Exponential Clustering and Mass Gap for Four-Dimensional $SU(N)$ Lattice Yang–Mills Theory

Via Balaban's Renormalization Group and Multiscale Correlator Decoupling

— a Conditional Clustering Theorem —*

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Abstract

We assemble a proof architecture for exponential clustering with a strictly positive mass gap for four-dimensional pure $SU(N)$ lattice Yang–Mills theory with Wilson's action,

$$\mathrm{Cov}_{\mu_\eta}(O(0), O(x)) \leq C\, e^{-m|x|/a_*}, \qquad m > 0, \quad a_* \sim \Lambda_{\mathrm{YM}}^{-1},$$

with constants uniform in lattice spacing η and physical volume L_{phys} — *conditionally* on three identified inputs. (1) Balaban's structural package (polymer decompositions, exponentially decaying activities), traceable per 2602.0069 v2 with the β_{LF} dichotomy attached. (2) A terminal-scale Kotecký–Preiss smallness bound, Hypothesis (H-KP): v1–v2 cited it as proved in an unpublished companion with no identifier; Version 3 retags it as a hypothesis until that companion exists and survives audit. (3) A localization hypothesis (H-LOC) for conditioned observables, made explicit for the first time in v3: the telescoping step compares the terminal clustering bound against $\tilde{O} = \mathbb{E}[O \,|\, \sigma_{a_*}]$, whose support is *not* local. The coupling control (Proposition 4.1) is proved by Cauchy bounds conditionally on the uniform-in-k analyticity radius of Balaban's discrete β-function and on the large-field penalty profile satisfying the (A_0, p_*) trichotomy of the audited ledger. What is unconditional and machine-verified: the multiscale telescoping identity (exact for any measure and any nested σ-algebra chain), the summation-over-scales arithmetic, the lattice-animal bounds, the implications KP⇒exponential clustering and clustering⇒spectral gap in exactly solvable settings, and the coupling-control recursion.

*__Version 3.__ This is an audited replacement of v2 within the programme-wide verification pass (2602.0041 v3, 2602.0051–0057 v2, 2602.0063 v3, 2602.0069/0070/0072/0073/0077 v2, 2602.0087 v3). Principal changes: (i) **F-KP**: the Kotecký–Preiss bound (Theorem 5.3), the sole load-bearing infrared input, was cited as "proved in Paper [KP]", an unpublished companion with no identifier; it is retagged here as Hypothesis (H-KP) until that companion exists and survives audit. (ii) **F-P0**: the large-field closure inequality $e^{-p_0(g)} \leq g^4$ (Eq. (10) of v2) is a condition on the penalty profile (A_0, p_*) — the trichotomy of the audited ledger ((H-θ)/F-SQRT, 2602.0073/0077 v2) — and is now stated as such; consequently Proposition 4.1 (coupling control) is conditional, and v2's claim that it is "the first published proof" of Balaban's Theorem 2 is softened. (iii) **F-LOC** (new): Theorem 7.1 applied the terminal clustering bound to $\tilde{O} = \mathbb{E}[O \,|\, \sigma_{a_*}]$ as if $\mathrm{supp}\tilde{O} = \mathrm{supp}O$; conditional expectations delocalize, and the required exponential localization is now Hypothesis (H-LOC), exhibited quantitatively in the companion suite. (iv) Lemma 6.2's per-scale cluster expansion with holes is tagged as a traceable import with a decoupling load ((H-DEC)-scale analogue). All exact content (telescoping identity, scale-summation arithmetic, lattice animal bounds, KP⇒clustering, clustering⇒gap, coupling recursion) is machine-verified in the companion suite (`verification/2602-0088/`, 7/7 PASS). No statement of this paper is unconditional.

We verify OS0, OS2, OS3 unconditionally at the lattice level and OS4 conditionally; OS1 (full $O(4)$ covariance) is not established here — its natural conditional supplier is 2602.0087 v3 via 2602.0063 v3. All adjudicable content is verified in a deterministic companion suite.

1 Introduction

The Yang–Mills existence and mass gap problem [16] asks for a rigorous construction of four-dimensional quantum gauge theory with a spectral gap. This paper is the assembly capstone of the 2602-series: it consumes the structural package audited in [2], the coupling-control technology audited in [21], and (for the missing axiom OS1) interfaces with [23, 24]. In v1–v2 the main theorem was stated unconditionally; the programme-wide audit found that its three load-bearing inputs are, respectively, traceable-with-load, unpublished, and unstated. This version attaches the loads explicitly.

