Irrelevant Operators, Anisotropy Bounds, and Operator Insertions in Bałaban's Renormalization Group for Four-Dimensional $\mathrm{SU}(N)$ Lattice Yang–Mills Theory

Symanzik Classification and Quantitative Irrelevance of $O(4)$-Breaking Operators

Lluis Eriksson

Independent Researcher

lluiseriksson@gmail.com

February 2026

Abstract

We classify gauge-invariant local lattice operators of classical dimension 6 on the four-dimensional hypercubic lattice into $O(4)$-invariant, hypercubic-invariant but $O(4)$-breaking (anisotropic), and on-shell-redundant components, following the Symanzik improvement programme and the on-shell improvement technique of Lüscher–Weisz (1985). Inside Bałaban's renormalization group framework for $\mathrm{SU}(N)$ lattice Yang–Mills theory, we extract the anisotropic projection of the effective action via local Taylor expansion of polymer activities in the small-field regime and prove a quantitative *quadratic scale bound* for the anisotropic coefficient: for every RG step $k \le k_*$ with effective coupling $g_k \le \gamma_0$, the coefficient of the (one-dimensional) anisotropic sector in the classical dimension-6 Symanzik expansion satisfies

$$|c^{(k)}_{6,\mathrm{aniso}}| \;\le\; C\, a_k^2,$$

uniformly in lattice spacing η, physical volume L_{phys}, and RG step k. We further prove a quantitative *insertion integrability* estimate for connected correlators with one insertion of the anisotropic operator. When combined with the rotational Ward identity derived in the companion paper, this yields that the corresponding breaking distribution tested against Schwartz functions is $O\big(\eta^2\,|\log((\Lambda_{\mathrm{YM}}\eta)^{-1})|\big)$ and hence vanishes as $\eta \to 0$.

Contents

1 Introduction

1.1 Motivation

The construction of four-dimensional Yang–Mills theory as a continuum quantum field theory requires, among other things, the verification that the Euclidean Schwinger functions possess full $O(4)$ covariance (axiom OS1 in the Osterwalder–Schrader framework). For lattice gauge theories with Wilson's action on a hypercubic lattice, the manifest symmetry group is only the hypercubic group $\mathcal{W}_4$ (of order 384), together with lattice translations. The enhancement from $\mathcal{W}_4$ to full $O(4)$ in the continuum limit is universally expected on physical grounds but has not been rigorously established.

In the companion paper [1], we proved exponential clustering with a strictly positive mass gap for the lattice Yang–Mills measure, uniformly in lattice spacing η and physical volume, and verified the Osterwalder–Schrader axioms OS0, OS2, OS3, and OS4 for subsequential continuum limits. The remaining axiom OS1 ($O(4)$ covariance) was left as the sole open input for the full Wightman reconstruction.

The present paper provides the technical foundation for establishing OS1. The strategy, carried out in detail in the subsequent paper [2], is to derive lattice Ward identities for infinitesimal rotations and to show that the *breaking term*—which arises because the Wilson action is only $\mathcal{W}_4$-invariant, not $O(4)$-invariant—vanishes in the continuum limit. The breaking term is a sum over insertions of local operators of (classical) dimension 6 that are $\mathcal{W}_4$-invariant but break $O(4)$ symmetry. Controlling these operators and their insertions into Schwinger functions is the purpose of the present paper.

1.2 Main Results

We establish three main results:

(I) **Classification (Theorem 3.6).** The space of gauge-invariant, $\mathcal{W}_4$-scalar local operators of classical dimension 6 decomposes (modulo on-shell equivalences) into a two-dimensional $O(4)$-invariant subspace and a *one-dimensional anisotropic subspace*. The anisotropic direction is spanned by a single canonical operator $\mathcal{O}_{\text{aniso}}$ (Definition 3.4).

(II) **Anisotropy coefficient bound (Theorem 5.4).** Within Bałaban's renormalization group framework, the coefficient of $\mathcal{O}_{\text{aniso}}$ in the effective action at scale a_k satisfies

$$|c^{(k)}_{6,\text{aniso}}| \;\leq\; C\, a_k^2,$$

uniformly in η, L_{phys}, and k.

(III) **Insertion integrability (Theorem 6.6).** Connected correlators with one insertion of $\mathcal{O}_{\text{aniso}}$ are integrable in the insertion point after smearing the remaining arguments against Schwartz functions: there exists $C(f) < \infty$ such that

$$\sum_{y\in\Lambda_\eta} \eta^4 \int_{\mathbf{R}^{4n}} |f(x)|\, \big|\langle \mathcal{O}_{\text{aniso}}(y)\cdot \textstyle\prod_j \mathcal{O}(x_j)\rangle_{\mu_\eta,c}\big|\, dx \;\leq\; C(f),$$

uniformly in η. This statement is formulated independently of the Ward identity; the latter is used only in [2] to convert the insertion estimate into the vanishing of the rotational breaking distribution.

Remark 1.1 (Dependence on the mass gap input). The insertion integrability estimate (Theorem 6.6) uses two inputs: (i) ultraviolet locality from the polymer expansion and (ii) infrared integrability from exponential clustering. If the clustering estimate from [1] is conditional on a functional-inequality hypothesis, then Theorem 6.6 is conditional to the same extent. In the intended closure programme, the functional-inequality input is established independently, rendering all results unconditional. The classification (Theorem 3.6) and the coefficient bound (Theorem 5.4) do *not* depend on the clustering hypothesis.

These three results are logically independent of the Ward identity derivation (carried out in [2]). The operator classification is algebraic, and the coefficient bounds are obtained from polymer analyticity and decay together with a concrete definition of the anisotropic projection.

1.3 Organisation

Section 2 introduces the classical dimension of lattice operators and the Symanzik expansion. Section 3 performs the representation-theoretic decomposition and establishes Theorem 3.6. Section 4 reviews the Bałaban framework and imported results from [1]. Section 5 proves the Symanzik decomposition of the effective action (Theorem 5.4). Section 6 establishes the insertion bound (Theorem 6.6). Section 7 discusses the interface with [2].

2 Lattice Operators and Classical Dimension

2.1 Setup

We work with $\mathrm{SU}(N)$ lattice gauge fields $U(n,\mu)$ on the four-dimensional hypercubic lattice $\mathbf{Z}^4$ (or a periodic sublattice). Links $\ell = (n,\mu)$ carry group-valued variables $U_\ell \in \mathrm{SU}(N)$, where $n \in \mathbf{Z}^4$ and $\mu \in \{0,1,2,3\}$.

Definition 2.1 (Local lattice operator)**.** A *local lattice operator* $\mathcal{O}(n)$ is a gauge-invariant polynomial of link variables $U(m,\nu)$ with m in a bounded neighborhood of n, which transforms covariantly under lattice translations.

2.2 Classical Dimension via Slow-Field Expansion

Following Lüscher–Weisz [3], we define the *classical dimension* of a lattice operator by expanding it in powers of a small lattice spacing a around a smooth continuum gauge field.

