Gradient Flow Observables and Ultraviolet Closure without Blocking-Map Hypotheses for Lattice Yang–Mills Theory in Four Dimensions

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Abstract

We prove that the continuum limit of pure $SU(N)$ lattice Yang–Mills theory in four Euclidean dimensions exists at fixed finite volume and fixed physical flow time $t > 0$ for *scale-consistent* families of gradient-flow observables. The Yang–Mills gradient flow (Wilson flow) of Lüscher replaces the geometric blocking map of Bałaban [1] as the regularization of observables: at RG scale k (lattice spacing $a_k = a_0\mathfrak{L}^{-k}$), we evaluate the flow at lattice time $\tau_k = t/a_k^2$, keeping the physical smoothing radius $\sqrt{8t}$ fixed. The smoothing properties of the flow — governed by a discrete heat kernel with Gaussian decay on the link graph — guarantee that the Doob influence seminorm of flowed observables decays as $O(1/\tau_k) = O(a_k^2) = O(\mathfrak{L}^{-2k})$, yielding geometric summability of the RG–Cauchy series for the measure-change increments. No oscillation-summability hypothesis on any blocking map (cf. Assumption A of [15]) is required. The construction is conditional on a single remaining hypothesis: an L^1 scale-consistency condition on the observable family across RG scales (Assumption A), which is verified automatically for all standard physical observables (Wilson loops, Polyakov loops, action density) by discretization-error estimates in the small-field region combined with Bałaban's large-field suppression; see Remark 4.3. The resulting continuum state is gauge-invariant, Euclidean-covariant, and positive. We further show that the gradient flow commutes with Euclidean time-reflection, a structural property relevant for reflection-positivity constructions in the Osterwalder–Schrader framework.

Contents

1 Introduction

1.1 Motivation and context

The construction of a four-dimensional quantum Yang–Mills theory satisfying the Wightman axioms and exhibiting a mass gap is one of the seven Clay Millennium Prize Problems [17]. A central difficulty is the *ultraviolet problem*: showing that physical quantities have well-defined limits as the lattice spacing is removed.

Bałaban's renormalization group (RG) program [2, 3, 4, 5, 6] provides rigorous control of the effective action at all scales, but does not by itself define a continuum limit for physical observables. The companion paper [15] established this continuum limit for *blocked observables* $F \circ Q_{\ell,k}$, conditional on a quantitative summability hypothesis (Assumption A of [15]) for the blocking map.

In the present paper we replace the geometric blocking by the **Yang–Mills gradient flow** (Wilson flow), introduced by Lüscher [18] and analysed perturbatively to all orders by Lüscher and Weisz [19]. This substitution achieves three goals simultaneously:

(i) The smoothing properties of the flow discharge the oscillation summability condition that was Assumption A of [15], replacing it with a provable theorem (Theorem 3.11). Bałaban's one-step block-averaging map $\mathfrak{q}_{k+1,k}$ is still used as a *technical intermediary* for the measure-change term β_k (Section 4.2), but no quantitative summability hypothesis is imposed on it: the needed bound follows unconditionally from the locality of $\mathfrak{q}_{k+1,k}$ (Theorem 4.2). The only remaining assumption is an L^1 scale-consistency condition on the observable family across RG scales (Assumption A), which holds automatically for standard physical observables (Remark 4.3).

(ii) The flow acts deterministically on each configuration and commutes with Euclidean time-reflection, providing a structural foundation for reflection positivity (Theorem 1.3).

(iii) The flowed field at positive flow time is a *renormalized* probe of the theory at the physical scale $\sqrt{8t}$, as shown perturbatively in [19].

1.2 Two-layer architecture

Our construction rests on a strict separation of concerns:

Layer 1 (Measure construction). Bałaban's program provides, for each lattice spacing $a_k = a_0\mathfrak{L}^{-k}$, a lattice measure μ_k with a controlled polymer representation of the effective action density. The per-link oscillation bounds, lattice-animal counting, and large-field suppression are all established in [2, 3, 4, 5, 6] and organized in [12, 13] (see also the expository accounts [9, 10, 11]).

Layer 2 (Observable regularization). The gradient flow $\mathcal{W}_\tau : U \mapsto V_\tau(U)$ defines a canonical smoothing of the gauge field at the physical scale $\sqrt{8t}$. At fixed RG scale k, physical observables are functions $F_k \circ \mathcal{W}_{\tau_k}$ where F_k is a bounded gauge-invariant function of the flowed configuration. Across scales, observables are organized as families $F = (F_k) \in \mathfrak{A}^t$.

The interaction between the two layers occurs through a single interface: the Doob-martingale covariance bound applied to the interpolating measure $\nu_{k,s}$ that arises in the Duhamel representation of the per-step increment δ_k.

1.3 Main results

Theorem 1.1 (UV closure for gradient-flow observable families)**.** *Let $G = SU(N)$, $N \geq 2$, on a finite four-dimensional torus $\mathbb{T}_L$. Fix a physical flow time $t > 0$ and set $\tau_k := t/a_k^2$. There exists $g_* > 0$ such that for every bare coupling $g_0 \in (0, g_*]$ and every family $F = (F_k) \in \mathfrak{A}^t$ satisfying the scale-consistency condition Assumption A, the sequence*

$$\omega_k(F_k \circ \mathcal{W}_{\tau_k}) \ := \ \int F_k\Big(V_{\tau_k}(U)\Big)\, d\mu_k(U), \qquad k = 1, 2, 3, \ldots, \tag{1}$$

converges. The limit

$$\omega_L^t(F) \ := \ \lim_{k\to\infty} \omega_k(F_k \circ \mathcal{W}_{\tau_k}) \tag{2}$$

defines a positive, gauge-invariant, Euclidean-covariant state on the C^*-algebra $\overline{\mathfrak{A}}^t$ (the completion with respect to $\|\cdot\|_\infty$). Moreover, for k large enough,

$$\Big|\omega_k(F_k \circ \mathcal{W}_{\tau_k}) - \omega_L^t(F)\Big| \ \leq \ C_t\, |||F|||\, \mathfrak{L}^{-2k}, \tag{3}$$

where C_t depends on t, N, L but not on k.

Remark 1.2. No hypothesis on any blocking map appears in the statement. The role previously played by Assumption A of [15] is now filled by the heat-kernel smoothing of the gradient flow (Theorem 3.11 below). The geometric decay $\mathfrak{L}^{-2k}$ arises because the lattice flow time $\tau_k = t/a_k^2$ grows as $\mathfrak{L}^{2k}$, so the Doob influence seminorm of the flowed observable decays as $O(1/\tau_k) = O(\mathfrak{L}^{-2k})$.

Proposition 1.3 (Flow–reflection commutation)**.** *The gradient flow commutes with Euclidean time-reflection: $\mathcal{W}_\tau \circ \Theta = \Theta \circ \mathcal{W}_\tau$ for every $\tau \geq 0$. Consequently, for every k and every bounded gauge-invariant function $F_k : G^{|\Lambda_k^1|} \to \mathbb{C}$,*

$$\omega_k\Big((\Theta(F_k \circ \mathcal{W}_{\tau_k}))^* \cdot (F_k \circ \mathcal{W}_{\tau_k})\Big) \ = \ \omega_k\Big((\Theta F_k \circ \mathcal{W}_{\tau_k})^* \cdot (F_k \circ \mathcal{W}_{\tau_k})\Big). \tag{4}$$

Remark 1.4 (Reflection positivity: status). The lattice RP inequality of Osterwalder–Seiler [23] states that $\int (\Theta A)^* A\, d\mu_k \geq 0$ for functions A depending only on links in the half-space $\{x_0 > 0\}$. Since the gradient flow is parabolic and diffuses information globally, the composed function $F_k \circ \mathcal{W}_{\tau_k}$ depends on *all* links and is therefore **not** directly admissible as a test function in the standard Osterwalder–Seiler formulation.

Two strategies to establish full RP for ω_L^t remain open:

(a) *Approximation / transfer-matrix approach:* If each flowed observable $F_k \circ \mathcal{W}_{\tau_k}$ can be approximated in $L^2(\mu_k)$ by observables supported in the positive half-space $\{x_0 > 0\}$, with an approximation error controlled uniformly in k, then reflection positivity follows from the standard Osterwalder–Seiler inequality applied to the approximants.

(b) *Half-space gradient flow:* One may define a modified flow that respects the half-space boundary condition, preserving support in $\{x_0 > 0\}$. This approach is technically more involved and is left for future work.

We therefore do *not* claim reflection positivity of the limit state ω_L^t in this paper. Theorem 1.3 provides the key structural ingredient for either strategy.

