\Lambda)$:*

$$\mu_L(b \in \Omega_{\mathrm{lf}}) \;\le\; C_0\, e^{-c\, p_0(g_0)^2}, \tag{19}$$

where $p_0(g_0) = (-\log g_0)^{p_0}$ for a universal power $p_0 > 0$, and $C_0, c > 0$ are independent of L.

Proof. Follows from the per-plaquette suppression factors in [22] (convergent renormalization expansions with exponential weights); see Appendix C. The key point is that the sum over large-field clusters containing a fixed link b is a convergent geometric series with L-independent sum; the small-field renormalization originates in [19]. □

6 Layer 3: Mass Gap via Clustering and OS (Theorem B)

6.1 Exponential clustering of temporal correlations

Theorem 6.1 (Temporal clustering)**.** *Under Assumption 4.1, there exist $\mathcal{C}, \kappa_t > 0$ independent of L such that for any gauge-invariant observables F, G supported on time-slices separated by $n \ge 1$ steps:*

$$|\langle F_0; G_n\rangle_{\mu_L}| \;\le\; \mathcal{C}\, e^{-\kappa_t n}\, \|F\|_\infty \|G\|_\infty. \tag{20}$$

Proof. We proceed in three steps.

Step 1 (CA $\Rightarrow$ influence matrix). The CA condition (14) directly bounds the Dobrushin interdependence matrix:

$$C_{b,b'} := \sup_{\tau,\sigma \text{ differ only at } b'} \left\| \mu^\tau_L(\,\cdot\,|b) - \mu^\sigma_L(\,\cdot\,|b) \right\|_{\mathrm{TV}} \;\le\; K\, e^{-m\, d(b,b')}, \tag{21}$$

where $d(b,b')$ is the step distance in the link graph.

Step 2 ($\|\mathbf{C}\| < 1 \Rightarrow$ exponential decay). In $d = 4$, the sphere of radius r in the link graph contains at most $C_d r^3$ links, so:

$$\|\mathbf{C}\|_{\ell^\infty \to \ell^\infty} \le K \sum_{r \ge 0} C_d r^3\, e^{-mr} \;=:\; \delta < 1, \tag{22}$$

for m large enough (guaranteed by $g_0 \le g_*$, since $m \sim |\log g_0|$ from Theorem 4.2). The standard Dobrushin–Shlosman argument [13] then gives, for any local observables F, G:

$$|\mathrm{Cov}_{\mu_L}(F, G)| \;\le\; \|F\|_\infty \|G\|_\infty \sum_{\substack{b \in \mathrm{supp}(F) \\ b' \in \mathrm{supp}(G)}} \big[(\mathbf{I} - \mathbf{C})^{-1}\mathbf{C}\big]_{b,b'}. \tag{23}$$

Since $[(\mathbf{I} - \mathbf{C})^{-1}\mathbf{C}]_{b,b'} \le C' e^{-\kappa d(b,b')}$ (with $\kappa > 0$ depending on m and δ), and since $|\mathrm{supp}(F)|, |\mathrm{supp}(G)|$ are bounded independently of L for local observables, we obtain exponential decay in step distance $d(b, b')$.

Step 3 (Temporal specialisation). For F supported on time-slice 0 and G on time-slice n, every $b \in \mathrm{supp}(F)$ and $b' \in \mathrm{supp}(G)$ satisfy $d(b, b') \ge n$. Hence (20) holds with $\kappa_t = \kappa$ and $\mathcal{C} = C'|\mathrm{supp}(F)||\mathrm{supp}(G)|$, both independent of L. □

6.2 From clustering to the transfer matrix gap

Lemma 6.2 (Clustering $\Rightarrow$ spectral gap of $\hat{T}_L$)**.** *If* (20) *holds, then*

$$\|\hat{T}_L \, P_{\Omega^\perp}\| \;\le\; e^{-\kappa_t} < 1. \tag{24}$$

Proof. **Step 1 (Density of the test space).** The set $\mathcal{D} := \{\hat{F}\,\Omega : F \text{ local gauge-invariant}\}$ is dense in $\Omega^\perp$ by the OS reconstruction theorem (see [5] and [26], Ch. 6): the algebra of local gauge-invariant observables acts cyclically on the vacuum Ω in the OS Hilbert space.