1.1 Strategy

Three ingredients: (i) Balaban's UV package (CMP 95–122) [4, 7, 8, 9, 10], imported per the audited bridge [2]; (ii) a terminal-scale polymer cluster expansion activated by a Kotecký–Preiss bound — Hypothesis (H-KP) below; (iii) a multiscale correlator decoupling identity (this paper; exact), with per-scale error bounds carrying a decoupling load.

1.2 Main Result

Hypothesis (H-KP) (terminal KP smallness; v2's Theorem 5.3). There exist $\gamma_0, \kappa, a > 0$ and $\delta \in (0,1)$ such that for all $\bar{g} \in (0, \gamma_0]$ and all $L_{\mathrm{phys}} > 0$ the terminal polymer activities satisfy

$$\sup_{\ell} \sum_{X \ni \ell} \left\| e^{R_*(X)} - 1 \right\|_\infty e^{a|X|} e^{\kappa d_{k_*}(X)} \ \le\ \delta, \tag{12}$$

with constants independent of L_{phys}. *Status:* v1–v2 cited this as "proved in Paper [KP]" — a companion with no public identifier. Until that derivation exists and survives audit, (12) is a hypothesis. Its intended source trail (Balaban's small-field activities [7], large-field localization [9, 10]) is recorded in Remark 5.4.

Hypothesis (H-LOC) (localization of conditioned observables; new in v3). For gauge-invariant local O with $\|O\|_\infty \le 1$ there is a decomposition $\mathbb{E}_{\mu_\eta}[O \,|\, \sigma_{a_*}] = \sum_Y \tilde{O}(Y)$ into terminal-scale local terms with $\|\tilde{O}(Y)\|_\infty \le C\, e^{-\kappa_{\mathrm{loc}} d_{k_*}(Y \cup \{x_O\})}$. *Status:* this is Balaban's composite (blocked) observable analysis; v1–v2 used it silently by writing $\mathrm{supp}\tilde{O} = \mathrm{supp} O$. The delocalization phenomenon and its exponential decay are exhibited exactly in a Gaussian toy in the suite (check 7).

Theorem 1.1 (Lattice Mass Gap — **conditional**). *Assume (H-KP), (H-LOC), and the coupling-control inputs of Proposition 4.1 ((H-Rβ) and (H-P0′)). Then for each $N \ge 2$ there is $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0]$, all $L_{\mathrm{phys}} > 0$, all gauge-invariant local O with $\|O\|_\infty \le 1$ and all $|x| \ge a_*$,*

$$\mathrm{Cov}_{\mu_\eta}(O(0), O(x)) \le C\, e^{-m|x|/a_*}, \tag{1}$$

with $m > 0$ and C independent of η and L_{phys}.

Corollary 1.2 (Physical Mass Gap — conditional)**.** *Under the same hypotheses, $\Delta_{\mathrm{phys}} \geq c_N \Lambda_{\mathrm{YM}} > 0$ along any sequence $(\eta_j, L_{\mathrm{phys},j})$ with $\eta_j \to 0$, $L_{\mathrm{phys},j} \to \infty$ (Remark 2.4).*

Remark 1.3. Throughout, $c_N > 0$ depends only on $SU(N)$. *Two routes to the gap now coexist in the programme:* the LSI route (2602.0053/0054 v2: conditional on (H-DOB-blk)+(H-P0)+volume window) and the present KP route (conditional on (H-KP)+(H-LOC)+profile). Neither is unconditional; they have different loads, and the ledger records both.

Theorem 1.4 (Continuum Limit and Reconstruction — conditionality per part)**.** (a) Subsequential convergence *(unconditional): the lattice Schwinger functions converge subsequentially to tempered distributions satisfying OS0, OS3, and translation+W_4 covariance (Proposition 8.3); OS2 holds by Osterwalder–Seiler. OS4 holds conditionally (it* is *Theorem 7.1).* (b) Reconstruction with distinguished time *(conditional on the Theorem 1.1 package): Hilbert space, positive Hamiltonian, unique vacuum, spectral gap $\geq c_N \Lambda_{\mathrm{YM}}$ via Lemma 8.2.* (c) Full Wightman theory *(additionally conditional on OS1): if $O(4)$ covariance holds in the limit — the conditional supplier in this programme is 2602.0087 v3 feeding 2602.0063 v3 (Prop. 5.3/Rmk 5.4 there) — the reconstructed theory is a non-trivial relativistic QFT with mass gap.*

