Retraction of Theorem 4.4 and Proposition 3.8, and Status Note (v2)

Almost Reflection Positivity for Gradient-Flow Observables via
Gaussian Localization in Lattice Yang–Mills Theory

Lluis Eriksson • ai.viXra:2602.0084 • July 2026

The principal results of this paper are withdrawn. They rest on three independent defects (A–C in §2–§4), two of which are outright false lemmas rather than gaps. A subsequent audit of the statements this note had initially listed as *surviving* produced four more (D–G in §6), three of them further false statements, none of them arising from the Wilson-flow gauge-mode obstruction that causes A–C. One of the false lemmas had already been refuted, with an explicit counterexample, in a companion paper of this same programme — in July 2026 — and the correction was never propagated here. **Nothing below shows that the conclusion of Theorem 4.4 is false**; it shows that it is unproved. Section 7 records the programme's three existing repair routes, which address defects A–C; §6 separately records the further reformulations that defects D–G would require, and those are not in hand. The v1 manuscript follows unmodified.

1 Status

Statement	Status as of v2
Definitions 2.2, 2.5, 2.6	Stand, as definitions
Theorem 4.1 (lattice RP, Osterwalder–Seiler)	Stands (cited, not ours)
Lemma 3.1 (heat-kernel domination)	**FALSE**, twice over (§3, §6.4)
Lemma A.1 (variance–oscillation bound)	**FALSE** (§4)
Theorem 5.1 (OS reconstruction, conditional)	**FALSE as stated** (§6.1)
Lemma 3.3 (graph distance vs. time sep.)	**FALSE** on the periodic torus; recoverable with a two-boundary buffer or a periodic distance (§6.2)
Lemma 3.5 (chain rule for oscillation)	**FALSE as printed**; recoverable by integrating the Jacobian along the single-link geodesic (§6.3)
Lemma 3.6 (Gaussian tail on the link graph)	Recoverable only after replacing its heat-kernel input and fixing the volume/time regime (§6.4)
Lemma 3.4 (trans-plane influence decay)	**Withdrawn**
Proposition 3.8 (half-space approximation)	**Withdrawn**
Theorem 4.4 (quantitative RP for flowed obs.)	**Withdrawn**; and see §6.2 on its *statement*, not only its proof
Proposition 5.3 (OS axioms for ω_L^t)	**Withdrawn** (§2)

2 Defect A: the object of the paper is not constructed

Section 2 states: "*The limit state ω_L^t on $\mathcal{A}$ exists by [1, Theorem 1.1]*", and the paper's opening sentence describes ω_L^t as "the continuum limit state constructed in [1]". Reference [1] is **ai.viXra:2602.0085**, whose Theorem 1.1 has been **withdrawn** in its own v2: the linearization identity underlying its ultraviolet estimate is false, because the unfixed Wilson flow has undamped gauge directions.

Consequently ω_L^t is not constructed by the cited result, and no alternative construction is provided in this paper. Every statement of this paper *about* ω_L^t — Theorem 4.4 and Proposition 5.3 in particular

— is a statement about an object whose construction has been retracted. Step 4 of the proof of Theorem 4.4 invokes [1, Theorem 1.1] directly, for the convergence $\omega_k^t((\Theta F)^*F) \to \omega_L^t((\Theta F)^*F)$.

3 Defect B: Lemma 3.1 is the withdrawn lemma, imported

Lemma 3.1 is stated here as "*Heat kernel domination — Paper 83, Lemma 3.6*", namely

$$\|X_\tau(\ell,\ell_0)\|_{\mathrm{op}} \le p_\tau(\ell,\ell_0), \qquad X_\tau(\ell,\ell_0) = \partial_{U_{\ell_0}} W_\tau(U)_\ell,$$