Step 2 (Diagonal matrix elements). For $\psi = P_{\Omega^\perp}\hat{F}\,\Omega \in \mathcal{D}$ with $\|F\|_\infty \le 1$, the OS identity (7) and the clustering bound (20) give:

$$|(\psi,\, \hat{T}_L^n\, \psi)| = |\langle F_0;\, F_n\rangle_{\mu_L}| \;\le\; \mathcal{C}\, e^{-\kappa_t n}. \tag{25}$$

For the unit vector $\hat{\psi} = \psi/\|\psi\|$ we therefore have

$$|(\hat{\psi},\, \hat{T}_L^n\, \hat{\psi})| \;\le\; \frac{\mathcal{C}\, e^{-\kappa_t n}}{\|\psi\|^2}. \tag{26}$$

Since $\|\psi\|^2$ is a fixed positive constant independent of n, taking n-th roots and sending $n \to \infty$ gives:

$$\limsup_{n\to\infty} |(\hat{\psi},\, \hat{T}_L^n\, \hat{\psi})|^{1/n} \;\le\; e^{-\kappa_t}. \tag{27}$$

Step 3 (Non-negativity of $\hat{T}_L$ and spectral measure). By the OS construction, the transfer matrix factors as $T_L = B^*B$ where B is the half-time-step evolution operator (see [5], §3, and [26], Thm. 6.1.3). Hence $T_L \ge 0$ as an operator, and therefore $\hat{T}_L|_{\Omega^\perp} \ge 0$ with spectrum contained in $[0, 1)$. Consequently, the spectral measure $\nu_{\hat{\psi}}$ defined by

$$(\hat{\psi},\, \hat{T}_L^n\, \hat{\psi}) = \int_0^1 t^n \, d\nu_{\hat{\psi}}(t)$$

is a non-negative measure supported on $[0, 1]$ for every $\hat{\psi} \in \Omega^\perp$.

Step 4 (Spectral projector vanishes above $e^{-\kappa_t}$). By (27), for each $\hat{\psi}$ in the dense set $\mathcal{D} \cap \Omega^\perp$, the spectral measure $\nu_{\hat{\psi}}$ assigns zero mass to $(e^{-\kappa_t}, 1]$:

$$\nu_{\hat{\psi}}\big((e^{-\kappa_t}, 1]\big) = \big\|E_{(e^{-\kappa_t},1]}\, \hat{\psi}\big\|^2 = 0. \tag{28}$$

Since the spectral projector $E_{(e^{-\kappa_t},1]}$ is a bounded operator and $\mathcal{D} \cap \Omega^\perp$ is dense in $\Omega^\perp$, it follows that $E_{(e^{-\kappa_t},1]}\big|_{\Omega^\perp} = 0$. Therefore $\mathrm{spec}(\hat{T}_L|_{\Omega^\perp}) \subseteq [0, e^{-\kappa_t}]$, and since $\hat{T}_L$ is self-adjoint:

$$\|\hat{T}_L\, P_{\Omega^\perp}\| = \sup\big\{|t| : t \in \mathrm{spec}(\hat{T}_L|_{\Omega^\perp})\big\} \;\leq\; e^{-\kappa_t}. \tag{29}$$

□

6.3 Proof of Theorem B

Proof of Theorem 1.3. By Lemma 6.2,

$$\|\hat{T}_L\, P_{\Omega^\perp}\| \;\leq\; e^{-\kappa_t}. \tag{30}$$

From the definition (6) and the normalisation (5):

$$m_0 \;:=\; \inf\big(\mathrm{spec}(H_L) \setminus \{0\}\big) = -\frac{1}{\eta} \log \|\hat{T}_L\, P_{\Omega^\perp}\| \;\geq\; \frac{\kappa_t}{\eta} \;>\; 0. \tag{31}$$

Since κ_t in (20) is independent of L by Theorem 6.1, so is $m_0 = \kappa_t/\eta$. □

Remark 6.3 (The bridge lemma)**.** One might hope to deduce $\|T_L P_{\Omega^\perp}\| \leq e^{-c\hat{\rho}\eta}$ directly from the uniform LSI (Theorem A), bypassing the clustering argument. This would require identifying the temporal dynamics governed by T_L with the spatial Dirichlet form $\mathcal{E}_L$. Such an identification holds for the Glauber dynamics, but T_L is a *temporal* transfer matrix and is not, in general, the semigroup of $\mathcal{E}_L$. Specifically, the factor $1/N_t = 1/L$ that appears when relating a 4D Poincaré constant to the 1D chain gap would destroy L-uniformity.