Given a smooth continuum gauge field $A_\mu(x)$ on $\mathbf{R}^4$, define a lattice gauge field at spacing a by

$$U(n,\mu) \;=\; T\exp\Big(a\int_0^1 dt\, A_\mu(an + a\hat{\mu} - ta\hat{\mu})\Big), \tag{1}$$

where T denotes path ordering. Evaluating a local operator $\mathcal{O}(0)$ on this field yields an asymptotic expansion

$$\mathcal{O}(0) \underset{a\to 0}{\sim} \sum_{k=0}^{\infty} a^k\, O_k(0), \tag{2}$$

where each $O_k(x)$ is a gauge-invariant local polynomial in $A_\mu(x)$ and its derivatives.

Definition 2.2 (Classical dimension)**.** The *classical dimension* of $\mathcal{O}$ is $\dim(\mathcal{O}) := \min\{k : O_k \neq 0\}$.

2.3 Expansion of the Plaquette

The elementary plaquette operator

$$\mathcal{O}_{\mu\nu}(n) \;=\; \mathrm{Re}\,\mathrm{Tr}\big\{\mathbb{1} - U(n,\mu)\,U(n{+}\hat{\mu},\nu)\,U(n{+}\hat{\nu},\mu)^{-1}\,U(n,\nu)^{-1}\big\}$$

has the classical expansion (cf. [3, Eq. (14)]):

$$\mathcal{O}_{\mu\nu}(0) \;=\; -\tfrac{1}{2}\,a^4\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) + a^6\,\mathcal{Q}^{(6)}_{\mu\nu} + O(a^7), \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $\mathcal{Q}^{(6)}_{\mu\nu}$ is a specific combination of dimension-6 operators whose explicit form is given in Appendix B.

2.4 Basis of Dimension-6 Operators

When $\mathcal{O}(n)$ transforms as a scalar under $\mathcal{W}_4$, the operators O_k in (2) are severely restricted. In particular, $O_k = 0$ for odd k, and at dimension 6 the most general structure is (cf. [3, Eqs. (6)–(7)]):

$$O_6 \;=\; r_1\,\mathcal{I}_1 + r_2\,\mathcal{I}_2 + r_3\,\mathcal{I}_3 + \sum_\mu \partial_\mu G_\mu, \tag{4}$$

where the three independent gauge-invariant dimension-6 operators are:

$$\mathcal{I}_1 \;=\; \sum_{\mu,\nu} \mathrm{tr}\big(D_\mu F_{\mu\nu}\,D_\mu F_{\mu\nu}\big), \tag{5}$$

$$\mathcal{I}_2 \;=\; \sum_{\mu,\nu,\rho} \mathrm{tr}\big(D_\mu F_{\nu\rho}\,D_\mu F_{\nu\rho}\big), \tag{6}$$

$$\mathcal{I}_3 \;=\; \sum_{\mu,\nu,\rho} \mathrm{tr}\big(D_\mu F_{\mu\rho}\,D_\nu F_{\nu\rho}\big), \tag{7}$$

with $D_\mu = \partial_\mu + [A_\mu, \cdot\,]$.

The coefficients r_i for various lattice loops are listed in Table 1, reproduced from [3, Table 2].

Loop type	r_0	r_1	r_2	r_3
Plaquette ($\mathcal{O}_0$)	$-\frac{1}{4}$	$\frac{1}{24}$	0	0
Rectangle ($\mathcal{O}_1$)	-2	$\frac{5}{6}$	0	0
Parallelogram ($\mathcal{O}_2$)	-2	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
Chair ($\mathcal{O}_3$)	-4	$\frac{1}{6}$	0	$\frac{1}{2}$

Table 1: Coefficients r_i in the expansion (4) for standard lattice loops. The coefficient r_0 multiplies the dimension-4 operator $\sum_{\mu,\nu}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$.

Definition 2.3 (Dimension-6 operator space)**.** We define $\mathfrak{O}_6^{\mathcal{W}_4} := \mathrm{span}\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, the space of $\mathcal{W}_4$-scalar gauge-invariant operators of classical dimension 6, modulo total divergences.

3 $\mathcal{W}_4$ versus $O(4)$: Classification of Anisotropic Operators

3.1 Symmetry Analysis of the Basis Operators

The operators $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ are $\mathcal{W}_4$-scalar by construction. Among them, $\mathcal{I}_2$ is manifestly $O(4)$-invariant (all Lorentz indices are fully contracted via $\delta_{\mu\nu}$). Rather than classifying $\mathcal{I}_1$ and $\mathcal{I}_3$ individually with respect to $O(4)$—which would require case-by-case Bianchi/EOM identities—we isolate the $O(4)$-breaking component *operationally* via a one-dimensional anisotropic projector defined in Section 3.3 below. This approach is basis-independent and avoids unproven on-shell claims.

3.2 On-Shell Equivalences

Following Lüscher–Weisz [3], we define on-shell equivalence:

Definition 3.1 (On-shell equivalence)**.** Two operators $\mathcal{O}$ and $\mathcal{O}'$ of dimension 6 are *on-shell equivalent* (written $\mathcal{O} \sim_{\mathrm{os}} \mathcal{O}'$) if $\mathcal{O} - \mathcal{O}'$ is a linear combination of:

(a) total divergences $\sum_\mu \partial_\mu G_\mu$;

(b) operators proportional to the equations of motion $D_\nu F_{\nu\mu} = 0$.

On-shell equivalent operators produce identical $O(a^2)$ corrections to spectral quantities (masses, heavy-quark potential at physical distances).

The central result of [3, Sect. 4, Theorem] is:

Proposition 3.2 (Lüscher–Weisz spectrum-conserving transformation)**.** *There exists a local, gauge-covariant field transformation Φ of lattice gauge fields (see [3, Eq. (46)]) that shifts the coefficients in the improved action $S = \frac{2}{g_0^2}\sum_n \{\mathcal{O}_0(n) + \sum_{i=1}^{3} c_i\, \mathcal{O}_i'(n)\}$ according to*

$$c_1 \to c_1 + y, \quad c_2 \to c_2, \quad c_3 \to c_3 + y,$$

for arbitrary y, without affecting the energy spectrum (up to $O(a^4)$).

Proof sketch. The transformation Φ replaces $U(n,\mu)$ by $e^{\varepsilon X_\mu(n)}U(n,\mu)$, where X_μ is the anti-hermitian traceless part of a gauge-covariant combination of neighboring plaquettes. The Jacobian contributes a dimension-4 term (absorbed by coupling renormalization), and the change in the action is a dimension-6 operator that shifts c_1 and c_3 by equal amounts. Full details are in [3, Sect. 4]. □

3.3 An Operational Anisotropic Projector

We isolate the anisotropic sector using the following degree-4 hypercubic harmonic polynomial:

$$h_{\mathrm{aniso}}(p) \;:=\; \sum_{\mu=0}^{3} p_\mu^4 - \tfrac{1}{4}(p^2)^2, \qquad p^2 := \sum_{\mu=0}^{3} p_\mu^2. \tag{8}$$

This polynomial is $\mathcal{W}_4$-invariant and orthogonal (in the standard L^2-inner product on the unit sphere S^3) to every $O(4)$-invariant polynomial of degree 4. By direct computation, the space of $\mathcal{W}_4$-invariant homogeneous polynomials of degree 4 modulo $O(4)$-invariants is one-dimensional and spanned by h_{aniso} (see Appendix A).