a bound on the columns of the Jacobian of the Wilson flow map. This is exactly the statement refuted in the v2 of ai.viXra:2602.0085, and the refutation applies verbatim here. At $U \equiv \mathbf{1}$ every plaquette is trivial, so the configuration is a global minimum of the Wilson action and a fixed point of the flow. For $\lambda : \Lambda_k^0 \to \mathfrak{su}(N)$ supported at a single vertex, the gauge variation $X^\lambda(e) = \lambda(x) - \lambda(y)$, $e = (x,y)$, is supported on the $2d$ incident links and is *exactly stationary* under the flow, since the whole gauge orbit consists of minima. By linearity it is a signed sum of the $2d$ single-link variations $X_\tau(\cdot, e_i)$. Fix any one incident link e_*; the gauge mode has unit norm there for every τ, so Lemma 3.1 would force

$$1 = \left\|X_\tau^\lambda(e_*)\right\| \;\le\; \sum_{i=1}^{2d} p_\tau(e_*, e_i) \xrightarrow[\tau\to\infty]{} \frac{2d}{|\mathcal{G}_k|},$$

the limit because the scalar heat kernel on a finite connected graph converges to its uniform stationary value. This is impossible as soon as $|\mathcal{G}_k| > 2d$. (Both sides here are pointwise, as Lemma 3.1 is; the ℓ^2 form of the same contradiction is the one used against Proposition 3.9 in the v2 of ai.viXra:2602.0085, whose bound is an ℓ^2 column bound.) Lemma 3.4 is proved from Lemma 3.1 together with Lemma 3.6, and is therefore withdrawn.

4 Defect C: Lemma A.1 is false, and this programme had already said so

Appendix A asserts, under the heading "*Variance–oscillation bound (no product assumption)*", that for *any* probability measure μ on G^E and any $S \subset E$,

$$\mathrm{Var}_\mu\big(\Phi \mid \mathcal{G}_S\big) \;\le\; \tfrac14 \sum_{e\in E\setminus S} \mathrm{osc}_e(\Phi)^2,$$

with the proof step "*each Δ_i is the conditional expectation (given $\mathcal{F}_{i-1}$) of the difference obtained by varying U_{e_i} only, so $\|\Delta_i\|_{L^\infty} \le \frac12 \mathrm{osc}_{e_i}(\Phi)$*", and the closing remark "*no independence or product structure on μ is needed*".

The proof step is false. The martingale increment $\Delta_i = \mathbb{E}[\Phi \mid \mathcal{F}_i] - \mathbb{E}[\Phi \mid \mathcal{F}_{i-1}]$ is *not* obtained by varying U_{e_i} alone: conditioning on U_{e_i} also changes the conditional law of the remaining variables $U_{e_{i+1}}, \dots, U_{e_n}$. Fix $g_0 \ne g_1$ in G and a continuous $f : G \to [-1,1]$ with $f(g_0) = 1$, $f(g_1) = -1$ (the range restriction is what makes the oscillations below exactly 1). On $E = \{e_1, e_2\}$ let $\mu = \frac12\delta_{(g_0,g_0)} + \frac12\delta_{(g_1,g_1)}$ and $\Phi(U_1,U_2) = f(U_2)$. Then $\mathrm{osc}_{e_1}(\Phi) = 0$, while $\Delta_1 = \mathbb{E}[\Phi \mid U_{e_1}] - \mathbb{E}[\Phi] = f(U_{e_1})$ has modulus 1.

And so is the inequality itself. The example above refutes the proof step but not the conclusion: there $\mathrm{osc}_{e_2}(\Phi) = 2$ pays the whole variance, and the asserted bound holds with equality. A direct counterexample to the *inequality* is obtained from the same measure by spreading the observable over both coordinates. With μ as above and

$$\Phi(U_1,U_2) = \tfrac12\big(f(U_1) + f(U_2)\big),$$

Φ takes the values ± 1 with probability $\frac{1}{2}$ each under μ, so (taking $S = \emptyset$, for which $\mathcal{G}_S$ is trivial) $\mathrm{Var}_\mu(\Phi) = 1$. But varying either coordinate alone moves Φ by exactly 1, so $\mathrm{osc}_{e_1}(\Phi) = \mathrm{osc}_{e_2}(\Phi) = 1$ and the asserted bound reads

$$1 = \mathrm{Var}_\mu(\Phi) \ \leq\ \tfrac{1}{4}\big(1^2 + 1^2\big) = \tfrac{1}{2},$$

which is false. Lemma A.1 is therefore false as stated, not merely unproved.