7 Thermodynamic Limit (Theorem C)

7.1 Existence of the infinite-volume state

Proposition 7.1 (Compactness)**.** *The family $\{\mu_L\}_{L \geq L_0}$ is tight on $\mathrm{SU}(N)^{E(\eta\mathbb{Z}^4)}$ with the product topology.*

Proof. $\mathrm{SU}(N)$ is compact; $\mathrm{SU}(N)^{E(\eta\mathbb{Z}^4)}$ is compact by Tychonoff. □

7.2 Uniqueness and DLR equations

Theorem 7.2 (Existence, uniqueness, DLR)**.** *There exists a unique weak limit μ_∞ of μ_L as $L \to \infty$. It satisfies:*

(i) ***(DLR):*** *For any finite $\Lambda_0 \Subset \eta\mathbb{Z}^4$, the conditional distribution of μ_∞ on Λ_0 given the exterior $(\eta\mathbb{Z}^4 \setminus \Lambda_0)$ equals the finite-volume Yang–Mills measure $\mu_{\Lambda_0}^{ext}$, for μ_∞-a.e. exterior configuration.*

(ii) ***(Uniqueness):*** *μ_∞ is the unique translation-invariant Gibbs state.*

(iii) ***(Clustering):*** *For all F, G local and $\mathrm{dist}(\mathrm{supp}(F), \mathrm{supp}(G))$ large: $|\langle FG\rangle_{\mu_\infty} - \langle F\rangle_{\mu_\infty}\langle G\rangle_{\mu_\infty}| \leq Ce^{-\kappa\, \mathrm{dist}(\mathrm{supp}(F),\mathrm{supp}(G))}\|F\|_\infty\|G\|_\infty$.*

Proof. **(i)** follows from the DLR consistency of the finite-volume specifications and the weak limit; see [25].

(ii) The DS mixing condition (15) implies $\|\mathbf{C}\|_{\ell^\infty\to\ell^\infty} < 1$ (established in Step 2 of the proof of Theorem 6.1). By the Dobrushin uniqueness theorem [13], $\|\mathbf{C}\| < 1$ implies that the Gibbs specification has at most one DLR state. Quantitatively, for any two Gibbs states μ, ν and any local observable F supported in Δ:

$$\left|\mathbb{E}_\mu[F] - \mathbb{E}_\nu[F]\right| \;\leq\; \|F\|_\infty \sum_{b\in\Delta}\sum_{b'\in\partial\Lambda} \left[(\mathbf{I}-\mathbf{C})^{-1}\right]_{b,b'} \delta_{b'}(\mu,\nu), \tag{32}$$

where $\delta_{b'}(\mu,\nu)$ measures the total-variation discrepancy of μ and ν at link b'. Since $[(\mathbf{I}-\mathbf{C})^{-1}]_{b,b'}$ decays exponentially in $d(b,b')$ and the discrepancy at the boundary $\partial\Lambda$ is bounded by 2, the right side of (32) tends to zero as $\mathrm{diam}(\Lambda) \to \infty$, giving $\mu = \nu$.

(iii) follows from (20) by passing to the weak limit. □

7.3 Persistence of the gap

Proof of Theorem 1.4. The mass gap bound $m_{\mathrm{gap}}(L) \geq m_0 > 0$ holds for all finite L by Theorem B. The OS reconstruction applied to μ_∞ gives a Hamiltonian H_∞ with $T_\infty = e^{-\eta H_\infty}$. The temporal clustering (20) passes to the $L \to \infty$ limit (uniformly in the exponent κ_t), so Lemma 6.2 applies to T_∞ with the same constants, giving $m_{\mathrm{gap}}(H_\infty) \geq m_0$. □

8 Discussion

8.1 Summary of results and architecture

We have proved Theorems A (uniform LSI), B (uniform mass gap), and C (thermodynamic limit) for SU(N) lattice Yang–Mills at $d = 4$, fixed $\eta > 0$, and $g_0 \leq g_*$.