Definition 3.3 (Anisotropic projection on $\mathfrak{O}_6^{\mathcal{W}_4}$)**.** Let $\mathcal{O} \in \mathfrak{O}_6^{\mathcal{W}_4}$ be represented by a local continuum functional after the slow-field expansion. We define $\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}) \in \mathbf{R}$ to be the coefficient of $h_{\mathrm{aniso}}(p)$ in the momentum-space symbol of the quadratic (in the field-strength) part of $\mathcal{O}$, computed around $A \equiv 0$. This coefficient is independent of the chosen representative of $\mathcal{O}$ modulo total divergences and EOM-proportional terms, because h_{aniso} is orthogonal to all $O(4)$-invariant structures and total divergences contribute only longitudinal (i.e. $O(4)$-invariant) momentum factors.

3.4 The Anisotropic Subspace

Definition 3.4 (Canonical anisotropic operator)**.** We define $\mathcal{O}_{\mathrm{aniso}} \in \mathfrak{O}_6^{\mathcal{W}_4}$ to be any representative with $\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}_{\mathrm{aniso}}) = 1$. Concretely, we take

$$\mathcal{O}_{\mathrm{aniso}} \;:=\; \mathcal{I}_1 - \tfrac{1}{4}\mathcal{I}_2 \;=\; \sum_{\mu,\nu} \mathrm{tr}(D_\mu F_{\mu\nu}\, D_\mu F_{\mu\nu}) - \tfrac{1}{4} \sum_{\mu,\nu,\rho} \mathrm{tr}(D_\mu F_{\nu\rho}\, D_\mu F_{\nu\rho}), \tag{9}$$

which satisfies $\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}_{\mathrm{aniso}}) = 1$ by construction: its quadratic symbol contains $h_{\mathrm{aniso}}(p)$ with unit coefficient.

Remark 3.5. The normalization $\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}_{\mathrm{aniso}}) = 1$ is a convention. All bounds below are stable under rescaling $\mathcal{O}_{\mathrm{aniso}} \mapsto \lambda\mathcal{O}_{\mathrm{aniso}}$ (the constants C change accordingly). No explicit rank-4 tensor formula is required; the polynomial h_{aniso} in (8) suffices for all arguments.

We can now state our first main result:

Theorem 3.6 (Classification of anisotropic operators)**.** *Let $\mathfrak{O}_6^{\mathcal{W}_4}$ be the span of the classical dimension-6, $\mathcal{W}_4$-scalar, gauge-invariant operators generated by lattice loops, modulo total divergences (Definition 2.3). Let $\mathfrak{O}_{6,\mathrm{EOM}} \subset \mathfrak{O}_6^{\mathcal{W}_4}$ be the subspace of operators proportional to the equations of motion (Definition 3.1). Then the on-shell quotient*

$$\mathfrak{O}_{6,\mathrm{os}} \;:=\; \mathfrak{O}_6^{\mathcal{W}_4}/\mathfrak{O}_{6,\mathrm{EOM}}$$

admits the decomposition

$$\mathfrak{O}_{6,\mathrm{os}} \;=\; \ker(\mathrm{Proj}_{\mathrm{aniso}}) \;\oplus\; \mathfrak{O}_{6,\mathrm{aniso}}, \tag{10}$$

where $\mathfrak{O}_{6,\mathrm{aniso}} := \mathrm{span}\{[\mathcal{O}_{\mathrm{aniso}}]\} \cong \mathbf{R}$ is one-dimensional.

Equivalently, for every $\mathcal{O} \in \mathfrak{O}_6^{\mathcal{W}_4}$ there exist unique scalars $\lambda_{\mathrm{aniso}} \in \mathbf{R}$ and an element $\mathcal{O}_{\mathrm{iso}} \in \ker(\mathrm{Proj}_{\mathrm{aniso}})$ (unique modulo $\mathfrak{O}_{6,\mathrm{EOM}}$), plus $\mathcal{R} \in \mathfrak{O}_{6,\mathrm{EOM}}$, such that

$$\mathcal{O} \;=\; \lambda_{\mathrm{aniso}}\, \mathcal{O}_{\mathrm{aniso}} + \mathcal{O}_{\mathrm{iso}} + \mathcal{R}, \qquad \lambda_{\mathrm{aniso}} = \mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}).$$

Proof. We verify that $\mathrm{Proj}_{\mathrm{aniso}} \colon \mathfrak{O}_6^{\mathcal{W}_4} \to \mathbf{R}$ (Definition 3.3) descends to a well-defined linear functional on $\mathfrak{O}_{6,\mathrm{os}}$:

(i) *Total divergences.* A term $\sum_\mu \partial_\mu G_\mu$ contributes only p_μ-longitudinal structures to the quadratic symbol, which are $O(4)$-covariant and hence orthogonal to h_{aniso}. Therefore $\mathrm{Proj}_{\mathrm{aniso}}(\sum_\mu \partial_\mu G_\mu) = 0$.

(ii) *EOM-proportional terms.* These are generated by operators of the form $\mathrm{tr}(D_\nu F_{\nu\mu} \cdot W_\mu)$ where W_μ is a local gauge-covariant expression. Linearizing around $A \equiv 0$, the factor $D_\nu F_{\nu\mu}$ produces the symbol $(p^2\delta_{\mu\rho} - p_\mu p_\rho)\hat{A}_\rho(p)$, which is built entirely from $O(4)$-covariant structures (p^2 and the transverse projector). Therefore the quadratic symbol of any EOM-proportional operator is $O(4)$-covariant, and its h_{aniso}-component vanishes: $\mathrm{Proj}_{\mathrm{aniso}}(\mathrm{EOM}) = 0$.

Hence $\mathrm{Proj}_{\mathrm{aniso}}$ descends to a linear functional on $\mathfrak{O}_{6,\mathrm{os}}$.

Since $\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{O}_{\mathrm{aniso}}) = 1 \neq 0$ (Definition 3.4), the functional $\mathrm{Proj}_{\mathrm{aniso}}$ is surjective. The decomposition (10) is then the standard splitting of a vector space by a nonzero linear functional: $\mathfrak{O}_{6,\mathrm{os}} = \ker(\mathrm{Proj}_{\mathrm{aniso}}) \oplus \mathbf{R} \cdot [\mathcal{O}_{\mathrm{aniso}}]$. □

3.5 Representation-Theoretic Interpretation

The anisotropic projector defined in Section 3.3 is implemented through the harmonic polynomial h_{aniso} in (8). The representation-theoretic content reduces to the following fact:

Proposition 3.7 (Uniqueness of the hypercubic harmonic)**.** *The space of $\mathcal{W}_4$-invariant homogeneous polynomials of degree 4 on $\mathbf{R}^4$, modulo $O(4)$-invariant polynomials, is one-dimensional and spanned by $h_{\mathrm{aniso}}(p) = \sum_\mu p_\mu^4 - \frac{1}{4}(p^2)^2$.*

Proof. Every $\mathcal{W}_4$-invariant homogeneous polynomial of degree 4 is a linear combination of $\sum_\mu p_\mu^4$ and $(p^2)^2$ (the only two independent $\mathcal{W}_4$-invariant monomials at this degree). The subspace of $O(4)$-invariant polynomials of degree 4 is spanned by $(p^2)^2$. The quotient is therefore one-dimensional, generated by $\sum_\mu p_\mu^4 - \frac{1}{4}(p^2)^2 = h_{\mathrm{aniso}}(p)$ (the coefficient $\frac{1}{4}$ ensures orthogonality on S^3). □

4 Bałaban's Framework: Review and Imported Results

We import the following results from [1] and the original Bałaban programme [8, 9, 10].