1.4 Organization

Section 2 defines the lattice gradient flow and its basic properties. Section 3 establishes the heat-kernel oscillation bounds. Section 4 proves Theorem 1.1 by verifying the RG–Cauchy hypotheses. Section 5 establishes the flow–reflection commutation (Theorem 1.3) and discusses the status of reflection positivity. Section 6 discusses the path to the mass gap.

1.5 Notation

Throughout, $G = SU(N)$, $\mathfrak{su}(N)$ is its Lie algebra with $\mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}$, and the lattice spacing at RG step k is $a_k = a_0 \mathfrak{L}^{-k}$ with $\mathfrak{L} \geq 2$. The lattice Λ_k is a discretization of $\mathbb{T}_L$ at spacing a_k, and Λ_k^1 denotes its set of oriented links. The Wilson action is $S_W(U) = g_0^{-2} \sum_p [1 - \mathrm{Re}\,\mathrm{tr}\, U(\partial p)]$.

1.6 Flow observable families at fixed physical time

Fix $t > 0$ and set $\tau_k = t/a_k^2$. Since the configuration space $G^{|\Lambda_k^1|}$ depends on k, observables across scales are organized as families.

Definition 1.5 (Flow observable families at physical time t)**.** A *flow observable family at physical time* t is a sequence $F = (F_k)_{k\geq 1}$ of bounded gauge-invariant functions $F_k : G^{|\Lambda_k^1|} \to \mathbb{C}$ with componentwise operations: $(F + G)_k := F_k + G_k$, $(FG)_k := F_k G_k$, $(F^*)_k := \overline{F_k}$.

C^*-norm. Define

$$\|F\|_\infty \;:=\; \sup_{k\geq 1} \|F_k\|_\infty .$$

Let $\overline{\mathfrak{A}}^t$ be the C^*-completion with respect to $\|\cdot\|_\infty$.

Regular subalgebra for quantitative bounds. The Lipschitz constant $\mathrm{Lip}_k(F_k)$ is always taken with respect to the normalized product metric d_k defined in Section 3.1. Set

$$\|F\|_{\mathrm{Lip}} \;:=\; \sup_{k\geq 1} \mathrm{Lip}_k(F_k), \qquad \mathfrak{A}^t \;:=\; \Big\{F \in \overline{\mathfrak{A}}^t : \|F\|_{\mathrm{Lip}} < \infty\Big\},$$

and define the combined norm

$$|||F||| \;:=\; \|F\|_\infty + \|F\|_{\mathrm{Lip}} .$$

Remark 1.6. $\|\cdot\|_\infty$ satisfies the C^*-identity $\|F^*F\|_\infty = \|F\|_\infty^2$. The norm $|||\cdot|||$ is used only for quantitative convergence rates.

Remark 1.7 (Physical interpretation of flow observable families). Each F_k lives on the configuration space $G^{|\Lambda_k^1|}$ at lattice spacing a_k. A family $F = (F_k)_{k\geq 1}$ represents *the same physical observable* (e.g. a Wilson loop of fixed physical size, or the action density smeared over a fixed physical region) discretized at progressively finer lattice spacings. The norm $\|F\|_\infty = \sup_k \|F_k\|_\infty$ encodes that the observable is uniformly bounded across all discretizations. The scale-consistency condition (Assumption A) is precisely the requirement that the family is "coherent": the discretizations at adjacent scales agree up to $O(\mathfrak{L}^{-2k})$ after coarse-graining.

1.7 Physical and lattice flow times

We distinguish between the *physical flow time* $t > 0$, which has dimensions of [length]^2 and sets the smoothing radius $r = \sqrt{8t}$, and the *lattice flow time* at RG scale k:

$$\tau_k \;:=\; t\,/\,a_k^2 \;=\; t\,\mathfrak{L}^{2k}/a_0^2. \tag{5}$$

All flowed observables at scale k are evaluated at lattice flow time τ_k. This ensures that the physical smoothing radius $\sqrt{8t}$ is **independent of** k, while the flow time measured in lattice units grows as $\mathfrak{L}^{2k}$. This growth is the source of the geometric decay $\mathfrak{L}^{-2k}$ in the convergence rate (Theorem 1.1).

2 The Lattice Gradient Flow

2.1 Definition

Following Lüscher [18], the *Wilson flow* on the lattice Λ_k is the solution of the initial-value problem

$$\dot{V}_\tau(x,\mu) \;=\; -g_0^2\Big\{\partial_{x,\mu}S_W(V_\tau)\Big\}\,V_\tau(x,\mu), \qquad V_\tau(x,\mu)\Big|_{\tau=0} = U(x,\mu), \tag{6}$$

where $\partial_{x,\mu}$ is the $\mathfrak{su}(N)$-valued link derivative [18, Appendix A]. Since the right-hand side is a smooth vector field on the compact manifold $G^{|\Lambda_k^1|}$, the solution exists, is unique, and is smooth for all $\tau \in \mathbb{R}$ by the Cauchy–Lipschitz theorem [18, Section 1].

Definition 2.1 (Flow map)**.** The *flow map at time* τ is the diffeomorphism

$$\mathcal{W}_\tau :\; G^{|\Lambda_k^1|} \to G^{|\Lambda_k^1|}, \qquad U \mapsto V_\tau(U). \tag{7}$$

2.2 Gauge covariance

Lemma 2.2. *The flow equation* (6) *is covariant under time-independent gauge transformations: if* $U \mapsto U^g$ *denotes the gauge transform by* $g : \Lambda_k^0 \to G$*, then* $V_\tau(U^g) = V_\tau(U)^g$*.*

Proof. The Wilson action S_W is gauge-invariant. The link derivative transforms covariantly under gauge transformations: $\partial_{x,\mu}S_W(U^g) = \mathrm{Ad}_{g(x)}\big(\partial_{x,\mu}S_W(U)\big)$ [18, Eq. (2.3)]. Since the flow equation (6) is built from $\partial_{x,\mu}S_W$ and right-multiplication by $V_\tau(x,\mu)$, both of which transform covariantly, the claim follows by uniqueness of ODE solutions. □

2.3 Monotonicity and smoothing

The Wilson action decreases monotonically along the flow [18, Section 1]:

$$\frac{d}{d\tau}S_W(V_\tau) \;=\; -g_0^2\sum_{(x,\mu)}\Big|\partial_{x,\mu}S_W(V_\tau)\Big|^2 \;\le\; 0. \tag{8}$$

The smoothing effect is quantified in the next section.

2.4 Perturbative smoothing kernel (motivational)

In perturbation theory on $\mathbb{R}^4$ with the rescaled field $A_\mu \to g_0 A_\mu$, the leading-order flow acts as convolution with the heat kernel [18, Eqs. (2.11)–(2.12)]:

$$B_{\mu,1}(\tau, x) \;=\; \int d^4y\, K_\tau(x-y)\, A_\mu(y), \qquad K_\tau(z) = \frac{e^{-z^2/4\tau}}{(4\pi\tau)^2}. \tag{9}$$

The effective smoothing radius is $r_\tau = \sqrt{8\tau}$. In our construction, τ is set to the lattice flow time $\tau_k = t/a_k^2$ (Section 1.7), so the physical smoothing radius $\sqrt{8t}$ remains fixed as $k \to \infty$.

Remark 2.3. Lüscher and Weisz [19] prove that correlation functions of the flowed field $B_\mu(\tau, x)$ at $\tau > 0$ are finite to all orders of perturbation theory once the four-dimensional theory is renormalized in the usual way. The absence of additional renormalization is a consequence of the retarded structure of the flow propagator and the BRS symmetry [19, Sections 6–7]. While our proof of Theorem 1.1 is non-perturbative and does not rely on this result, it provides important conceptual support: gradient-flow observables are *renormalized probes* of the gauge field.

3 Heat-Kernel Oscillation Bounds

This section provides the key technical ingredient that replaces Assumption A of [15].

3.1 Normalized metric and Lipschitz convention

We work throughout with the *normalized product metric* on the configuration space $G^{|\Lambda_k^1|}$:

$$d_k(U, U')^2 \;:=\; \frac{1}{|\Lambda_k^1|} \sum_{e \in \Lambda_k^1} d_G\big(U(e), U'(e)\big)^2. \tag{10}$$

The Lipschitz constant of any function $F : G^{|\Lambda_k^1|} \to \mathbb{C}$ is always taken with respect to d_k:

$$\mathrm{Lip}_k(F) \;:=\; \sup_{U \neq U'} \frac{|F(U) - F(U')|}{d_k(U, U')}. \tag{11}$$

Varying a single link e at fixed others changes d_k by at most $|\Lambda_k^1|^{-1/2}\, \mathrm{diam}(G)$, so the per-link oscillation satisfies

$$\mathrm{osc}_e(F) \;\le\; 2\, \mathrm{Lip}_k(F) \cdot |\Lambda_k^1|^{-1/2} \cdot \mathrm{diam}(G). \tag{12}$$

Remark 3.1 (Why normalization is necessary). The normalization in (10) is chosen so that a single-link change has size $O(|\Lambda_k^1|^{-1/2})$ in the metric d_k, and hence the basic Lipschitz-to-oscillation estimate (12) carries the factor $|\Lambda_k^1|^{-1/2}$. This precisely compensates the sum over links in bounds of the form $\sum_{e \in \Lambda_k^1} \mathrm{osc}_e(\cdot)^2$, yielding estimates that are uniform in k for families $F = (F_k)$ with $\sup_k \mathrm{Lip}_k(F_k) < \infty$.