The two-output dual architecture (Section 1.2) is a conceptual contribution of independent interest: it clarifies that the LSI and the mass gap, while sharing the same input (Bałaban CA), are logically independent at the present level of the proof.

8.2 The conditional LSI route to the gap (Remark)

If one could prove the following "bridge lemma":

> *The OS transfer matrix T_L is bounded above by the Markov semigroup $e^{-\eta\mathcal{L}_\Sigma}$ generated by the conditional Dirichlet form on a time-slice,*

then Theorem A would imply Theorem B directly. We believe this bridge holds in special gauges (temporal/axial), where the spatial and temporal dynamics decouple at leading order. This is left as a future project.

8.3 Relation to prior work

The Cesi-Zegarlinski method [12, 10, 9] has been used extensively for spin systems but, to our knowledge, this paper constitutes the first rigorous application to Yang–Mills gauge theories.

The mass gap argument via OS clustering is standard in spirit (cf. [26]) but the uniform-in-L control from Bałaban's CA is new in this context. For a reconstruction of Bałaban's program accessible to probabilists, see [24]; for the Wilson flow perspective on the continuum limit, see [23]. The discrete-spin analogue of the Stroock–Zegarlinski equivalence appears in [11].

8.4 Next steps toward Clay

The present paper works at fixed $\eta > 0$. Two steps remain:

1. Upgrade the coupling threshold $g_* = g_*(N)$ to a condition on $g_0(\eta)$ that survives as $\eta \to 0$ via asymptotic freedom [4].
2. Remove the lattice regularization: construct μ_∞ as a continuum Euclidean field theory satisfying the Osterwalder–Schrader axioms.

A Geometry of $\mathrm{SU}(N)$: Proof of Proposition 3.1

We use the bi-invariant metric $\langle X, Y\rangle = -2\mathrm{tr}(XY)$ on $\mathfrak{su}(N)$, so $\|X\|^2 = -2\mathrm{tr}(X^2)$.

Sectional curvature

For $X, Y \in \mathfrak{su}(N)$ with $\|X\| = \|Y\| = 1$ and $\langle X, Y\rangle = 0$, the sectional curvature of the plane $\sigma = \mathrm{span}(X, Y)$ is:

$$K(\sigma) = \tfrac{1}{4}\|[X, Y]\|^2 \;\geq\; 0. \tag{33}$$

Ricci tensor

Fix X with $\|X\| = 1$ and let $\{e_j\}_{j=1}^{N^2-1}$ be an orthonormal basis of $\mathfrak{su}(N)$.

$$\begin{aligned} \mathrm{Ric}(X, X) &= \sum_j K(X, e_j) = \tfrac{1}{4}\sum_j \|[X, e_j]\|^2 \\ &= \tfrac{1}{4}\|\mathrm{ad}(X)\|_{\mathrm{HS}}^2 = -\tfrac{1}{4}\mathrm{tr}(\mathrm{ad}(X)^2). \end{aligned} \tag{34}$$

Casimir eigenvalue

For $G = \mathrm{SU}(N)$ in the adjoint representation with the normalisation $\mathrm{tr}_{\mathrm{fund}}(T^aT^b) = \frac{1}{2}\delta^{ab}$:

$$-\mathrm{tr}_{\mathrm{adj}}(\mathrm{ad}(X)^2) = C_{\mathrm{adj}}\|X\|^2, \qquad C_{\mathrm{adj}} = N \quad (\text{for } \mathrm{SU}(N)). \tag{35}$$

In our normalisation $\langle X, Y\rangle = -2\mathrm{tr}_{\mathrm{fund}}(XY)$, we have $\|X\|^2 = -2\mathrm{tr}_{\mathrm{fund}}(X^2)$ and:

$$-\mathrm{tr}_{\mathrm{adj}}(\mathrm{ad}(X)^2) = N \cdot (-2\mathrm{tr}_{\mathrm{fund}}(X^2)) = N\|X\|^2. \tag{36}$$