4.1 RG Setup

Fix $N \geq 2$, $G = \mathrm{SU}(N)$, lattice spacing $\eta > 0$, physical volume $L_{\mathrm{phys}} > 0$, and blocking factor $L \geq 2$. The k-th RG scale is $a_k = L^k \eta$, and the terminal step is $k_* = \lfloor \log_L((\Lambda_{\mathrm{YM}}\eta)^{-1}) \rfloor$, so that $a_* = L^{k_*}\eta \sim \Lambda_{\mathrm{YM}}^{-1}$.

4.2 Effective Action and Polymer Decomposition

Theorem 4.1 (Bałaban's inductive representation, [1, Thm. 2.1])**.** *For $g_k \in (0, \gamma]$ for all $k \leq K \leq k_*$, the effective action at step k has the form*

$$A_k \;=\; -g_k^{-2}\, A(U_k) + \mathbf{E}_k(U_k) + \mathbf{R}_k(U_k) + \mathbf{B}_k(U_k) - E_k, \tag{11}$$

where $\mathbf{E}_k$, $\mathbf{R}_k$, $\mathbf{B}_k$ are polymer activities with the bounds stated below.

Proposition 4.2 (Polymer bounds, [1, Prop. 3.2])**.** *There exist $E_0, \kappa > 0$ independent of $k, \eta, L_{\mathrm{phys}}$ such that:*

$$\|\mathbf{E}_k(X)\|_\infty \leq E_0\, e^{-\kappa\, d_k(X)}, \tag{12}$$

$$\|\mathbf{R}_k(X)\|_\infty \leq e^{-p_0(g_k)}\, e^{-\kappa\, d_k(X)}. \tag{13}$$

Each polymer activity $\mathbf{E}_k(X)$ depends only on the gauge field restricted to links in X.

Proposition 4.3 (Coupling control, [1, Prop. 3.4])**.** *There exists $\gamma_0 \in (0, \gamma)$ such that for $g_0(\eta) \le \gamma_0$:*

$$g_k \le g_0 \le \gamma_0 < \gamma \quad \forall\, k = 0, \ldots, k_*.$$

Moreover, $g_k^{-2} \ge g_0^{-2} + k b_0/2$.

4.3 Analyticity of Polymer Activities

Proposition 4.4 (Analyticity in the small-field domain, [1, Prop. 3.5])**.** *In the small-field domain (where $\chi_k = 1$), the polymer activities $\mathbf{E}_k(X)$ are analytic functions of the link variables U_ℓ for $\ell \in X$, with a radius of analyticity $R_k > 0$ (in $\mathfrak{su}(N)$ coordinates around the identity) that depends only on γ and L, not on k.*

4.4 Exponential Clustering

Theorem 4.5 (Mass gap bound, [1, Thm. 6.1])**.** *For all gauge-invariant local observables $\mathcal{O}$ with $\|\mathcal{O}\|_\infty \le 1$ and all $|x| \ge a_*$:*

$$\left|\mathrm{Cov}_{\mu_\eta}(\mathcal{O}(0), \mathcal{O}(x))\right| \;\le\; C\, e^{-m\,|x|/a_*},$$

with $m > 0$ and C independent of η and L_{phys}.

5 Symanzik Decomposition of the Effective Action

5.1 Taylor Expansion of Polymer Activities

We now extract local operator content from the polymer activities $\mathbf{E}_k(X)$.

Lemma 5.1 (Polymer Taylor expansion)**.** *Let $\mathbf{E}_k(X)$ be a polymer activity in the small-field domain, viewed as a function of the link variables $U_\ell = e^{a_k \omega_\ell}$ for $\ell \in X$, where $\omega_\ell \in \mathfrak{su}(N)$ with $\|\omega_\ell\| \le R_k/a_k$. Then for any multi-index $\alpha = (\alpha_{\ell_1}, \ldots, \alpha_{\ell_n})$ with $|\alpha| \le M$:*

$$\left|\frac{\partial^{|\alpha|}\mathbf{E}_k(X)}{\partial\omega_{\ell_1}^{\alpha_{\ell_1}} \cdots \partial\omega_{\ell_n}^{\alpha_{\ell_n}}}\right| \;\le\; |\alpha|!\, R_k^{-|\alpha|}\, E_0\, e^{-\kappa\, d_k(X)} \prod_{i=1}^{n} \mathbf{1}_{\ell_i \in X}, \tag{14}$$

where $R_k > 0$ is the analyticity radius from Proposition 4.4.

Proof. This is a direct application of the Cauchy estimate for analytic functions of several variables: for f analytic in a polydisc of radius R with $|f| \le M$, we have $|\partial^\alpha f(0)| \le |\alpha|!\, R^{-|\alpha|}\, M$. Applied to $\mathbf{E}_k(X)$ viewed as a function of ω_ℓ, $\ell \in X$, with $M = E_0\, e^{-\kappa d_k(X)}$ and radius R_k (Proposition 4.4), the bound (14) follows. □

5.2 Local Operator Extraction

For a polymer $\mathbf{E}_k(X)$ centered at a point $x_0 \in X$, we define the *local jet of order* Δ by Taylor-expanding in the link variables ω_ℓ and collecting terms of homogeneity degree Δ in the "slow field" approximation (where $\omega_\ell \approx a_k\, A_\mu(x_0)$ for $\ell = (x_0, \mu)$ plus gradient corrections).

Definition 5.2 (Local jet extraction and anisotropic coefficient)**.** Fix a scale k and write $U_\ell = \exp(a_k \omega_\ell)$ on the small-field domain, with $\omega_\ell \in \mathfrak{su}(N)$. The *degree-6 jet extraction map* $\mathcal{J}_{k,6}$ applied to a polymer activity $\mathbf{E}_k(X)$ at basepoint $x_0 \in X$ is defined by:

(i) compute the Taylor polynomial of $\mathbf{E}_k(X)$ in the variables ω_ℓ ($\ell \in X$) up to total order 6 at $\omega = 0$;

(ii) identify ω_ℓ with the continuum field components and first derivatives at x_0 via the standard slow-field correspondence $\omega_{(x_0,\mu)} \leftrightarrow A_\mu(x_0)$, $\partial_\nu\omega_{(x_0,\mu)} \leftrightarrow \partial_\nu A_\mu(x_0)$, etc.;

(iii) discard total-divergence terms.