2 Setup and Notation

Unchanged from v2 (transcribed there in full; only the items used below are restated). $\Lambda_\eta = (\eta\mathbb{Z}/L_{\mathrm{phys}}\mathbb{Z})^4$; Wilson measure $d\mu_\eta = Z_\eta^{-1} e^{-g_0(\eta)^{-2}A(U)} \prod_\ell dU_\ell$ with $g_0(\eta)^{-2} = b_0 \log((\Lambda_{\mathrm{YM}}\eta)^{-1}) + O(\log\log)$, $b_0 = 11N/48\pi^2$. RG scales $a_k = L^k \eta$; terminal step $k_* = \lfloor \log_L(\Lambda_{\mathrm{YM}}\eta)^{-1} \rfloor$, $a_* = L^{k_*}\eta \sim \Lambda_{\mathrm{YM}}^{-1}$ (v2 Eq. (3), Remark 2.1). Local observables, Schwinger functions and the thermodynamic-limit bookkeeping are v2's Definitions 2.2–2.3 and Remark 2.4; the audit found Remark 2.4 sound *given* the volume-uniform clustering it invokes — i.e. it inherits the conditionality of Theorem 7.1.

3 Balaban's Renormalization Group Framework

Theorem 3.1 (inductive representation), Proposition 3.2 (polymer bounds, with the large-field factor $e^{-p_0(g_k)}$, $p_0(g) = (A_0 \log g^{-2})^{p_*}$), Proposition 3.3 (UV stability), Proposition 3.4 (R-operation), Remark 3.5 (exponentiation) and Proposition 3.6 (analyticity of the discrete β-function in $h = g_k^2$) are imports from [4, 7, 8, 9, 10, 6], retained with v2's numbering. *Audited status* [2]: the small-field package is traceable to the primary sources; the large-field suppression carries the β_{LF} dichotomy (the $2/(1+\beta_{\mathrm{LF}})$ reading), and the exponent profile (A_0, p_*) in p_0 is *not pinned* by [10], Eq. (1.100) — it is exactly the ledger's (H-θ) profile datum. The uniform-in-k analyticity radius R_β of Proposition 3.6 is the load (H-Rβ) (v2's own Remark 9.1 identified it as the non-trivial point).

4 Coupling Control

Proposition 4.1 (Coupling Control — conditional on (H-Rβ) and (H-P0$'$))**.** *Assume* (H-Rβ)*: β_{k+1} is analytic in $h = g_k^2 \in (0, \gamma^2]$ with radius R_β and bound M_β uniform in k; and* (H-P0$'$)*: the penalty profile satisfies $e^{-p_0(g)} \leq g^4$ for all $g \leq \gamma_0$, i.e. $(A_0\ell)^{p_*} \geq 4\ell + \log C_{\mathrm{lf}}$ with $\ell = \log g^{-2}$. Then there exists $\gamma_0 \in (0, \gamma)$ such that $g_0(\eta) \leq \gamma_0$ implies $g_k \leq g_0 \leq \gamma_0 < \gamma$ and $g_k^{-2} \geq g_0^{-2} + k\, b_0/2$ for all $k \leq k_*$.*

Proof (as in v2, loads attached). Induction on k. The inductive hypothesis activates the imports of §3. Large-field remainder: $|r_k^{(\mathrm{lf})}| \le C_{\mathrm{lf}} e^{-p_0(g_k)} \le g_k^4$ *by (H-P0′)* — v2 asserted this "for g_k sufficiently small" without the profile condition; the trichotomy is: false for every small g if $p_* < 1$; equivalent to $A_0 \ge 4$ (asymptotically) if $p_* = 1$; true eventually if $p_* > 1$, with onset scale $\ell_* = (4/A_0^{p_*})^{1/(p_*-1)}$ which is astronomically large for moderate (A_0, p_*) — e.g. $p_* = 1.1$, $A_0 = 1$ gives $\ell_* = 4^{10} \approx 1.05 \times 10^6$, i.e. $g^2 \le e^{-10^6}$ [suite check 6]. Small-field remainder: the Cauchy estimate in h gives $|r_k^{(\mathrm{sf})}| \le (M_\beta/R_\beta) g_k^2$ by (H-Rβ). Closure: $|r_k| \le C g_k^2$ with $C\gamma_0^2 < b_0/2$ yields $g_{k+1}^{-2} \ge g_k^{-2} + b_0/2$; the recursion arithmetic is exact [suite check 6]. □

Remark 4.2 (v2's Remark 4.2, softened). The argument is not circular in structure. But v2's claim that this is "the first published proof" of Balaban's Theorem 2 overstates: what is proved here is the *closure implication* (H-Rβ)+(H-P0′)⇒coupling control. The two hypotheses are the content of Balaban's announcement, and pinning them to [6, 10] remains open (interface task, same class as 2602.0077 v2's Assumption A task).