This is not a new observation. It is the two-spin counterexample recorded in the v2 of **ai.viXra:2602.0070**, and it is the reason the v2 of **ai.viXra:2602.0077** withdrew its own Lemma 1.5 — which asserted the same oscillation control "for any probability measure" by the same martingale-increment argument. Those corrections were made in July 2026. **The present paper was submitted in February 2026 and never replaced, so the correction never reached it.**

The measure to which Lemma A.1 is applied here is μ_k, the Wilson measure at scale k (Remark A.2). It is not a product measure — that is the entire content of an interacting lattice gauge theory. Proposition 3.8 derives its L^2 error bound from Lemma A.1 and is therefore withdrawn on this ground alone, independently of §3.

5 What falls, and what does not

Theorem 4.4 fails through three independent routes: its Step 1 rests on Proposition 3.8, which rests on both defective lemmas; its Step 4 rests on the withdrawn Theorem 1.1 of [1]; and its subject ω_L^t is not constructed. Proposition 5.3 is withdrawn for the same reason as its subject.

Exactly two things are untouched:

- **Definitions 2.2, 2.5, 2.6**, as definitions.
- **Theorem 4.1**: exact reflection positivity of the Wilson measure at the lattice level, cited from Osterwalder–Seiler. Not a result of this paper, and not affected by anything here.

Everything else that a first reading of this note listed as surviving did not survive the audit of §6.

The conclusion of Theorem 4.4 is not refuted. Almost-reflection-positivity for flowed observables, with a defect exponentially small in the buffer distance, may well hold. Nothing above bears on its truth.

6 Defects D–G: the survivors, audited

An earlier draft of this note carried a list of results "untouched" by defects A–C. Each had been cleared by checking what it *depends on*. None had had its proof read. Reading them produced four more defects, three of them false statements. None arises from the Wilson-flow gauge-mode obstruction of §3: D is an error about semigroups, E about the periodic geometry of the torus, F about differentiating a nonlinear map, G about the long-time behaviour of a heat kernel on a finite graph. (F does concern the flow map, but its error is generic to nonlinear maps and owes nothing to the special structure of the Wilson flow.) They are recorded in full, because a retraction that quietly shortens its own survivors list between drafts is worth nothing.

6.1 D. Theorem 5.1 is false as stated

The theorem assumes reflection positivity on $\mathcal{E}_+$ and a strongly continuous semigroup $\{\alpha_\tau\}$ of $*$-endomorphisms with (i) $\alpha_\tau(\mathcal{E}_+) \subset \mathcal{E}_+$, (ii) $\omega \circ \alpha_\tau = \omega$, (iii) $\Theta \circ \alpha_\tau = \alpha_\tau \circ \Theta$; and concludes, in item 3, a *self-adjoint* $H \geq 0$ with $T_\tau = e^{-\tau H}$. Step 5 of the proof deduces this from Hille–Yosida.

Hille–Yosida gives no such thing. It produces a generator $-H$ with H accretive; it does not make H self-adjoint. Self-adjointness of the OS Hamiltonian comes from the *reflection* relation between Θ and the evolution, typically $\Theta\alpha_\tau = \alpha_{-\tau}\Theta$ in a two-sided setting, which yields $\langle F, T_\tau G\rangle_{\mathrm{OS}} = \langle T_\tau F, G\rangle_{\mathrm{OS}}$. Hypothesis (iii) is *commutation*, a different relation, and it does not give symmetry.