The result $\mathcal{J}_{k,6}\mathbf{E}_k(X) \in \mathfrak{O}_6^{\mathcal{W}_4}$ is a well-defined element of the dimension-6 operator space.

The *anisotropic coefficient at scale* k is

$$c_{6,\mathrm{aniso}}^{(k)} := \frac{1}{|\Lambda_{a_k}|} \sum_{x_0 \in \Lambda_{a_k}} \sum_{X \ni x_0} \mathrm{Proj}_{\mathrm{aniso}}\big(\mathcal{J}_{k,6}\mathbf{E}_k(X)\big), \tag{15}$$

where $\mathrm{Proj}_{\mathrm{aniso}}$ is from Definition 3.3. By the properties of $\mathrm{Proj}_{\mathrm{aniso}}$ (independence of on-shell representative), this definition is unambiguous.

5.3 Large-Field Contributions

Lemma 5.3 (Large-field remainder)**.** *The contribution of large-field polymers $\mathbf{R}_k(X)$ and boundary polymers $\mathbf{B}_k(X)$ to any dimension-6 local coefficient is bounded by*

$$\big|c_{6,\alpha}^{(k)}\big|_{\textit{large-field}} \le C\, e^{-p_0(g_k)},$$

which is super-polynomially small in g_k^{-1} and hence negligible compared to the small-field contribution.

Proof. By (13), $\|\mathbf{R}_k(X)\|_\infty \le e^{-p_0(g_k)-\kappa d_k(X)}$. The Taylor expansion of $\mathbf{R}_k(X)$ at any order Δ is bounded by $C_\Delta\, R_k^{-\Delta}\, e^{-p_0(g_k)}$ by Lemma 5.1. Summing over connected polymers $X \ni x_0$ using the lattice animal bound (Lemma D.1) gives a finite constant times $e^{-p_0(g_k)}$. □

5.4 Main Bound

Theorem 5.4 (Anisotropy coefficient bound)**.** *For each $N \ge 2$, there exists a constant $C < \infty$ (depending on N, L, γ_0 but not on $k, \eta, L_{\mathrm{phys}}$) such that for all $k \le k_*$ and $\eta \in (0, \eta_0]$:*

$$\big|c_{6,\mathrm{aniso}}^{(k)}\big| \le C\, a_k^2. \tag{16}$$

Proof. By Definition 5.2 and linearity of $\mathrm{Proj}_{\mathrm{aniso}}$, it suffices to bound $|\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{J}_{k,6}\mathbf{E}_k(X))|$ for each polymer X and then sum over X.

Step 1: per-polymer jet bound (small-field). In the small-field domain, Lemma 5.1 provides Cauchy bounds for all derivatives of $\mathbf{E}_k(X)$ in the coordinates ω_ℓ.

The jet extraction map $\mathcal{J}_{k,6}$ is a fixed linear combination of finitely many partial derivatives of total order ≤ 6. Each occurrence of ω_ℓ in the Taylor polynomial carries one explicit factor of a_k from the parameterization $U_\ell = \exp(a_k\omega_\ell)$. A term of homogeneity degree d in ω therefore carries a_k^d. Since the leading (dimension-4) term corresponds to $d = 4$, the dimension-6 jet carries $a_k^6/a_k^4 = a_k^2$ relative to the leading term.

Applying $\mathrm{Proj}_{\mathrm{aniso}}$ (a bounded linear functional on the finite-dimensional space $\mathfrak{O}_6^{\mathcal{W}_4}$) and using the Cauchy bound (14), we obtain: there exists $C_6 < \infty$ (depending on N, L, γ_0, but not on k, η, or L_{phys}) such that

$$\left|\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{J}_{k,6}\mathbf{E}_k(X))\right| \;\le\; C_6\, a_k^2\, \Xi_k\, e^{-\kappa\, d_k(X)}, \tag{17}$$

where $\Xi_k := 6!\, R_k^{-6}\, E_0$ and the factor R_k^{-6} comes from the Cauchy estimate with analyticity radius R_k (Proposition 4.4). Since R_k depends only on γ and L (not on k), Ξ_k is uniformly bounded.

Step 2: polymer summation (uniform in volume). Summing (17) over connected polymers containing a fixed basepoint x_0:

$$\sum_{X\ni x_0} \left|\mathrm{Proj}_{\mathrm{aniso}}(\mathcal{J}_{k,6}\mathbf{E}_k(X))\right| \;\le\; C_6\, a_k^2\, \Xi_k \sum_{X\ni x_0} e^{-\kappa\, d_k(X)} \;\le\; C_6\, a_k^2\, \Xi_k\, C_{\mathrm{anim}},$$

by the lattice animal bound (Lemma D.1). Averaging over $x_0 \in \Lambda_{a_k}$ gives $|c_{6,\mathrm{aniso}}^{(k)}|_{\text{small-field}} \le C'\, a_k^2$.

Step 3: large-field contribution. By Lemma 5.3, the large-field polymers $\mathbf{R}_k$ and boundary terms $\mathbf{B}_k$ contribute at most $O(e^{-p_0(g_k)})$ to any dimension-6 coefficient, which is negligible compared to a_k^2 for $g_k \le \gamma_0$.

Uniformity. All constants (E_0, κ, R_k, C_{anim}) are independent of k, η, and L_{phys} by Propositions 4.2 and 4.4 and Lemma D.1. In particular, Ξ_k is uniformly bounded because R_k depends only on γ and L, not on k (Proposition 4.4).

Combining the small-field bound $O(a_k^2)$ from Steps 1–2 with the negligible large-field remainder from Step 3 yields (16). □

Corollary 5.5 (UV and terminal-scale formulations)**.** *Since $a_k = L^k\eta$, Theorem 5.4 gives*

$$\left|c_{6,\mathrm{aniso}}^{(k)}\right| \;\le\; C\, L^{2k}\, \eta^2 \qquad (k \le k_*). \tag{18}$$

In particular, using $k_* = \lfloor \log_L((\Lambda_{\mathrm{YM}}\eta)^{-1})\rfloor$ and $a_* = L^{k_*}\eta$, we have $a_* \le \Lambda_{\mathrm{YM}}^{-1} < L\, a_*$ and therefore

$$\left|c_{6,\mathrm{aniso}}^{(k)}\right| \;\le\; C\, a_k^2 \;\le\; C\, a_*^2 \;\le\; C\, \Lambda_{\mathrm{YM}}^{-2},$$

uniformly in η and $k \le k_*$.

6 Operator Insertions and Correlation Bounds

The final ingredient needed for the rotational symmetry restoration argument (to be carried out in [2]) is a bound on the effect of inserting the anisotropic operator into lattice Schwinger functions.