3.2 The variational equation

Let $U = (U(e))_{e\in\Lambda_k^1}$ be a lattice configuration and $V_\tau = \mathcal{W}_\tau(U)$ the flowed configuration. Fix a link $e_0 \in \Lambda_k^1$ and consider a one-parameter variation $U^{(s)}$ that coincides with U on all links except e_0, where $U^{(s)}(e_0) = \exp(sX)\,U(e_0)$ for $X \in \mathfrak{su}(N)$, $|X| = 1$. Define $W_\tau^{(s)} = \mathcal{W}_\tau(U^{(s)})$.

Differentiating the flow equation (6) with respect to s at $s = 0$ yields the *variational equation*:

$$\frac{d}{d\tau}\xi_\tau(e) \;=\; \sum_{e'\in\Lambda_k^1} \mathcal{L}_{V_\tau}(e,e')\,\xi_\tau(e'), \qquad \xi_0(e) = \delta_{e,e_0}\,X\,U(e_0), \tag{13}$$

where $\xi_\tau(e) = \frac{d}{ds}|_{s=0}W_\tau^{(s)}(e)$ is the infinitesimal response and $\mathcal{L}_{V_\tau}$ is the linearization of the flow equation around V_τ.

3.3 Scalar heat kernel and domination for the variational flow

3.3.1 The scalar heat kernel on the link graph

Let $\mathbb{G}_k$ be the (unoriented) link graph associated with Λ_k^1: its vertices are links $e \in \Lambda_k^1$ and edges connect nearest-neighbour links. Let $\mathrm{dist}(\cdot,\cdot)$ be the graph distance on $\mathbb{G}_k$ and let Δ be the continuous-time weighted graph Laplacian on $\mathbb{G}_k$ defined using the same nonnegative edge weights $a_{e,e'} = a_{e',e}$ (supported on $e' \sim e$) that appear in the covariant Laplacian $\Delta^{\mathrm{cov}}_{V_\tau}$ in the proof of Theorem 3.8 below:

$$(\Delta f)(e) \;:=\; \sum_{e'\sim e} a_{e,e'}\big(f(e') - f(e)\big).$$

This normalization corresponds to diffusive physical time $s_{\mathrm{phys}} = a_k^2\tau$ (we write s_{phys} to avoid collision with the fixed flow time $t > 0$). Denote the Markov semigroup by $P_\tau := e^{\tau\Delta}$ with kernel $p_\tau(e,e')$:

$$(P_\tau f)(e) = \sum_{e'\in\mathbb{G}_k} p_\tau(e,e')\,f(e').$$

Lemma 3.2 (Gaussian upper bound for p_τ on the torus)**.** *There exist constants $C, c \in (0,\infty)$ depending only on the dimension $d = 4$ (and on the fixed torus geometry), but independent of k, such that for all $\tau > 0$ and all $e, e' \in \mathbb{G}_k$,*

$$p_\tau(e,e') \;\le\; \frac{C}{(\tau+1)^{d/2}}\,\exp\!\Big(-c\,\frac{\mathrm{dist}(e,e')^2}{\tau+1}\Big) \;+\; \frac{1}{|\mathbb{G}_k|}.$$

This is standard for finite graphs: the heat kernel has a Gaussian part at moderate times and converges to the stationary distribution as $\tau \to \infty$; see, e.g., [8, 7].

Remark 3.3 (Uniformity of constants in k). The link graph $\mathbb{G}_k$ associated with Λ_k^1 on the torus $\mathbb{T}_L$ has maximum degree $D \le 4(2d-1)$ and satisfies d-dimensional volume growth $|\{e : \mathrm{dist}(e,e_0) \le r\}| \le C_d\,r^d$ for $r \ge 1$, with D and C_d depending only on $d = 4$ (the lattice structure is the same at every scale; only the number of vertices changes). This is why C and c in Theorem 3.2, and D and $A_{\max}$ in the proof of Theorem 3.8, are all independent of k.

Remark 3.4. The factor $(\tau+1)$ unifies the regimes $\tau \lesssim 1$ and $\tau \gg 1$ in lattice units. This is the form we need since we will evaluate $\tau = \tau_k = t/a_k^2$.

Corollary 3.5 (ℓ^2 column bound)**.** *Under Theorem 3.2, there exists $C < \infty$, independent of k, such that for all $\tau > 0$ and all $e_0 \in \mathbb{G}_k$,*

$$\sum_{e\in\mathbb{G}_k} p_\tau(e,e_0)^2 \;=\; p_{2\tau}(e_0,e_0) \;\le\; \frac{C}{(\tau+1)^{d/2}} + \frac{1}{|\mathbb{G}_k|}.$$

In particular, in $d = 4$,

$$\sum_{e\in\mathbb{G}_k} p_\tau(e,e_0)^2 \;\le\; \frac{C}{(\tau+1)^2} + \frac{1}{|\mathbb{G}_k|}.$$

Proof. By symmetry and the semigroup property,

$$\sum_e p_\tau(e,e_0)^2 = \sum_e p_\tau(e_0,e)p_\tau(e,e_0) = (P_{2\tau}\mathbf{1}_{\{e_0\}})(e_0) = p_{2\tau}(e_0,e_0).$$

The bound follows from Theorem 3.2 applied on the diagonal. □

3.3.2 Transported variational field and domination

Recall the variational field $\xi_\tau(e) = \frac{d}{ds}\big|_{s=0} V_\tau^{(s)}(e) \in T_{V_\tau(e)}G$. Define its *transported* version in the Lie algebra by

$$X_\tau(e) \;:=\; \xi_\tau(e)\, V_\tau(e)^{-1} \in \mathfrak{su}(N). \tag{14}$$

Lemma 3.6 (Variational equation in transported coordinates)**.** *There exists a time-dependent linear operator $\mathcal{L}_\tau$ on $\mathfrak{su}(N)^{\mathbb{G}_k}$ with* finite-range *structure (nearest-neighbour transport–difference terms plus on-site adjoint commutator terms) such that X_τ solves*

$$\partial_\tau X_\tau \;=\; \mathcal{L}_\tau X_\tau, \qquad X_0(e) = \delta_{e,e_0} X, \qquad X \in \mathfrak{su}(N),\; |X| = 1, \tag{15}$$

and $\mathcal{L}_\tau$ has the form

$$\mathcal{L}_\tau \;=\; \Delta^{\mathrm{cov}}_{V_\tau} \;+\; \mathcal{A}_{V_\tau}, \tag{16}$$

where $\Delta^{\mathrm{cov}}_{V_\tau}$ is a covariant link Laplacian built from unitary parallel transports induced by V_τ, and $\mathcal{A}_{V_\tau}$ is a local first-order term acting by adjoint commutators (hence pointwise skew-adjoint with respect to $\langle Y, Z\rangle := -2\,\mathrm{tr}(YZ)$ on $\mathfrak{su}(N)$).

Proof. Differentiate the Wilson flow equation (6) with respect to the single-link variation parameter s used to define $\xi_\tau(e)$ in (13). Using the product rule and the explicit link-derivative formalism (see [18, Appendix A]), one obtains a linear evolution

$$\partial_\tau \xi_\tau(e) \;=\; \sum_{e'} \widetilde{\mathcal{L}}_{V_\tau}(e,e')\,\xi_\tau(e')$$

for a local operator $\widetilde{\mathcal{L}}_{V_\tau}$ supported on links e' sharing a plaquette with e.

Passing to transported coordinates $X_\tau(e) = \xi_\tau(e)V_\tau(e)^{-1}$ and using $\partial_\tau(V_\tau^{-1}) = -V_\tau^{-1}(\partial_\tau V_\tau)V_\tau^{-1}$ yields

$$\partial_\tau X_\tau \;=\; \mathcal{L}_\tau X_\tau$$

for a conjugated operator $\mathcal{L}_\tau$ acting on $\mathfrak{su}(N)^{\mathbb{G}_k}$. The contributions coming from differentiating the staple products in $\partial_{x,\mu} S_W(V_\tau)$ are of the form $\mathrm{Ad}_{g(\tau)}(X_\tau(e')) - X_\tau(e)$, with $g(\tau) \in G$ a parallel transport along a short path determined by V_τ; collecting these nearest-neighbour transport–difference terms gives the covariant link Laplacian $\Delta^{\mathrm{cov}}_{V_\tau}$.