What (i)–(iii) do give is stronger than a contraction and useless for the conclusion: since α_τ is a $*$-endomorphism, (iii) and (ii) give

$$\|T_\tau[F]\|^2_{\mathrm{OS}} = \omega\big((\Theta\alpha_\tau F)^*\alpha_\tau F\big) = \omega\big(\alpha_\tau((\Theta F)^*F)\big) = \omega\big((\Theta F)^*F\big) = \|[F]\|^2_{\mathrm{OS}},$$

so T_τ is an *isometry.* A semigroup of isometries fixing Ω_{vac} has no decay to extract.

Counterexample. Take $\mathcal{A} = \mathcal{E}_+ = M_2(\mathbb{C})$, $\Theta = \mathrm{id}$, $\omega = \frac{1}{2}\,\mathrm{tr}$, and $\alpha_\tau(F) = U_\tau^* F U_\tau$ with $U_\tau = e^{i\tau K}$ for a self-adjoint non-scalar K. Reflection positivity holds ($\omega(F^*F) \geq 0$), and (i)–(iii) hold. The OS space is $M_2(\mathbb{C})$ with the Hilbert–Schmidt inner product, and $T_\tau = e^{-i\tau\,\mathrm{ad}_K}$ is a *unitary group.* It equals $e^{-\tau H}$ with $H \geq 0$ self-adjoint only if $H = 0$, i.e. only if K is scalar. Conclusion 3 of Theorem 5.1 fails. Separability (conclusion 1) also does not follow for an arbitrary $*$-algebra.

What survives of it. The final sentence of the theorem — that applying the construction to the lattice Wilson measure μ_k yields a lattice Hilbert space and $H_k \geq 0$ unconditionally — is true, but *not by this theorem.* It is the Osterwalder–Seiler transfer-matrix result already quoted here as Theorem 4.1, where self-adjointness and positivity of the transfer operator are established by the reflection structure, not by Hille–Yosida. Theorem 5.1 is recoverable by assuming directly that the induced T_τ are self-adjoint contractions, or by replacing (iii) with the reflection relation above and proving symmetry from it. Either way a new proof is required, and the statement as printed must not be used.

6.2 E. Lemma 3.3 is false on the periodic torus

Remark 2.1 fixes time coordinates on $\mathbb{T}^4_L$ by the representative $x_0 \in (-L/2, L/2]$, and $x_0(\ell)$ is the minimum of the representatives of the two endpoints of ℓ. Lemma 3.3 asserts that adjacent links satisfy $|x_0(\ell) - x_0(\ell')| \leq a_k$, its proof reasoning that "the time coordinates of the endpoints differ by at most a_k". Across the periodic cut they do not.

Let v be a lattice site whose time representative x_* is *maximal* in $(-L/2, L/2]$. (One should not assume $L/2$ is itself a site: if the number of temporal sites is odd, that representative does not occur. Maximality is all that is needed, and it holds for either parity.) The spatial link ℓ_s based at v has both endpoints at time x_*, so $x_0(\ell_s) = x_*$. The temporal link ℓ_t leaving v in the positive time direction wraps across the cut, since $x_* + a_k > L/2$ by maximality, so its other endpoint has representative $x_* + a_k - L$ and $x_0(\ell_t) = x_* + a_k - L$. The two links share v, hence are adjacent, yet

$$|x_0(\ell_s) - x_0(\ell_t)| = L - a_k.$$

The consequence drawn from the lemma, $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \geq |x_0(\ell) - x_0(\ell_0)|/a_k$, fails with it: these two links are at graph distance 1.

The underlying issue is not a slip in a two-line proof. **On a torus the positive-time region has two boundaries**, the plane $x_0 = 0$ and the cut $x_0 = \pm L/2$; the support condition $x_0 > \delta$ separates an observable from the first and not from the second. Recovery requires a two-sided buffer, $\delta < x_0(\ell) < L/2 - \delta'$, or a formulation in terms of periodic distance and distance to the complement of the positive region. **This bears on the *statement* of Theorem 4.4, not only on its proof**: its support hypothesis $\{x_0 > \delta_0\}$ is, as written, insufficient on the torus.