6.1 Lattice Representative of $\mathcal{O}_{\mathrm{aniso}}$

Definition 6.1 (Lattice representative and normalization of $\mathcal{O}_{\mathrm{aniso}}$)**.** In the insertion estimates we use a *fixed* gauge-invariant lattice observable $\widetilde{\mathcal{O}}_{\mathrm{aniso}}(y)$ supported in a bounded neighborhood of y on the terminal lattice (spacing a_*), chosen so that its slow-field expansion at spacing a_* has leading classical dimension-6 term proportional to the continuum operator $\mathcal{O}_{\text{aniso}}$ from Definition 3.4. We then set

$$\mathcal{O}_{\text{aniso}}(y) \;:=\; a_*^{-6}\,\widetilde{\mathcal{O}_{\text{aniso}}}(y), \tag{19}$$

so that $\mathcal{O}_{\text{aniso}}$ is normalized as a dimension-6 *density*. Since $\mathrm{SU}(N)$ is compact and $\widetilde{\mathcal{O}_{\text{aniso}}}$ depends on finitely many link variables, there exists $C_{\mathcal{O}_{\text{aniso}}} < \infty$ such that

$$\|\mathcal{O}_{\text{aniso}}\|_\infty \;\le\; C_{\mathcal{O}_{\text{aniso}}}\, a_*^{-6}. \tag{20}$$

6.2 Smeared Insertion Functional (Ward-Independent)

Definition 6.2 (Smeared insertion functional)**.** Fix $n \ge 1$ and a bounded gauge-invariant local observable $\mathcal{O}$ with $\|\mathcal{O}\|_\infty \le 1$. For a test function $f \in \mathcal{S}(\mathbf{R}^{4n})$, define

$$\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f) \;:=\; \sum_{y\in\Lambda_\eta} \eta^4 \int_{\mathbf{R}^{4n}} |f(x_1,\dots,x_n)|\,\big|\big\langle \mathcal{O}_{\text{aniso}}(y)\cdot\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\big\rangle_{\mu_\eta,c}\big|\,dx_1\cdots dx_n. \tag{21}$$

Remark 6.3 (Interface with the Ward identity)*.* The functional $\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f)$ controls the absolute value of the insertion sum without reference to any Ward identity. In [2], the rotational Ward identity produces a breaking distribution of the form $\eta^2\, g_0(\eta)^{-2} \times$ (signed insertion sum). A uniform bound $\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f) \le C(f)$ then immediately yields that the breaking distribution is $O(\eta^2\, g_0(\eta)^{-2}) = O\big(\eta^2\,|\log((\Lambda_{\text{YM}}\eta)^{-1})|\big) \to 0$.

6.3 UV Control of Insertions

Lemma 6.4 (Local (UV) insertion bound)**.** *There exist constants $C_n < \infty$ and $\kappa' > 0$ (independent of η and L_{phys}) such that for all $y \in \Lambda_\eta$ and all configurations $x_1,\dots,x_n$:*

$$\Big|\Big\langle \mathcal{O}_{\text{aniso}}(y)\cdot\prod_{j=1}^{n}\mathcal{O}(x_j)\Big\rangle_{\mu_\eta,c}\Big| \;\le\; C_n\, a_*^{-6}\, \exp\!\Big(-\kappa'\,\frac{\min_j |y-x_j|}{a_*}\Big). \tag{22}$$

Proof. By Definition 6.1, the insertion observable satisfies $\|\mathcal{O}_{\text{aniso}}\|_\infty \le C_{\mathcal{O}_{\text{aniso}}}\, a_*^{-6}$, which accounts for the prefactor a_*^{-6}.

The connected correlator is a sum over connected polymer clusters in the terminal expansion that link $\operatorname{supp}(\mathcal{O}_{\text{aniso}}(y))$ to $\operatorname{supp}(\prod_{j=1}^{n}\mathcal{O}(x_j))$. Let $S_{\mathcal{O}_{\text{aniso}}}$ and $S_{\mathcal{O}}$ denote the fixed support radii (in terminal lattice units) of $\widetilde{\mathcal{O}_{\text{aniso}}}$ and $\mathcal{O}$ respectively. Then

$$\operatorname{dist}\!\big(\operatorname{supp}(\mathcal{O}_{\text{aniso}}(y)),\, \operatorname{supp}(\textstyle\prod_j \mathcal{O}(x_j))\big) \;\ge\; \min_j |y-x_j| - (S_{\mathcal{O}_{\text{aniso}}} + S_{\mathcal{O}})\, a_*.$$

Any connected cluster spanning this distance has tree diameter at least the left-hand side divided by a_*, and is suppressed by $e^{-\kappa d_{k_*}(X)}$ (Proposition 4.2). Absorbing the fixed $O(1)$ correction $(S_{\mathcal{O}_{\text{aniso}}} + S_{\mathcal{O}})$ and the lattice-animal summation into the constant C_n yields (22) with $\kappa' < \kappa$. □

6.4 IR Control of Insertions

Lemma 6.5 (Infrared decay via clustering)**.** *Assume the exponential clustering estimate of Theorem 4.5. Then there exist $C_n < \infty$ and $m > 0$ such that for all y and $x_1, \dots, x_n$:*

$$\Big|\Big\langle \mathcal{O}_{\text{aniso}}(y) \cdot \prod_{j=1}^{n} \mathcal{O}(x_j) \Big\rangle_{\mu_{\eta,c}}\Big| \;\le\; C_n\, a_*^{-6}\, \|\mathcal{O}\|_\infty^n \exp\!\Big(-m\, \frac{\min_j |y - x_j|}{a_*}\Big). \tag{23}$$

Proof. The factor a_*^{-6} enters through $\|\mathcal{O}_{\text{aniso}}\|_\infty \le C_{\mathcal{O}_{\text{aniso}}}\, a_*^{-6}$ (Definition 6.1).

Let $S_{\mathcal{O}}$ denote the fixed lattice support radius of $\mathcal{O}$ and $S_{\mathcal{O}_{\text{aniso}}}$ that of $\widetilde{\mathcal{O}_{\text{aniso}}}$, so that each observable depends only on links within the respective radius (in terminal lattice units). Define

$$r(y; x_1, \dots, x_n) \;:=\; \min_{1 \le j \le n} \operatorname{dist}\!\big(\operatorname{supp}(\mathcal{O}_{\text{aniso}}(y)),\, \operatorname{supp}(\mathcal{O}(x_j))\big).$$

Then $r(y; x) \ge \min_j |y - x_j| - (S_{\mathcal{O}_{\text{aniso}}} + S_{\mathcal{O}})\, a_*$.

By the exponential clustering bound (Theorem 4.5) applied iteratively along a telescoping decomposition linking $\mathcal{O}_{\text{aniso}}(y)$ to $\prod_j \mathcal{O}(x_j)$ (or equivalently via the cluster expansion of the terminal measure, Corollary 5.3 of [1]):

$$\Big|\Big\langle \mathcal{O}_{\text{aniso}}(y) \cdot \textstyle\prod_j \mathcal{O}(x_j) \Big\rangle_{\mu_{\eta,c}}\Big| \;\le\; C_n'\, \|\mathcal{O}_{\text{aniso}}\|_\infty\, \|\mathcal{O}\|_\infty^n\, e^{-m\, r(y;x)/a_*}.$$

Absorbing the fixed support radii and $\|\mathcal{O}_{\text{aniso}}\|_\infty \le C_{\mathcal{O}_{\text{aniso}}}\, a_*^{-6}$ into C_n yields (23). □

6.5 Main Insertion Bound

Theorem 6.6 (Insertion integrability)**.** *Assume the exponential clustering estimate of Theorem 4.5. Then for every $n \ge 1$, every gauge-invariant local observable $\mathcal{O}$ with $\|\mathcal{O}\|_\infty \le 1$, and every $f \in \mathcal{S}(\mathbf{R}^{4n})$, there exists $C(f) < \infty$ (depending on n, $\|\mathcal{O}\|_\infty$, $\|f\|_{L^1(\mathbf{R}^{4n})}$, and Λ_{YM}) such that for all $\eta \in (0, \eta_0]$:*

$$\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f) \;\le\; C(f), \tag{24}$$

where $\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f)$ is from Definition 6.2.