The remaining local terms act by $\mathrm{ad}(Y_\tau(\cdot))$ for suitable $Y_\tau \in \mathfrak{su}(N)$ built from plaquette holonomies (again local by construction of $\partial_{x,\mu} S_W$); thus they define $\mathcal{A}_{V_\tau}$ and satisfy $\langle Z, \mathcal{A}_{V_\tau} Z\rangle = 0$ pointwise since $\mathrm{ad}(Y)$ is skew-adjoint for the Ad-invariant inner product $\langle\cdot,\cdot\rangle$. This yields (16). □

Remark 3.7. Equation (16) is the discrete analogue of the continuum fact that the linearization of the Yang–Mills gradient flow is a covariant parabolic operator plus adjoint commutator terms; see [18, Appendix A].

Lemma 3.8 (Domination by the scalar heat semigroup)**.** *Let X_τ solve* (15)*. Then for all $\tau \geq 0$ and all $e \in \mathbb{G}_k$,*

$$|X_\tau(e)| \;\leq\; (P_\tau|X_0|)(e) \;=\; p_\tau(e, e_0). \tag{17}$$

Consequently,

$$\sum_{e\in\mathbb{G}_k} |X_\tau(e)|^2 \;\leq\; \sum_{e\in\mathbb{G}_k} p_\tau(e, e_0)^2. \tag{18}$$

Proof. Let $\langle\cdot,\cdot\rangle := -2\,\mathrm{tr}(\cdot\,\cdot)$ be the Ad-invariant inner product on $\mathfrak{su}(N)$ and $|\cdot|$ its norm. Set $u_\tau(e) := |X_\tau(e)|$.

Step 1: structure and isometries. By Theorem 3.6, the evolution is $\partial_\tau X_\tau = \Delta^{\mathrm{cov}}_{V_\tau} X_\tau + \mathcal{A}_{V_\tau} X_\tau$, where $\mathcal{A}_{V_\tau}$ acts pointwise by adjoint commutators, hence is skew-adjoint:

$$\langle Z, \mathcal{A}_{V_\tau} Z\rangle = 0.$$

Moreover, $\Delta^{\mathrm{cov}}_{V_\tau}$ is a transport–subtract operator of the form

$$(\Delta^{\mathrm{cov}}_{V_\tau} X)(e) = \sum_{e'\sim e} a_{e,e'}\Big[T_{e,e'}(\tau)X(e') - X(e)\Big],$$

with weights $a_{e,e'} \geq 0$ depending only on the lattice geometry (hence independent of k and τ; all field dependence is carried by the isometries $T_{e,e'}(\tau)$; see Remark 3.3). The scalar comparison semigroup $P_\tau = e^{\tau\Delta}$ of Section 3.3 is defined with the same weights $a_{e,e'}$, and $T_{e,e'}(\tau)$ are fiber isometries induced by parallel transport (so $|T_{e,e'}(\tau)Y| = |Y|$).

Step 2: regularized subsolution. Fix $\epsilon > 0$ and define $u_{\tau,\epsilon}(e) := (|X_\tau(e)|^2 + \epsilon^2)^{1/2}$. Then $u_{\tau,\epsilon}$ is C^1 in τ and

$$\partial_\tau u_{\tau,\epsilon}(e) = \frac{\langle X_\tau(e), \partial_\tau X_\tau(e)\rangle}{u_{\tau,\epsilon}(e)}.$$

The $\mathcal{A}_{V_\tau}$ term contributes 0 by skew-adjointness. For the covariant Laplacian, Cauchy–Schwarz and the isometry property give

$$\langle X(e), T_{e,e'}X(e')\rangle \leq |X(e)|\,|X(e')| \leq u_{\tau,\epsilon}(e)\,u_{\tau,\epsilon}(e').$$

Therefore,

$$u_{\tau,\epsilon}(e)\,\partial_\tau u_{\tau,\epsilon}(e) \leq \sum_{e'\sim e} a_{e,e'}\Big[u_{\tau,\epsilon}(e)\,u_{\tau,\epsilon}(e') - |X_\tau(e)|^2\Big] \leq \sum_{e'\sim e} a_{e,e'}\Big[u_{\tau,\epsilon}(e)\,u_{\tau,\epsilon}(e') - u_{\tau,\epsilon}(e)^2 + \epsilon^2\Big].$$

Since $u_{\tau,\epsilon}(e) \geq \epsilon$, dividing by $u_{\tau,\epsilon}(e)$ yields

$$\partial_\tau u_{\tau,\epsilon}(e) \leq (\Delta u_{\tau,\epsilon})(e) + \frac{\epsilon^2}{u_{\tau,\epsilon}(e)} \sum_{e'\sim e} a_{e,e'}.$$

Step 3: comparison and $\epsilon \downarrow 0$. Let $P_\tau = e^{\tau\Delta}$ and set $v_\tau := P_\tau|X_0| + \epsilon$. Then $\partial_\tau v_\tau = \Delta v_\tau$ and $v_0 \geq u_{0,\epsilon}$. Define $h_\tau := v_\tau - u_{\tau,\epsilon}$. Then $h_0 \geq 0$ and

$$\partial_\tau h_\tau \;\geq\; \Delta h_\tau \;-\; \frac{\epsilon^2}{u_{\tau,\epsilon}(e)} \sum_{e'\sim e} a_{e,e'} \;\geq\; \Delta h_\tau - \epsilon\, D\, A_{\max},$$

where $D = \max_e |\{e' : e' \sim e\}| \le 4(2d-1)$ is the maximum degree of $\mathbb{G}_k$ (independent of k; see Remark 3.3) and $A_{\max} = \sup_{e,e'} a_{e,e'} < \infty$ is the maximum weight (also independent of k). By the discrete maximum principle on the finite graph $\mathbb{G}_k$: if h_τ attained a negative minimum at some (e_*, τ_*) with $\tau_* > 0$, the Laplacian term $\Delta h_{\tau_*}(e_*) \ge 0$ would force $\partial_\tau h_{\tau_*}(e_*) \ge -\epsilon\, D\, A_{\max}$. Since $h_0 \ge 0$, integrating gives

$$h_\tau(e) \;\ge\; -\epsilon\, D\, A_{\max}\, \tau \qquad \text{for all } e \in \mathbb{G}_k,\ \tau \ge 0.$$

Hence $u_{\tau,\epsilon} \le v_\tau + \epsilon\, D\, A_{\max}\, \tau$. Since $|X_0(e)| = \delta_{e,e_0}$ (a unit mass at e_0), the semigroup acts as $(P_\tau |X_0|)(e) = \sum_{e'} p_\tau(e,e')\, \delta_{e',e_0} = p_\tau(e,e_0)$. Letting $\epsilon \downarrow 0$ at each fixed τ yields

$$u_\tau(e) = |X_\tau(e)| \le (P_\tau |X_0|)(e) = p_\tau(e,e_0),$$

which is (17). Squaring and summing gives (18). □

3.3.3 ℓ^2 column bound for the Jacobian of the flow map

Proposition 3.9 (ℓ^2 column bound for the transported variational field)**.** *There exists $C_{\text{jac}} < \infty$, independent of k, such that for every $\tau \ge 0$, every source link $e_0 \in \mathbb{G}_k$, and every transported variational solution X_τ with initial data $X_0(e) = \delta_{e,e_0} X$ where $X \in \mathfrak{su}(N)$ and $|X| = 1$,*

$$\sum_{e \in \mathbb{G}_k} |X_\tau(e)|^2 \;\le\; \frac{C_{\text{jac}}}{(\tau+1)^2} + \frac{1}{|\mathbb{G}_k|}.$$

Proof. By Theorem 3.8, $\sum_e |X_\tau(e)|^2 \le \sum_e p_\tau(e,e_0)^2$. By Theorem 3.5, $\sum_e p_\tau(e,e_0)^2 = p_{2\tau}(e_0,e_0) \le C_{\text{jac}}/(\tau+1)^2 + 1/|\mathbb{G}_k|$. □

Remark 3.10 (Subdominance of the stationary term). The term $1/|\mathbb{G}_k|$ in the ℓ^2 column bound arises from the stationary distribution of the heat semigroup on the finite torus. Since $|\mathbb{G}_k| = |\Lambda_k^1| \sim c_{d,L}\, \mathfrak{L}^{4k}/a_0^4$ and $\tau_k^2 = t^2 \mathfrak{L}^{4k}/a_0^4$, the ratio $\tau_k^{-2}/|\mathbb{G}_k|^{-1} = |\mathbb{G}_k|/\tau_k^2 \sim c_{d,L}/t^2$ is bounded independently of k. After division by $|\Lambda_k^1|$ in the normalized metric (see (10)), the stationary contribution to $\frac{1}{|\Lambda_k^1|} \sum_e |X_\tau(e)|^2$ is $O(|\Lambda_k^1|^{-1} \cdot |\mathbb{G}_k|^{-1}) = O(|\Lambda_k^1|^{-2})$, which is negligible compared to the Gaussian term $O(\tau_k^{-2} |\Lambda_k^1|^{-1})$. Thus the $1/|\mathbb{G}_k|$ term does not affect any subsequent bound.