The manuscript is aware of this phenomenon elsewhere — Remark 5.2 observes that on $\mathbb{T}^4_L$ "literal time translation by large τ can wrap around and need not preserve the half-space algebra" — and does not apply the same care in Lemma 3.3.

6.3 F. Lemma 3.5 is false as printed

The proof of Lemma 3.5 bounds, for U, U' differing only at ℓ_0,

$$d_k\big(W_{\tau_k}(U), W_{\tau_k}(U')\big) \le \frac{1}{|\Lambda_k^1|^{1/2}} \Big(\sum_\ell \|X_{\tau_k}(\ell,\ell_0)\|_{\mathrm{op}}^2\, d_G(U_{\ell_0}, U'_{\ell_0})^2\Big)^{1/2},$$

where $X_{\tau_k}(\ell,\ell_0)$ is, by Definition 2.3, the differential of the flow *at a configuration*. This is the estimate valid for a linear map, applied to a nonlinear one. For a nonlinear map the mean-value inequality requires the derivative along the connecting path, not at one endpoint: writing $U^{(s)}$ for the geodesic in the ℓ_0 variable from U_{ℓ_0} to U'_{ℓ_0}, of length ρ, the correct bound carries $\int_0^\rho \|X_{\tau_k}(\ell,\ell_0)(U^{(s)})\|_{\mathrm{op}}\, ds$, or the cruder $\rho \sup_V \|X_{\tau_k}(\ell,\ell_0)(V)\|_{\mathrm{op}}$. Neither the integral nor the supremum appears; the supremum taken at the end of the proof is over the endpoint U_{ℓ_0} only.

In fairness this defect alone is repairable, and would have been invisible in the original architecture: the (false) Lemma 3.1 asserted a bound on $\|X_\tau(\ell,\ell_0)\|_{\mathrm{op}}$ uniform in the configuration, so inserting the supremum would have cost nothing. It is recorded because the lemma as printed does not hold, and because the repair is no longer available once Lemma 3.1 is gone.

6.4 G. The scalar bound in Lemma 3.1 omits the stationary term

Independently of the gauge modes of §3, the right-hand side of Lemma 3.1 is itself false for large τ. It asserts, for the heat kernel of a finite graph and *all* $\tau > 0$,

$$p_\tau(\ell,\ell_0) \le \frac{C}{(\tau+1)^{d/2}} \exp\Big(-\frac{c\, \mathrm{dist}_{\mathrm{gr}}(\ell,\ell_0)^2}{\tau+1}\Big).$$

On a finite connected graph $p_\tau(\ell,\ell_0) \to 1/|\mathcal{G}_k| > 0$ as $\tau \to \infty$, while the right-hand side tends to 0. The bound therefore fails at large times before any question about the Wilson flow arises.

The companion paper ai.viXra:2602.0085 carries the corresponding bound *with* the stationary term $1/|\mathcal{G}_k|$, and devotes its Remark 3.10 to the size of that term. The present paper dropped it. Consequently Lemma 3.6, whose proof consumes this bound shell by shell, cannot be recovered merely by renaming its kernel: recovery requires restoring the stationary term, or restricting to times before mixing, or using the correct periodic kernel with its images — and, for localization against one half of the torus, handling the two boundaries of §6.2.

7 Three repairs, all of which exist in this programme

(a) **For Lemma A.1.** The programme's own repair is the Dobrushin-type decoupling hypothesis (H-DEC/AT) of ai.viXra:2602.0070 and 2602.0072 v2, under which the increment is controlled by the oscillations with an influence-leakage constant. ai.viXra:2602.0077 v2 already adopted it. Applying it here converts Proposition 3.8 into a conditional statement with an explicit constant.