5 Terminal Scale Cluster Expansion and Exponential Clustering

The terminal measure μ_{a_*}, the reference measure $\nu_{\bar g}$, the activities $z(X) = e^{R_*(X)} - 1$ and Remark 5.2 (relation to Balaban's activities) are as in v2. Proposition 5.1 (intensive oscillation bound, via the lattice-animal Lemma C.1) survives as stated [suite check 3 verifies the animal bound].

Theorem 5.3 of v2 is (H-KP) (see §1.2); Remark 5.4's audit trail (activities [7] Eq. (2.11)–(2.38); large-field [10] Eq. (1.98)–(1.100)) is retained as the intended derivation route, *not* as a proof.

Theorem 5.1 (Terminal exponential clustering — conditional on (H-KP))**.** *Assume (12). Then there are $C < \infty$, $m_* > 0$ independent of L_{phys} with* $\mathrm{Cov}_{\mu_{a_*}}(F, G) \le C\|F\|_\infty \|G\|_\infty e^{-m_* \mathrm{dist}(\mathrm{supp}F, \mathrm{supp}G)}$ *for all bounded gauge-invariant F, G.*

The implication KP⇒clustering (Appendix A, Proposition A.1) is standard [3] and is verified exactly on a solvable gas in the suite [check 4, volume-uniformity included].

6 Multiscale Correlator Decoupling

Proposition 6.1 (Multiscale Telescoping — exact, unconditional)**.** *For bounded F, G:*

$$\mathrm{Cov}_{\mu_\eta}(F, G) = \mathrm{Cov}_{\mu_{a_*}}(\tilde F, \tilde G) + \sum_{k=0}^{k_*-1} \mathcal{R}_k(F, G), \tag{13}$$

$\tilde F = \mathbb{E}_{\mu_\eta}[F|\sigma_{a_*}]$, $\mathcal{R}_k = \mathbb{E}_{\mu_{a_{k+1}}}[\mathrm{Cov}_{\mu_{a_k}|\sigma_{a_{k+1}}}(F_k, G_k)]$.

Proof. Iterated law of total covariance along $\sigma_{a_0} \supset \cdots \supset \sigma_{a_{k_*}}$; exact for any probability measure and any nested chain [suite check 1]. □

Lemma 6.2 (Single-Scale Error — traceable import with decoupling load)**.** *For F, G with supports separated by R: $|\mathcal{R}_k(F, G)| \le C\|F\|_\infty \|G\|_\infty e^{-\kappa R/a_k}$, with C, κ independent of $k, \eta, R, L_{\mathrm{phys}}$.*

Status. The proof route (random-walk decay of the fluctuation propagator [4, 5], Lemma B.1; large-field factor e^{-p_0}; "cluster expansion with holes" [8, 10], [12] App. F) is a per-scale conditional-measure expansion: its convergence *uniformly over the conditioning coarse field* is the scale-k analogue of the UV circuit's decoupling load (H-DEC/AT) (2602.0070/0072 v2). v3 tags it accordingly rather than citing it as settled.

Theorem 6.3 (UV Suppression — arithmetic exact given Lemma 6.2)**.** *For supports separated by $R \geq a_*$: $\sum_{k<k_*} |\mathcal{R}_k| \leq C' \|F\|_\infty \|G\|_\infty e^{-c_0 R/a_*}$, uniformly in η, R, L_{phys}, with $c_0 = \kappa L$.*

The summation $\sum_{j\geq 1} \exp(-\kappa(R/a_*)L^j) \leq \text{const}(\kappa, L)$, uniform in k_*, is exact arithmetic [suite check 2].

7 Assembly: The Mass Gap Bound

Theorem 7.1 (Mass Gap Bound — conditional on (H-KP), (H-LOC), Lemma 6.2, Proposition 4.1)**.** *For $\|O\|_\infty \leq 1$ and $|x| \geq a_*$:* $\text{Cov}_{\mu_\eta}(O(0), O(x)) \leq Ce^{-m|x|/a_*}$ *with* $m = \min(m_*, c_0) > 0$*, uniformly in η and L_{phys}.*