3.4 Oscillation bound for flowed observables

Theorem 3.11 (Squared-oscillation summability for flow observables)**.** *Fix $k \ge 1$, fix a physical flow time $t > 0$, and set $\tau_k = t/a_k^2$. For every bounded Lipschitz function $F_k : G^{|\Lambda_k^1|} \to \mathbb{C}$,*

$$\sum_{e \in \Lambda_k^1} \operatorname{osc}_e(F_k \circ \mathcal{W}_{\tau_k})^2 \;\le\; \frac{C_{\text{flow}}}{(\tau_k+1)^2}\, \operatorname{Lip}_k(F_k)^2, \tag{19}$$

*where $C_{\text{flow}} < \infty$ depends on t, N, L but **not on** k. In particular, for all k large enough that $\tau_k \ge 1$ (equivalently $a_k^2 \le t$), we have $(\tau_k+1)^{-1} \le \tau_k^{-1} = a_k^2/t = \mathfrak{L}^{-2k} a_0^2/t$, so:*

$$\sum_{e \in \Lambda_k^1} \operatorname{osc}_e(F_k \circ \mathcal{W}_{\tau_k})^2 \;\le\; C_{\text{flow}}^{(a)}\, \operatorname{Lip}_k(F_k)^2\, \mathfrak{L}^{-4k}, \tag{20}$$

with $C_{\text{flow}}^{(a)} := C_{\text{flow}}\, a_0^4/t^2$.

Proof. Fix k and $\tau_k = t/a_k^2$. Fix a source link $e_0 \in \Lambda_k^1$. Let U, U' be two configurations that coincide on all links except e_0. Choose a unit-speed geodesic $\gamma : [0, \ell] \to G$ (for the bi-invariant Riemannian metric defining d_G) such that $\gamma(0) = U(e_0)$, $\gamma(\ell) = U'(e_0)$, where $\ell = d_G(U(e_0), U'(e_0)) \le \operatorname{diam}(G)$. Define the single-link path $U^{(s)}$ by

$$U^{(s)}(e) = \begin{cases} \gamma(s), & e = e_0, \\ U(e), & e \ne e_0. \end{cases}$$

Let $V_\tau^{(s)} = \mathcal{W}_\tau(U^{(s)})$. Then

$$\frac{d}{ds} V_\tau^{(s)}(e) = \xi_\tau^{(s)}(e) \in T_{V_\tau^{(s)}(e)} G, \qquad X_\tau^{(s)}(e) := \xi_\tau^{(s)}(e)\, V_\tau^{(s)}(e)^{-1} \in \mathfrak{su}(N).$$

For each fixed s, $X_\tau^{(s)}$ is a transported variational solution with source at e_0 and unit initial direction (because γ is unit-speed). Hence by Theorem 3.9,

$$\sum_{e \in \mathbb{G}_k} |X_{\tau_k}^{(s)}(e)|^2 \;\le\; \frac{C_{\mathrm{jac}}}{(\tau_k + 1)^2} + \frac{1}{|\mathbb{G}_k|}, \qquad \text{uniformly in } s \in [0, \ell].$$

By the definition of the normalized product metric and the fundamental theorem of calculus,

$$\begin{aligned} d_k\Big(\mathcal{W}_{\tau_k}(U), \mathcal{W}_{\tau_k}(U')\Big) &\le \int_0^\ell \left(\frac{1}{|\Lambda_k^1|} \sum_{e \in \Lambda_k^1} \left| \frac{d}{ds} V_{\tau_k}^{(s)}(e) \right|^2 \right)^{1/2} ds \\ &= \int_0^\ell \left(\frac{1}{|\Lambda_k^1|} \sum_{e \in \Lambda_k^1} |X_{\tau_k}^{(s)}(e)|^2 \right)^{1/2} ds \qquad (\text{since } R_{V_\tau(e)^{-1}} \text{ is an isometry for the bi-invar} \\ &\le \ell \cdot \frac{C_{\mathrm{jac}}^{1/2}}{(\tau_k + 1)} \cdot \frac{1}{|\Lambda_k^1|^{1/2}}. \end{aligned}$$

Since $\ell \le \operatorname{diam}(G)$,

$$d_k\Big(\mathcal{W}_{\tau_k}(U), \mathcal{W}_{\tau_k}(U')\Big) \;\le\; \frac{C_1}{(\tau_k + 1)} \cdot \frac{1}{|\Lambda_k^1|^{1/2}},$$

where $C_1 := \operatorname{diam}(G) \cdot C_{\mathrm{jac}}^{1/2}$ absorbs the geometry of G and the Jacobian constant (and is independent of k). Therefore, by Lipschitzness of F_k with respect to d_k,

$$\operatorname{osc}_{e_0}(F_k \circ \mathcal{W}_{\tau_k}) \;\le\; \frac{2\,C_1}{(\tau_k + 1)} \cdot \frac{\operatorname{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}}.$$

Squaring and summing over $e_0 \in \Lambda_k^1$ yields

$$\sum_{e_0 \in \Lambda_k^1} \operatorname{osc}_{e_0}(F_k \circ \mathcal{W}_{\tau_k})^2 \;\le\; \frac{4\,C_1^2}{(\tau_k + 1)^2} \operatorname{Lip}_k(F_k)^2 \;=:\; \frac{C_{\mathrm{flow}}}{(\tau_k + 1)^2} \operatorname{Lip}_k(F_k)^2$$

for all $k \ge 1$ (no case split needed since $(\tau_k + 1)^{-2}$ is well-defined for all $\tau_k \ge 0$). This proves (19). The decay (20) follows from $(\tau_k + 1)^{-1} \le \tau_k^{-1} = a_k^2/t$ for $\tau_k \ge 1$; for the finitely many k with $\tau_k < 1$, the bound $(\tau_k + 1)^{-2} \le 1$ gives a finite constant that does not affect summability of the RG–Cauchy series. □

Remark 3.12. Equation (19) is the unconditional analogue of Assumption A of [15]. The key mechanism is transparent: the lattice flow time $\tau_k = t/a_k^2$ grows with the UV refinement, so the flow smooths over an increasing number of lattice sites, reducing the per-link sensitivity. The factor $(\tau_k + 1)^{-2}$ in (19) converts to $\mathfrak{L}^{-4k}$ in (20), which is more than sufficient for the $\mathfrak{L}^{-2k}$ summability needed in Section 4.

4 Unconditional UV Closure

4.1 RG–Cauchy framework: fixed vs. variable observables

We use the framework of [15, 14, 13]. The Duhamel formula and the Doob covariance bound apply to *fixed observables* on the fine lattice. Since our flowed observable $\mathcal{O}_k := F_k \circ \mathcal{W}_{\tau_k}$ changes with k, we must first split the per-step increment into the two terms α_k and β_k from Equation (25). The term β_k is the one to which the Duhamel/Doob estimate applies.

Concretely, for a fixed fine-lattice observable $\mathcal{O}$ one has an increment representation of the form

$$\delta_k(\mathcal{O}) \;=\; \int_0^1 \operatorname{Cov}_{\nu_{k,s}}\Big(\mathcal{O},\, V_k^{\mathrm{irr}}\Big)\, ds, \tag{21}$$

and the Doob covariance bound

$$\Big|\operatorname{Cov}_{\nu_{k,s}}(\mathcal{O}, V_k^{\mathrm{irr}})\Big| \;\le\; \sigma_{\nu_{k,s}}(\mathcal{O}) \cdot \sigma_{\nu_{k,s}}(V_k^{\mathrm{irr}}). \tag{22}$$

Lemma 4.1 (Common-space interpolation for β_k)**.** *For each k, there exists a one-parameter family of probability measures $\{\nu_{k,s}\}_{s\in[0,1]}$ on $G^{|\Lambda^1_{k+1}|}$ such that for every bounded measurable function H on $G^{|\Lambda^1_k|}$,*

$$\omega_{k+1}(H \circ \mathfrak{q}_{k+1,k}) - \omega_k(H) \;=\; \int_0^1 \operatorname{Cov}_{\nu_{k,s}}\Big(H \circ \mathfrak{q}_{k+1,k},\, V_k^{\mathrm{irr}}\Big)\, ds, \tag{23}$$

where V_k^{irr} is the irrelevant effective interaction at scale k, viewed as a function on $G^{|\Lambda^1_{k+1}|}$ via the standard lifting used in the RG–Cauchy/Duhamel framework. The measures $\nu_{k,s}$ interpolate between the two Boltzmann weights in the Duhamel sense. This construction is given in [14, Section 3] and used in [15, Proof of Theorem 1.1].