Proof. By Lemmas 6.4 and 6.5, there exist $C_n < \infty$ and $m' > 0$ (set $m' := \min(\kappa', m)$) such that for all y and $x = (x_1, \dots, x_n)$:

$$\Big|\Big\langle \mathcal{O}_{\text{aniso}}(y) \cdot \textstyle\prod_j \mathcal{O}(x_j) \Big\rangle_{\mu_{\eta,c}}\Big| \;\le\; C_n\, a_*^{-6} \exp\!\Big(-m'\, \tfrac{\min_j |y - x_j|}{a_*}\Big),$$

where the prefactor a_*^{-6} is the normalization from Definition 6.1.

Fix $x \in \mathbf{R}^{4n}$ and sum over $y \in \Lambda_\eta$. Using $e^{-m' \min_j |y - x_j|/a_*} \le \sum_{j=1}^{n} e^{-m'|y - x_j|/a_*}$, the lattice sum satisfies

$$\sum_{y \in \Lambda_\eta} \eta^4 \exp\!\Big(-m'\, \frac{\min_j |y - x_j|}{a_*}\Big) \;\le\; \sum_{j=1}^{n} \sum_{y \in \Lambda_\eta} \eta^4 \exp\!\Big(-m'\, \frac{|y - x_j|}{a_*}\Big) \;\le\; n \int_{\mathbf{R}^4} \exp\!\Big(-m'\, \frac{|z|}{a_*}\Big)\, dz \;=\; n\, C_0\, a_*^4$$

where $C_0 = 2\pi^2 \int_0^\infty e^{-m' r}\, r^3\, dr = 12\pi^2/(m')^4$.

Inserting into Definition 6.2:

$$\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f) \;\le\; C_n\, a_*^{-6} \cdot n\, C_0\, a_*^4 \cdot \|f\|_{L^1(\mathbf{R}^{4n})} \;=\; n\, C_n\, C_0\, a_*^{-2}\, \|f\|_{L^1}.$$

Since $a_* = L^{k_*}\eta$ with $k_* = \lfloor \log_L((\Lambda_{\text{YM}}\eta)^{-1}) \rfloor$, we have $a_* \le \Lambda_{\text{YM}}^{-1} < L\, a_*$, and therefore $a_*^{-2} \le L^2\, \Lambda_{\text{YM}}^2$ uniformly in η. This gives $\mathcal{I}^{\eta}_{\mathcal{O}_{\text{aniso}}}(f) \le C(f)$ with $C(f) = n\, C_n\, C_0\, L^2\, \Lambda_{\text{YM}}^2\, \|f\|_{L^1}$. □

7 Discussion

7.1 Summary of Results

This paper establishes three results that together provide the complete quantitative control of $O(4)$-breaking lattice artifacts needed for the restoration of rotational symmetry in the continuum limit:

(I) The space of anisotropic operators is one-dimensional (Theorem 3.6), so only a single coefficient needs to be controlled.

(II) This coefficient satisfies the quadratic scale bound $|c_{6,\text{aniso}}^{(k)}| \le C\, a_k^2$ uniformly in all parameters (Theorem 5.4).

(III) Connected correlators with one insertion of $\mathcal{O}_{\text{aniso}}$ are integrable in the insertion point after smearing the remaining arguments against Schwartz functions (Theorem 6.6).

7.2 Interface with the Companion Paper

The companion paper [2] derives a lattice Ward identity for infinitesimal rotations and shows that the breaking term takes the schematic form

$$\langle L_{\mu\nu}\mathcal{S}_n^\eta, f\rangle \;=\; \eta^2\, g_0(\eta)^{-2}\, \big(\text{signed insertion of } \mathcal{O}_{\text{aniso}}\big) \;+\; \big(O(4)\text{-invariant cancellations}\big), \tag{25}$$

where $L_{\mu\nu}$ is the orbital angular momentum generator. Theorem 6.6 provides a uniform bound on the absolute value of the insertion sum. Using $g_0(\eta)^{-2} \sim b_0 \log((\Lambda_{\text{YM}}\eta)^{-1})$, the right-hand side of (25) is bounded by $O\big(\eta^2\, |\log((\Lambda_{\text{YM}}\eta)^{-1})|\big) \to 0$ as $\eta \to 0$ (since $\eta^2 \log((\Lambda_{\text{YM}}\eta)^{-1}) \to 0$), yielding $L_{\mu\nu}\mathcal{S}_n = 0$ for any subsequential continuum limit.

7.3 Extensions

- **Other gauge groups.** The classification (Theorem 3.6) applies to any compact simple gauge group: the operators $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ are universal. The coefficient bounds (Theorem 5.4) hold whenever Bałaban's framework applies, i.e., whenever $b_0 > 0$ (asymptotic freedom).

- **Improved actions.** For actions already improved à la Symanzik ($c_{6,\text{aniso}}^{(\text{tree})} = 0$), the bounds improve to $O\big(\eta^2\, g_0(\eta)^2\big)$, i.e., the anisotropy is suppressed by an additional power of the coupling.

- **Fermions.** The classification extends to gauge-fermion theories with the same dimension-6 operators in the pure gauge sector. The fermionic operators of dimension 6 (involving $\bar\psi D^3 \psi$ etc.) require separate analysis.

A Representation Theory of $\mathcal{W}_4$

The representation-theoretic content used in this paper reduces to the existence and uniqueness (up to scaling) of a $\mathcal{W}_4$-invariant degree-4 harmonic polynomial that is not $O(4)$-invariant. An explicit choice is h_{aniso} from equation (8).

The hyperoctahedral group $\mathcal{W}_4$ is the symmetry group of the four-dimensional hypercube. It has order $|\mathcal{W}_4| = 2^4 \cdot 4! = 384$ and is isomorphic to the wreath product $\mathbf{Z}_2 \wr S_4$.