(b) **For Lemma 3.1.** The routes recorded in ai.viXra:2602.0085 v2: prove the oscillation bound for the damped flow $\alpha_0 > 0$ (for gauge-invariant F the composed observable is unchanged, so the bound transfers), or estimate transverse to the gauge orbit on the quotient.

(c) **For the subject.** Repair of Theorem 1.1 of ai.viXra:2602.0085, on which the existence of ω_L^t depends.

Each needs a new proof. Note that (a) and (b) are independent: fixing either one alone leaves Proposition 3.8 unproved.

8 On how this happened

The mathematical defects in §3 and §4 are ordinary errors. What is worth recording is §4's history: the claim was refuted inside this programme, in a companion paper, and the refutation did not propagate to a paper that depended on it. The papers were corrected; the dependency graph between them was not maintained. That is the same failure that produced the archive's metadata errata — no manifest, no propagation step — and it is being addressed by a replacement ledger that records, for each record, what depends on it.

A second lesson is recorded because it was learned the hard way in the drafting of this very note. Theorem 5.1 appeared in the first draft's table as a survivor. It had been cleared by checking what it *depends on*, not by reading its proof; the defect in §6 sits in that proof and has nothing to do with the gradient flow. In a retraction, the list of survivors deserves exactly the adversarial scrutiny given to the list of casualties — a survivor is a positive claim being made under the author's name at the moment when the reader's trust is lowest, and it is the one place where the temptation to salvage is strongest.

Provenance. Defect B follows from an external adversarial review of ai.viXra:2602.0085, verified independently against that manuscript. Defects A and C were found on 2026-07-29 by tracing the dependencies of that retraction through the companion papers, exactly the propagation step whose absence is described in §8. A further round of the same external review supplied Defect D, corrected the norm mismatch in the displayed contradiction of §3, and pointed out that the original counterexample in §4 refuted the proof step of Lemma A.1 without refuting its conclusion; the direct counterexample to the inequality was added in response, and each point was verified against the manuscript before being adopted. A later survivor audit, in the same review, supplied defects E–G: the periodic-cut counterexample to Lemma 3.3, the missing path integration in Lemma 3.5, and the omitted stationary term in the finite-graph heat-kernel bound; a subsequent pass replaced the witness of §6.2 by the maximal-representative site, so that it no longer presumes $L/2$ to be a lattice point. Each was checked directly against the unchanged v1 proof before inclusion. It is recorded that three of the four survivor-audit defects came from outside. No compiled theorem in the Lean core of `THE-ERIKSSON-PROGRAMME` discharges any hypothesis of this paper, and none is affected by this retraction. Remark 4.3 of the v1 manuscript records that the Hamiltonian used in the companion papers is reconstructed from exact lattice reflection positivity (Theorem 4.1), not from the almost-RP estimate withdrawn here; that statement is unaffected, and limits how far this retraction propagates.

Almost Reflection Positivity for Gradient-Flow Observables via Gaussian Localization in Lattice Yang–Mills Theory

Lluis Eriksson

Independent Researcher

lluiseriksson@gmail.com

February 2026

Abstract

Let ω_L^t be the continuum limit state constructed in [1] for Wilson-gradient-flow observables in $SU(N)$ lattice Yang–Mills theory on the four-torus $\mathbb{T}_L^4$. Although the gradient flow is parabolic and propagates information globally [9], we prove a quantitative *Gaussian localization* statement: for observables supported in $\{x_0 > \delta\}$, the influence of links in $\{x_0 < 0\}$ on flowed observables decays as $\exp(-c\,\delta^2/t)$.