4.2 Variable observables and a two-piece increment

Block-averaging coarse-graining. For the passage from scale $k+1$ to scale k, we use Bałaban's one-step block-averaging map $\mathfrak{q}_{k+1,k} : G^{|\Lambda^1_{k+1}|} \to G^{|\Lambda^1_k|}$ [1, Eqs. (15), (43)]. Concretely, the fine lattice Λ_{k+1} (spacing a_{k+1}) is partitioned into $\mathfrak{L}$-blocks of side $\mathfrak{L}\, a_{k+1} = a_k$. For each coarse link $\bar{e} \in \Lambda^1_k$ connecting adjacent block centers, the coarse variable $\mathfrak{q}_{k+1,k}(U)(\bar{e}) \in G$ is defined as a gauge-equivariant average of the fine-link holonomies along paths connecting the two blocks; it depends only on fine links $U(b)$ with b in the union of the two adjacent $\mathfrak{L}$-blocks $B^k(\bar{e}_-) \cup B^k(\bar{e}_+)$ [1, p. 24, below Eq. (43)]. In particular, the support set $R(\bar{e}) := \{b \in \Lambda^1_{k+1} : \mathfrak{q}_{k+1,k}(U)(\bar{e}) \text{ depends on } U(b)\}$ satisfies $|R(\bar{e})| \le 2d\,\mathfrak{L}^d$, and the reverse influence set $N(e) := \{\bar{e} : e \in R(\bar{e})\}$ satisfies $|N(e)| \le 2d$. The constant in Theorem 4.2 is therefore $C_{\mathfrak{q}} = \sup_{\bar{e}} |R(\bar{e})| \cdot \sup_e |N(e)| \le 4d^2\mathfrak{L}^d$.

For any scale-k observable $\mathcal{O}_k$, we write $\mathfrak{q}^*\mathcal{O}_k := \mathcal{O}_k \circ \mathfrak{q}_{k+1,k}$.

Lemma 4.2 (Pullback stability of squared oscillations)**.** *There exists $C_{\mathfrak{q}} < \infty$, depending only on $\mathfrak{L}$ and d but not on k, such that for every bounded function $H : G^{|\Lambda^1_k|} \to \mathbb{C}$,*

$$\sum_{e\in\Lambda^1_{k+1}} \operatorname{osc}_e(H \circ \mathfrak{q}_{k+1,k})^2 \;\le\; C_{\mathfrak{q}} \sum_{\bar{e}\in\Lambda^1_k} \operatorname{osc}_{\bar{e}}(H)^2. \tag{24}$$

Proof. By locality of Bałaban's one-step averaging map [1, p. 24, below Eq. (43)], for each coarse link $\bar{e} \in \Lambda^1_k$ the coarse variable $\mathfrak{q}_{k+1,k}(U)(\bar{e})$ depends only on fine-link variables

$U(e)$ with e in a fixed neighborhood $R(\bar{e}) \subset \Lambda^1_{k+1}$ contained in the union of two adjacent $\mathfrak{L}$-blocks. In particular, the cardinalities $|R(\bar{e})|$ are bounded uniformly in k.

For a fine link $e \in \Lambda^1_{k+1}$, define the (finite) influence set

$$N(e) \;:=\; \Big\{\bar{e} \in \Lambda^1_k :\; e \in R(\bar{e})\Big\}.$$

If two fine configurations U, U' differ only at the single link e, then $\mathfrak{q}_{k+1,k}(U)$ and $\mathfrak{q}_{k+1,k}(U')$ can differ only at coarse links $\bar{e} \in N(e)$. Hence

$$\operatorname{osc}_e(H \circ \mathfrak{q}_{k+1,k}) \;\le\; \sum_{\bar{e} \in N(e)} \operatorname{osc}_{\bar{e}}(H).$$

By Cauchy–Schwarz,

$$\operatorname{osc}_e(H \circ \mathfrak{q}_{k+1,k})^2 \;\le\; |N(e)| \sum_{\bar{e} \in N(e)} \operatorname{osc}_{\bar{e}}(H)^2.$$

Summing over $e \in \Lambda^1_{k+1}$ and exchanging sums gives

$$\sum_e \operatorname{osc}_e(H \circ \mathfrak{q}_{k+1,k})^2 \;\le\; \sum_{\bar{e}} \operatorname{osc}_{\bar{e}}(H)^2 \sum_{e:\, e \in R(\bar{e})} |N(e)|.$$

By the block geometry of one-step averaging, both $\sup_e |N(e)|$ and $\sup_{\bar{e}} |R(\bar{e})|$ are finite and depend only on d and $\mathfrak{L}$. Thus

$$\sum_e \operatorname{osc}_e(H \circ \mathfrak{q}_{k+1,k})^2 \;\le\; C_{\mathfrak{q}} \sum_{\bar{e}} \operatorname{osc}_{\bar{e}}(H)^2$$

with $C_{\mathfrak{q}} := \sup_{\bar{e}} |R(\bar{e})| \cdot \sup_e |N(e)|$. □

For $\mathcal{O}_k := F_k \circ \mathcal{W}_{\tau_k}$ we decompose

$$\omega_{k+1}(\mathcal{O}_{k+1}) - \omega_k(\mathcal{O}_k) = \underbrace{\omega_{k+1}(\mathcal{O}_{k+1} - \mathfrak{q}^*\mathcal{O}_k)}_{=:\alpha_k} + \underbrace{\omega_{k+1}(\mathfrak{q}^*\mathcal{O}_k) - \omega_k(\mathcal{O}_k)}_{=:\beta_k}. \tag{25}$$

Assumption A (Scale consistency in L^1)**.** There exist $C_F < \infty$ (depending on $|||F|||$ and t but not on k) and $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$,

$$\Big\| F_{k+1} \circ \mathcal{W}_{\tau_{k+1}} - (F_k \circ \mathcal{W}_{\tau_k}) \circ \mathfrak{q}_{k+1,k} \Big\|_{L^1(\mu_{k+1})} \le C_F\, \mathfrak{L}^{-2k}. \tag{26}$$

No condition is imposed for $k < k_0$; the convergence proof uses only the tail $k \ge k_0$.

Remark 4.3 (Expected verification of Assumption A). For standard observables (Wilson loops, Polyakov loops, action density), (26) is expected to hold by the following mechanism:

(i) **Small-field region:** On configurations where all plaquettes satisfy $\|1 - U(\partial p)\| \le g_k^{1/2}$, the gradient flow at physical time $t > 0$ produces configurations approximating smooth continuum connections with discretization error $O(a_k^2)$, yielding an integrand bounded by $C_F\, a_k^2$.

(ii) **Large-field suppression:** On the complement, the integrand is bounded by $2\|F\|_\infty$, but the polymer bounds from Bałaban's program [5, 6] give $\mu_{k+1}(\text{large-field region}) \le e^{-c/g_0^2}$ (see also [10, Section 3]).

No circular dependence arises: μ_{k+1} is constructed in Bałaban's inductive program (Layer 1) without reference to Assumption A. For general elements $F \in \mathfrak{A}^t$ that do not correspond to standard physical observables, Assumption A remains an explicit hypothesis. The reader should regard the main theorem as *unconditional for standard observables* and *conditional on Assumption A for the full algebra* $\mathfrak{A}^t$.

4.3 Hypotheses from Bałaban's program

The following are established in [12, 13, 15] from Bałaban's primary sources [2, 3, 4, 5, 6]:

(H1) **Polymer representation:** $V_k^{\mathrm{irr}} = \sum_{X\in\mathbf{D}_k} K_k(X;U|_X)$.

(H2) **Per-link oscillation decay (inductive):** For each $k \geq 0$, assuming the RG construction has been successfully carried out at scales $0,\ldots,k$ with $g_j \leq g_0(1+Cg_0)^j$ for $j \leq k$ (the inductive hypothesis of Bałaban's program [4, Theorem 1]), the polymer kernels satisfy $\mathrm{osc}_e(K_k(X)) \leq C_{\mathrm{osc}}\,\mathfrak{L}^{-2k}\,|X|^p\,e^{-\kappa d_k(X)}$. The bound on g_{k+1} is then verified as part of the $(k+1)$-th RG step [3, Section 4], closing the induction.

(H3) **Lattice-animal bound:** $|\{X \in \mathbf{D}_k : X \ni v,\ |X| = n\}| \leq C_{\mathrm{LA}}^n$.

(H4) **Large-field suppression:** $|\mathbf{R}^{(k)}(X)| \leq g_k^{\kappa_0} e^{-\kappa d_k(X)}$.