Proof (v2's, with the F-LOC gap repaired by hypothesis). Decompose via (13). *IR term:* v2 applied Theorem 5.1 to $\tilde{O}$ writing "supp$\tilde{O} \subseteq$ supp$O$" — false in general: $\mathbb{E}[O|\sigma_{a_*}]$ depends on *all* coarse links [suite check 7 exhibits this exactly in a Gaussian chain: the coefficient of a distant block is nonzero and decays exponentially]. Under (H-LOC), $\tilde{O} = \sum_Y \tilde{O}(Y)$ with exponentially localized terms; applying Theorem 5.1 termwise and summing the double polymer sum (lattice-animal arithmetic, Lemma C.1) gives the IR bound with m_* degraded to $\min(m_*, \kappa_{\text{loc}})/2$. *UV term:* Theorem 6.3. Combine. □

8 Osterwalder–Schrader Reconstruction and Non-Triviality

The axiom table of v2 is amended in one column: OS4 is *conditional* (it is Theorem 7.1); OS0 (temperedness; L^∞+Banach–Alaoglu), OS2 (Osterwalder–Seiler [14], unconditional at the lattice level), OS3 (automatic) and Proposition 8.3 (translation+W_4 covariance of subsequential limits) survive as in v2. Remark 8.4 (O(4) restoration not proved here) was already honest; v3 adds the programme wiring: the conditional supplier of OS1 is 2602.0087 v3 (one-dimensional anisotropic sector, $|c_{6,\text{aniso}}| \leq Ca_k^2$) feeding 2602.0063 v3 (Prop. 5.3/Rmk 5.4), under the ledger loads recorded there. Lemma 8.2 (exponential clustering⇒spectral gap of the transfer matrix) is unconditional given its hypotheses and is verified exactly on an explicit transfer matrix in the suite [check 5]. Proposition 8.5 (vacuum uniqueness) and Theorem 8.7 (non-triviality, tree-level $\bar{g}^4$ lower bound with polymer remainder $O(\bar{g}^6)$) inherit conditionality through Theorem 7.1 and (H-KP) respectively.

Remark 8.1 (Status of the Clay Millennium Problem claim — restated)*.* v2's status box implied the lattice half was settled. Corrected statement: *conditionally on (H-KP), (H-LOC), (H-Rβ), (H-P0$'$) and the Lemma 6.2 import*, this paper assembles uniform lattice clustering, OS0/2/3/4, vacuum uniqueness, non-triviality, and gapped reconstruction with distinguished time; OS1 remains open (conditional supplier: 2602.0087 v3 → 2602.0063 v3). No part of the Clay problem is claimed unconditionally by this programme.

9 Discussion

The two-route ledger. The programme now contains two independent conditional routes to the gap: LSI (2602.0053/0054 v2; loads (H-DOB-blk), (H-P0), volume window $L \le e^{C/g^2}$) and KP-terminal (this paper; loads (H-KP), (H-LOC), profile (H-P0′), (H-Rβ), Lemma 6.2). The loads are non-identical — notably the KP route claims volume uniformity without a window, but only because (H-KP) itself asserts L_{phys}-independence; nothing is gained for free. *Audit queue.* Two companions remain unaudited and identifier-less: Paper [KP] (the intended derivation of (12)) and the rotational Ward identity companion cited by 2602.0087. They are the natural next targets; this paper's Theorem 1.1 upgrades automatically if Paper [KP] survives audit.

Note added in v3

Findings F-KP (deferral of the load-bearing KP bound to an unpublished companion, presented as proof), F-P0 (profile condition stated as smallness), F-LOC (support of conditioned observables), plus the Lemma 6.2 decoupling tag and the softening of Remark 4.2. What is exact survives verbatim and is machine-verified: Proposition 6.1, Theorem 6.3's arithmetic, Lemma C.1, Proposition A.1 and Lemma 8.2 in solvable settings, and the coupling recursion. Item numbering of v2 is preserved. Suite: `verify_2602_0088.py`, 7 deterministic checks, exit 0 iff 7/7.

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[20] L. Eriksson, ai.viXra:2602.0053/0054 **v2** (LSI route: conditional gap, volume window).

[21] L. Eriksson, ai.viXra:2602.0073 **v2** (Cauchy bounds, (H-θ)/F-SQRT trichotomy).

[22] L. Eriksson, ai.viXra:2602.0077 **v2** (UV circuit assembly; (H-DEC/AT)).

[23] L. Eriksson, ai.viXra:2602.0087 **v3** (conditional $SO(4)$-restoration input).

[24] L. Eriksson, ai.viXra:2602.0063 **v3** (continuum capstone).

[25] L. Eriksson, rotational Ward identity companion — **unaudited, identifier pending**.