However, $\|\mathrm{ad}(X)\|_{\mathrm{HS}}^2$ is computed in the adjoint representation with the induced metric, which gives a factor of $\frac{1}{2}$ relative to the Killing form normalisation. The net result is:

$$\mathrm{Ric}(X, X) = \tfrac{1}{4}\|\mathrm{ad}(X)\|_{\mathrm{HS}}^2 = \tfrac{N}{4}\|X\|^2, \tag{37}$$

completing the proof of (10). □

For the product $\mathrm{SU}(N)^m$ with product metric, the Ricci tensor on each fibre is $\frac{N}{4}g$, so the product Ricci satisfies the same bound fibrewise.

B DLR Framework for Lattice Gauge Theory

We briefly recall the DLR formalism adapted to gauge theories. A *Gibbs specification* for Yang–Mills is the family $\{\gamma_\Delta\}_{\Delta\Subset\eta\mathbb{Z}^4}$ of kernels:

$$\gamma_\Delta(f \mid U_{\Delta^c}) = \frac{1}{Z_\Delta(U_{\Delta^c})} \int_{\mathcal{U}_\Delta} f(U)\, e^{-S_W(U\cup U_{\Delta^c})} \prod_{b\in E(\Delta)} dU_b. \tag{38}$$

A probability measure μ on $\mathcal{U}_{\eta\mathbb{Z}^4}$ is a *Gibbs state* if $\mu\gamma_\Delta = \mu$ for all Δ.

The existence of at least one Gibbs state follows from tightness (Proposition 7.1). Uniqueness under DS mixing follows from (32) by taking $\Lambda \to \infty$ [13].

The gauge invariance of S_W implies that every Gibbs state is invariant under local gauge transformations, and the reconstructed Hamiltonian H_∞ acts on the gauge-invariant subalgebra.

C Import Map from Bałaban's Infrastructure

Table 1: Results imported from Bałaban's papers.

Statement imported	Source	Used in
Propagator decay: $\|G_{xy}\| \le Ce^{-m\|x-y\|/\eta}$	CMP 95 [17], Prop. 1.2	Prop. 4.3, Remark 3.3
Analyticity of averaging operations	CMP 98 [18], Props. 4–7	§3.3
Convergent cluster expansion, effective action	CMP 116 [20], Thm. 1	Assumption 4.1
Large-field suppression $\varrho\|_{\Omega_{\rm lf}} \le e^{-cp_0^2\|\text{region}\|}$	CMP 122 [22]	Prop. 5.5
Convergent RG expansion, CA constants (K, m)	CMP 119 [21], Cor. 3	Assumption 4.1

D Cesi's Quasi-Factorisation: Precise Statement

We record the precise form of the two results from [12] that we use.

Proposition 2.1 of Cesi (Quasi-factorisation)

Let μ be a probability measure on $\Omega_1 \times \Omega_2$ and let μ_i denote the marginals. Let $\vartheta(\varepsilon)$ measure the weak dependence: if the conditional laws of μ on Ω_i given the other variable satisfy a ε-perturbation bound, then:

$$\mathrm{Ent}_\mu(f^2) \;\le\; \frac{1}{1-\vartheta(\varepsilon)} \big[\mu\,\mathrm{Ent}_1(f^2) + \mu\,\mathrm{Ent}_2(f^2)\big], \tag{39}$$

where Ent_i denotes entropy in the i-th variable. This is a purely algebraic identity; no topology on Ω_i is assumed.

Main theorem of §4 of Cesi (Geometric iteration)

By applying the quasi-factorisation at dyadic scales and iterating geometrically, Cesi proves: under DS mixing, the global LSI constant satisfies $\hat{\rho} \geq \rho_0/M(\kappa, d)$ where $M(\kappa, d) = \sum_{k\geq 0}(1 + C_{\mathrm{DS}})^k e^{-\kappa \cdot 2^k} < \infty$ for $\kappa > 0$. This M is independent of the volume.

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