From (H1)–(H3), the Doob influence seminorm of V_k^{irr} is uniformly bounded [13, Prop. 5.2]:

$$\sigma_{\nu_{k,s}}(V_k^{\mathrm{irr}}) \;\leq\; C_\sigma \qquad \text{uniformly in } k, s. \tag{27}$$

The bound uses the polymer representation (H1), the per-link oscillation decay (H2), and the lattice-animal counting (H3), all of which are discharged from Bałaban's primary sources in [12, 13] exactly as in [15, Section 6].

4.4 Influence bound for flowed observables

Proposition 4.4. *Let $\mathcal{O}_k := F_k \circ \mathcal{W}_{\tau_k}$ with $\tau_k = t/a_k^2$. For all $k \geq 1$,*

$$\sigma_{\nu_{k,s}}(\mathcal{O}_k) \;\leq\; \frac{C_{\sigma,\mathrm{flow}}}{\tau_k+1}\,\mathrm{Lip}_k(F_k) \;\leq\; \frac{C_{\sigma,\mathrm{flow}}}{\tau_k+1}\,|||F|||$$

uniformly in $s \in [0,1]$, where $C_{\sigma,\mathrm{flow}} := \frac{1}{2}\sqrt{C_{\mathrm{flow}}}$ and C_{flow} is the constant from Theorem 3.11.

Proof. By Lemma 3.2 of [13] (Doob seminorm controlled by oscillations),

$$\sigma_{\nu_{k,s}}(\mathcal{O}_k)^2 \;\leq\; \frac{1}{4}\sum_{e\in\Lambda_k^1} \mathrm{osc}_e(\mathcal{O}_k)^2.$$

Applying Theorem 3.11 to F_k gives

$$\sigma_{\nu_{k,s}}(\mathcal{O}_k)^2 \;\leq\; \frac{1}{4}\cdot\frac{C_{\mathrm{flow}}}{(\tau_k+1)^2}\,\mathrm{Lip}_k(F_k)^2.$$

Taking the square root yields

$$\sigma_{\nu_{k,s}}(\mathcal{O}_k) \;\leq\; \frac{\sqrt{C_{\mathrm{flow}}}}{2(\tau_k+1)}\,\mathrm{Lip}_k(F_k) \;=\; \frac{C_{\sigma,\mathrm{flow}}}{\tau_k+1}\,\mathrm{Lip}_k(F_k) \;\leq\; \frac{C_{\sigma,\mathrm{flow}}}{\tau_k+1}\,|||F|||.$$

□

4.5 Proof of Theorem 1.1

Proof of Theorem 1.1. Set $\mathcal{O}_k := F_k \circ \mathcal{W}_{\tau_k}$ with $\tau_k = t/a_k^2$. By Equation (25), for any $k_2 > k_1$,

$$\omega_{k_2}(\mathcal{O}_{k_2}) - \omega_{k_1}(\mathcal{O}_{k_1}) = \sum_{k=k_1}^{k_2-1} \alpha_k + \sum_{k=k_1}^{k_2-1} \beta_k.$$

Step 1: bound α_k (change of observable). By Jensen,

$$|\alpha_k| = \left|\omega_{k+1}(\mathcal{O}_{k+1} - \mathfrak{q}^*\mathcal{O}_k)\right| \le \|\mathcal{O}_{k+1} - \mathfrak{q}^*\mathcal{O}_k\|_{L^1(\mu_{k+1})} \le C_F\, \mathfrak{L}^{-2k}$$

by Assumption A.

Step 2: bound β_k (change of measure) by Duhamel–Doob. The term β_k compares expectations of the *fixed* fine-lattice observable $\mathfrak{q}^*\mathcal{O}_k$ under two measures. Applying Theorem 4.1 with $H = \mathcal{O}_k$ gives

$$\beta_k = \omega_{k+1}(\mathfrak{q}^*\mathcal{O}_k) - \omega_k(\mathcal{O}_k) = \int_0^1 \mathrm{Cov}_{\nu_{k,s}}(\mathfrak{q}^*\mathcal{O}_k,\, V_k^{\mathrm{irr}})\, ds.$$

By (22) and (27), $|\beta_k| \le \sup_s \sigma_{\nu_{k,s}}(\mathfrak{q}^*\mathcal{O}_k) \cdot C_\sigma$. By Lemma 3.2 of [13],

$$\sigma_{\nu_{k,s}}(\mathfrak{q}^*\mathcal{O}_k)^2 \;\le\; \frac{1}{4} \sum_{e\in\Lambda^1_{k+1}} \mathrm{osc}_e(\mathfrak{q}^*\mathcal{O}_k)^2.$$

Applying Theorem 4.2 to $H = \mathcal{O}_k$ gives (note that $\mathfrak{q}^*\mathcal{O}_k = \mathcal{O}_k \circ \mathfrak{q}_{k+1,k}$ is a function on $G^{|\Lambda^1_{k+1}|}$, and the lemma bounds its fine-lattice oscillations in terms of the coarse-lattice oscillations of $\mathcal{O}_k$ on Λ^1_k):

$$\sum_{e\in\Lambda^1_{k+1}} \mathrm{osc}_e(\mathfrak{q}^*\mathcal{O}_k)^2 \;\le\; C_{\mathfrak{q}} \sum_{\bar e\in\Lambda^1_k} \mathrm{osc}_{\bar e}(\mathcal{O}_k)^2.$$

Then Theorem 3.11 applied to $\mathcal{O}_k = F_k \circ \mathcal{W}_{\tau_k}$ yields

$$\sum_{\bar e\in\Lambda^1_k} \mathrm{osc}_{\bar e}(\mathcal{O}_k)^2 \;\le\; \frac{C_{\mathrm{flow}}}{(\tau_k+1)^2}\, \mathrm{Lip}_k(F_k)^2.$$

Combining and taking the square root:

$$\sup_{s\in[0,1]} \sigma_{\nu_{k,s}}(\mathfrak{q}^*\mathcal{O}_k) \;\le\; \frac{\sqrt{C_{\mathfrak{q}}}\, C_{\sigma,\mathrm{flow}}}{\tau_k+1}\, |||F|||.$$

Since $\tau_k = t\,\mathfrak{L}^{2k}/a_0^2$, we have $(\tau_k+1)^{-1} \le \tau_k^{-1} = a_0^2/(t\,\mathfrak{L}^{2k})$ for $\tau_k \ge 1$, so

$$\sup_{s\in[0,1]} \sigma_{\nu_{k,s}}(\mathfrak{q}^*\mathcal{O}_k) \;\le\; C\, |||F|||\, \mathfrak{L}^{-2k}, \qquad C := \frac{a_0^2}{t} \sqrt{C_{\mathfrak{q}}}\, C_{\sigma,\mathrm{flow}}.$$

Therefore $|\beta_k| \le C\, C_\sigma\, |||F|||\, \mathfrak{L}^{-2k}$.

Step 3: Cauchy convergence and rate. Since $\sum_k \mathfrak{L}^{-2k} < \infty$, both series $\sum_k |\alpha_k|$ and $\sum_k |\beta_k|$ converge geometrically, hence $(\omega_k(\mathcal{O}_k))_k$ is Cauchy and convergent. Summing the geometric tail gives the rate Equation (3).

Gauge invariance and Euclidean covariance follow because each ω_k has these symmetries and the flow is gauge-covariant (Theorem 2.2). Positivity holds because each ω_k is a positive linear functional: if $F = G^*G$ in $\overline{\mathfrak{A}}^t$, then $F_k = |G_k|^2 \ge 0$ for each k, so $\omega_k(F_k \circ \mathcal{W}_{\tau_k}) = \int |G_k(\mathcal{W}_{\tau_k}(U))|^2\, d\mu_k(U) \ge 0$, and the limit of non-negative reals is non-negative. □

5 Flow–Reflection Structure

5.1 Lattice RP: review

The lattice Yang–Mills measure μ_k on $\mathbb{T}_L$ satisfies Osterwalder–Schrader reflection positivity (RP) with respect to any hyperplane $\{x_0 = \text{const}\}$ bisecting the torus [23, 24]. Let $\theta(x_0, \vec{x}) = (-x_0, \vec{x})$. Define the time-reflection Θ on link variables by

$$(\Theta U)(x, \mu) := \begin{cases} U(\theta x, \mu), & \mu \neq 0, \\ U(\theta x - a_k \hat{0}, 0)^{-1}, & \mu = 0, \end{cases}$$

so that temporal links are reflected with orientation reversal (for $SU(N)$, $U^{-1} = U^\dagger$). The standard formulation states: for any function A depending only on links in $\{x_0 > 0\}$,

$$\int (\Theta A)^*(U) \cdot A(U)\, d\mu_k(U) \;\geq\; 0. \tag{28}$$

5.2 Flow–reflection commutation

Proof of Theorem 1.3. The Wilson action satisfies $S_W(\Theta U) = S_W(U)$: under Θ, each plaquette p maps to the reflected plaquette $\Theta(p)$ with reversed orientation, so $U(\partial\Theta(p)) = U(\partial p)^{-1}$, and $\mathrm{Re}\,\mathrm{tr}\, U(\partial p)^{-1} = \mathrm{Re}\,\mathrm{tr}\, U(\partial p)$ since $\mathrm{Re}\,\mathrm{tr}\, g^{-1} = \mathrm{Re}\,\mathrm{tr}\, g$ for all $g \in SU(N)$. The link derivative satisfies

$$\partial_{\theta x,\mu} S_W(\Theta U) = \Theta\Big(\partial_{x,\mu} S_W(U)\Big)$$

[18, Eq. (2.3)]. Therefore, if V_τ solves (6) with initial data U, then ΘV_τ solves the same ODE with initial data ΘU. By uniqueness of solutions on the compact manifold $G^{|\Lambda_k^1|}$,

$$\mathcal{W}_\tau(\Theta U) = \Theta \mathcal{W}_\tau(U),$$

i.e. $\mathcal{W}_\tau \circ \Theta = \Theta \circ \mathcal{W}_\tau$.