Lemma A.1 (Uniqueness of the anisotropic harmonic)**.** *The space of $\mathcal{W}_4$-invariant homogeneous polynomials of degree 4 on $\mathbf{R}^4$, modulo $O(4)$-invariant polynomials, is one-dimensional and is spanned by $h_{\text{aniso}}(p) = \sum_\mu p_\mu^4 - \frac{1}{4}(p^2)^2$.*

Proof. Every $\mathcal{W}_4$-invariant monomial of degree 4 is a linear combination of the elementary symmetric polynomials in $p_0^2, p_1^2, p_2^2, p_3^2$. At degree 4 in p, the independent $\mathcal{W}_4$-invariant monomials are $\sum_\mu p_\mu^4$ and $(\sum_\mu p_\mu^2)^2 = (p^2)^2$. The $O(4)$-invariant subspace at this degree is spanned by $(p^2)^2$ alone. The quotient is therefore one-dimensional, generated by $\sum_\mu p_\mu^4 \mod (p^2)^2$, which equals h_{aniso} after subtracting the $O(4)$-invariant part with the normalization that makes h_{aniso} traceless (i.e., orthogonal to $(p^2)^2$ on S^3). □

Remark A.2. No explicit character table or rank-4 tensor formula is needed for any result in this paper. The polynomial h_{aniso} and its uniqueness (Lemma A.1) are the only representation-theoretic inputs.

B Classical Expansions of Lattice Loops

We collect the classical expansions of the lattice loops used in Section 2, computed by the method of [3, Sect. 2].

B.1 Plaquette

The plaquette matrix $P_{\mu\nu}(0) = U(0,\mu)U(\hat{\mu},\nu)U(\hat{\nu},\mu)^{-1}U(0,\nu)^{-1} - \not{1}$ expands as [3, Eq. (16)]:

$$P_{\mu\nu}(0) = a^2 F_{\mu\nu} + \tfrac{1}{2}a^3(D_\mu + D_\nu)F_{\mu\nu} \\ + \tfrac{1}{6}a^4\left\{3F_{\mu\nu}F_{\mu\nu} + (D_\mu^2 + \tfrac{3}{2}D_\mu D_\nu + D_\nu^2)F_{\mu\nu}\right\} + O(a^5). \tag{26}$$

The plaquette trace is then $\operatorname{Re}\operatorname{Tr}(\not{1} - U(\partial p)) = -\frac{1}{2}a^4 \operatorname{tr}(F_{\mu\nu}^2) + O(a^6)$, confirming $\dim(\mathcal{O}_0) = 4$. The dimension-6 contributions are those listed in Table 1.

B.2 Rectangle (1×2)

The rectangle loop in the (μ,ν)-plane with sides a and $2a$ produces operators $\mathcal{O}_1' = \mathcal{O}_1 - 8\mathcal{O}_0$ of dimension 6, with coefficients $r_1 = \frac{5}{6}$, $r_2 = r_3 = 0$.

B.3 Parallelogram and Chair

Similarly, the parallelogram (three-dimensional) and chair (bent rectangle) loops produce the remaining entries in Table 1. These are computed using the expansion (26) for the component plaquettes and combining them multiplicatively.

C Cauchy Bounds on Polymer Derivatives

We provide a self-contained proof of Lemma 5.1.

Proof of Lemma 5.1. Let X be a connected polymer and $\ell_1, \dots, \ell_n$ links in X. Parameterize $U_{\ell_i} = e^{a_k \omega_{\ell_i}}$ with $\omega_{\ell_i} \in \mathfrak{su}(N)$.

By Proposition 4.4, the function $\omega \mapsto \mathbf{E}_k(X)(e^{a_k\omega})$ is analytic in a polydisc $\{\omega : \|\omega_\ell\| < R_k/a_k, \forall \ell \in X\}$ and satisfies $|\mathbf{E}_k(X)| \le E_0\, e^{-\kappa d_k(X)}$ throughout this domain.

The standard Cauchy estimate for analytic functions of several complex variables states: if f is analytic in a polydisc of radius R and $|f| \le M$ there, then

$$\Big|\frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(0)\Big| \;\le\; \frac{|\alpha|!}{R^{|\alpha|}}\, M.$$

Applying this with $z_i = \omega_{\ell_i}$, $R = R_k/a_k$, and $M = E_0\, e^{-\kappa d_k(X)}$ gives the bound (14) (after absorbing the factor $a_k^{|\alpha|}$ from the chain rule into a redefinition of R_k). $\square$

D Lattice Animal Bounds

Lemma D.1 (Lattice animal bound, [1, Lem. A.4])**.** *For $\kappa > \log(2de)$ (where $d = 4$ is the dimension):*

$$\sum_{\substack{X \ni 0 \\ X \text{ connected}}} e^{-\kappa d(X)} \;\le\; (1 - (2de)e^{-\kappa})^{-1} \;<\; \infty.$$

References

[1] L. Eriksson, Exponential clustering and mass gap for four-dimensional SU(N) lattice Yang–Mills theory, preprint (2026).

[2] L. Eriksson, Rotational symmetry restoration in the continuum limit of 4D SU(N) lattice Yang–Mills theory, in preparation (2026).

[3] M. Lüscher and P. Weisz, On-shell improved lattice gauge theories, *Commun. Math. Phys.* **97** (1985), 59–77.

[4] M. Lüscher and P. Weisz, Perturbative analysis of the gradient flow in non-abelian gauge theories, *JHEP* **1102** (2011), 051; arXiv:1101.0963.

[5] K. Symanzik, Continuum limit and improved action in lattice theories. I. Principles and ϕ^4 theory, *Nucl. Phys.* **B226** (1983), 187–204.

[6] K. Symanzik, Continuum limit and improved action in lattice theories. II. $O(N)$ nonlinear sigma model in perturbation theory, *Nucl. Phys.* **B226** (1983), 205–227.

[7] T. Reisz, A power counting theorem for Feynman integrals on the lattice, *Commun. Math. Phys.* **116** (1988), 81–126.

[8] T. Bałaban, Propagators and renormalization transformations for lattice gauge theories I, *Commun. Math. Phys.* **95** (1984), 17–40.

[9] T. Bałaban, Convergent renormalization expansions for lattice gauge theories, *Commun. Math. Phys.* **119** (1988), 243–285.

[10] T. Bałaban, Large field renormalization II, *Commun. Math. Phys.* **122** (1989), 355–392.

[11] T. Bałaban, Renormalization group approach to lattice gauge field theories I, *Commun. Math. Phys.* **109** (1987), 249–301.

[12] D. W. Stroock and B. Zegaliński, The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition, *Commun. Math. Phys.* **144** (1992), 303–323.

[13] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions II, *Commun. Math. Phys.* **42** (1975), 281–305.

[14] D. J. Gross and F. Wilczek, Ultraviolet behavior of non-abelian gauge theories, *Phys. Rev. Lett.* **30** (1973), 1343–1346.

[15] H. D. Politzer, Reliable perturbative results for strong interactions?, *Phys. Rev. Lett.* **30** (1973), 1346–1349.

[16] J. Dimock, The renormalization group according to Bałaban I: small fields, *Rev. Math. Phys.* **25** (2013), 1330010; arXiv:1108.1335.

[17] P. Weisz, Continuum limit improved lattice action for pure Yang-Mills theory I, *Nucl. Phys.* **B212** (1983), 1–17.

[18] G. Curci, P. Menotti, and G. Paffuti, Symanzik's improved Lagrangian for lattice gauge theory, *Phys. Lett.* **130B** (1983), 205–208.