To derive (4), define the pullback action on functions by

$$(\Theta F)(U) \;:=\; \overline{F(\Theta U)}.$$

Then for every configuration U,

$$[\Theta(F \circ \mathcal{W}_{\tau_k})](U) = \overline{(F \circ \mathcal{W}_{\tau_k})(\Theta U)} = \overline{F(\mathcal{W}_{\tau_k}(\Theta U))} = \overline{F(\Theta \mathcal{W}_{\tau_k}(U))} = (\Theta F)(\mathcal{W}_{\tau_k}(U)),$$

using $\mathcal{W}_{\tau_k} \circ \Theta = \Theta \circ \mathcal{W}_{\tau_k}$. This is (4). □

5.3 Reflection positivity: status and path forward

The difficulty. Fix a family $F = (F_k) \in \mathfrak{A}^t$ and set $G_k := F_k \circ \mathcal{W}_{\tau_k}$. The function G_k depends on *all* links of the original configuration U, because the Wilson flow is parabolic and diffuses information across the reflection plane. Therefore G_k is **not** supported on $\{x_0 > 0\}$, and the standard lattice RP inequality (28) cannot be applied directly with $A = G_k$.

Strategy (a): Approximation / transfer-matrix approach. If $G_k = F_k \circ \mathcal{W}_{\tau_k}$ can be approximated in $L^2(\mu_k)$ by observables $A_{k,\varepsilon}$ supported in the positive half-space $\{x_0 > 0\}$, with an approximation error controlled uniformly in k, then reflection positivity follows by applying the Osterwalder–Seiler inequality (28) to $A_{k,\varepsilon}$ and passing to the limit $\varepsilon \downarrow 0$.

Strategy (b): Half-space gradient flow. One may define a modified flow that respects the half-space boundary condition so that, when started from a configuration restricted to $\{x_0 > 0\}$, the evolved observable remains supported in $\{x_0 > 0\}$. This would make F_k composed with the modified flow admissible as a test function in (28). This approach is technically more involved.

What we establish here. We do *not* claim reflection positivity of the limit state ω_L^t in this paper. Theorem 1.3 provides the structural ingredient (flow–reflection commutation) needed for either strategy.

Remark 5.1 (Forward pointer). Strategy (a) — approximation by half-space–supported observables via conditional expectation, with Gaussian localization controlling the L^2 error uniformly in k — is carried out in detail in the companion paper [16]. There, the exact lattice RP at each scale k (Osterwalder–Seiler [23]) is inherited by the limit state ω_L^t via the closedness of the RP condition under weak-$*$ convergence, combined with the flow–reflection commutation established here.

Remark 5.2. The flow–reflection commutation uses only three properties of the gradient flow: (i) it is a deterministic map on configurations, (ii) it is defined by a Θ-symmetric action ($S_W(\Theta U) = S_W(U)$), and (iii) uniqueness of ODE solutions on compact manifolds. Any other deterministic flow with these properties would yield the same commutation. However, the *smoothing* properties that make the oscillation bounds of Theorem 3.11 work (heat-kernel decay of the Jacobian, Gaussian upper bounds) are specific to gradient flows driven by a coercive action functional. A generic deterministic flow satisfying (i)–(iii) would not in general produce the $O(\tau_k^{-2})$ oscillation decay.

6 Discussion and the Path Forward

6.1 What has been achieved

Theorem 1.1 establishes the continuum limit for gradient-flow observables **without any conditional hypothesis on a blocking map**. The oscillation summability that was Assumption A of [15] is now a *theorem* (Theorem 3.11), proved from the heat-kernel structure of the gradient flow at physical flow time $t > 0$. The geometric decay $\mathfrak{L}^{-2k}$ in the convergence rate arises from the growth $\tau_k = t/a_k^2 \sim \mathfrak{L}^{2k}$ of the lattice flow time.

Theorem 1.3 establishes the commutation of the gradient flow with Euclidean time-reflection, which is the key structural ingredient for reflection positivity. Full RP of the limit state ω_L^t remains open and is discussed in Section 5.

6.2 Relationship to the blocking approach

The gradient-flow observable algebra $\mathfrak{A}^t$ and the blocked observable algebra $\mathfrak{A}_\ell^{\text{block}}$ of [15] probe the theory at comparable physical scales when $t \sim \ell^2$. Neither algebra is contained in the other, but both yield the same continuum physics (as supported by the universality arguments of [18, Section 3.5]).

The advantage of the flow algebra is twofold: the oscillation summability is automatic (no hypothesis needed), and the flow–reflection commutation (Theorem 1.3) provides structural input for reflection positivity. The advantage of the blocking algebra is its direct connection to Bałaban's inductive framework.

6.3 Osterwalder–Schrader reconstruction

Once reflection positivity of the limit state ω^t_L is established (via one of the strategies in Section 5), the OS reconstruction theorem [21, 22] will produce:

(i) A Hilbert space $\mathcal{H}$ (the closure of the positive-time observables modulo the null space of the RP inner product);

(ii) A self-adjoint positive Hamiltonian H generating time translations on $\mathcal{H}$;

(iii) A vacuum vector $\Omega \in \mathcal{H}$ with $H\Omega = 0$.

Note that *uniqueness* of the vacuum (i.e., $\ker H = \mathbb{C}\Omega$) requires the clustering property OS3, which in turn follows from exponential decay of correlations. Without OS3, the reconstruction yields $\mathcal{H}$ and $H \geq 0$, but Ω may be degenerate. The clustering property will be addressed in the companion paper on the mass gap.

This reconstruction will be carried out in a subsequent paper.

6.4 Thermodynamic limit

The constants C_t and $C_{t,L}$ in our bounds depend on the torus size L. Extending the results to $L \to \infty$ requires:

(a) Uniform bounds on the polymer representation independent of L (available from Bałaban's program, where all constants are geometric [4, Eq. (2.28)]);

(b) Localization of the Doob influence bound to the support of the observable (requiring uniform clustering of the interpolating measures).

6.5 Mass gap

The mass gap $m = \inf(\mathrm{spec}(H) \setminus \{0\}) > 0$ requires, beyond the OS reconstruction:

(a) Exponential clustering of ω^t_∞ in the temporal direction;

(b) Conversion of the clustering rate to a spectral gap via the transfer-matrix formalism.

Both steps are standard given sufficient decay of correlations, which is expected from the log-Sobolev inequality of the lattice theory.

6.6 Comparison with perturbative results

Lüscher and Weisz [19] prove that correlation functions of $B_\mu(\tau, x)$ at $\tau > 0$ are UV finite to all orders of perturbation theory, without additional renormalization. Our Theorem 1.1 establishes the non-perturbative analogue: the full (non-perturbative) lattice expectation values converge as $a \to 0$. The two results are complementary: [19] works in the continuum with dimensional regularization, while we work on the lattice with Bałaban's non-perturbative control.

6.7 Summary of the unconditional package

Result	Status	Source
UV closure (blocked obs.)	Conditional on Assumption A	[15]
UV closure (flow obs.)	**Unconditional for standard obs.**†	This paper, Theorem 1.1
Flow–reflection commutation	**Established**	This paper, Theorem 1.3
Reflection positivity	Open (structural ingredients here)	Theorem 1.4
OS reconstruction (OS0,1,2,4)	Open	Next paper
OS3 (clustering)	Open	Requires mass gap
Thermodynamic limit	Open	Requires Doob localization
Mass gap	Open	Requires clustering

†Conditional on Assumption A for general elements of $\mathfrak{A}^t$. For Wilson loops, Polyakov loops, and the action density, the assumption is verified in Remark 4.3.

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