This yields an $L^2(\mu_k)$ approximation of flowed observables by half-space-supported conditional expectations, and hence a quantitative *almost*-reflection-positivity estimate for ω_L^t: for observables separated from the reflection plane by a buffer $\delta_0 > \sqrt{8t}$ [9], the RP defect is bounded by $C\exp(-c(\delta_0 - \sqrt{8t})^2/t)$. We record the standard Osterwalder–Schrader reconstruction as a conditional statement: exact reflection positivity on a positive-time algebra implies a Hilbert space, a vacuum, and a non-negative Hamiltonian.

1 Introduction

2 Setup and Notation

We adopt the setup of [1] throughout. In particular:

- $\mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$ is the d-dimensional torus of side L, with $d = 4$.
- Λ_k is the lattice (vertex set) on $\mathbb{T}_L^4$ with spacing $a_k > 0$, and Λ_k^1 denotes its set of *positively oriented* nearest-neighbour links $\ell = (x, \mu)$, $\mu \in \{0, 1, 2, 3\}$.
- $G = \mathrm{SU}(N)$ and gauge configurations are maps $U : \Lambda_k^1 \to G$.
- We extend U to negatively oriented links by the convention $U_{(x,-\mu)} := U^{-1}_{(x-a_k\hat{\mu},\mu)}$, so that link reversal corresponds to group inversion.
- μ_k is the Wilson lattice measure with action $S_W(U) = g_0^{-2} \sum_p \mathrm{Re}\,\mathrm{tr}(\mathbb{1} - U(p))$.
- $\mathcal{W}_{\tau_k}$ is the lattice Wilson flow at time $\tau_k = t/a_k^2$ [9], and $\overline{\mathfrak{A}}^t$ is the C^*-closure of the flow observable algebra ([1, Definition 2.1]).
- The limit state ω_L^t on $\overline{\mathfrak{A}}^t$ exists by [1, Theorem 1.1].
- For each k, we write $G_k := F_k \circ \mathcal{W}_{\tau_k}$ and define the prelimit functional $\omega_k^t(F) := \int G_k(U)\, d\mu_k(U)$.

Remark 2.1 (Time coordinate convention on $\mathbb{T}_L^4$)**.** *Whenever we compare time coordinates to a threshold (e.g. $x_0(\ell) > 0$ or $x_0(\ell) > \delta$), we fix the representative $x_0 \in (-L/2, L/2]$. In particular, we implicitly assume $\delta < L/2$ so that the regions $\{x_0 > 0\}$ and $\{x_0 > \delta\}$ are separated subsets of the chosen fundamental domain.*

Definition 2.2 (Link graph and graph distance)**.** *Fix k and consider the undirected graph $\mathcal{G}_k = (V_k, E_k)$ with vertex set $V_k := \Lambda_k^1$ (positively oriented links). Two vertices $\ell, \ell' \in \Lambda_k^1$ are adjacent (i.e. $\{\ell, \ell'\} \in E_k$) if the corresponding links share at least one endpoint in Λ_k. We write $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)$ for the associated graph distance.*

Definition 2.3 (Single-link derivatives and Jacobian columns)**.** *Fix a bi-invariant Riemannian metric on $G = \mathrm{SU}(N)$ and let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denote the induced inner product on $\mathfrak{g} := \mathrm{Lie}(G)$ under the standard identification $T_{\mathbb{1}} G \simeq \mathfrak{g}$.*

Let $H : G^{\Lambda_k^1} \to G$ be a differentiable map. For a link $\ell_0 \in \Lambda_k^1$, define the single-link directional derivative *at U in direction $\xi \in \mathfrak{g}$ by*

$$(D_{\ell_0} H)(U)[\xi] := \frac{d}{ds}\bigg|_{s=0} H\big(U^{(\ell_0,s)}\big)\, H(U)^{-1} \ \in \ \mathfrak{g},$$

where $U^{(\ell_0,s)}$ agrees with U on all links except ℓ_0, and $U_{\ell_0}^{(\ell_0,s)} := \exp(s\xi)\, U_{\ell_0}$. Thus $(D_{\ell_0} H)(U)$ is a linear map $\mathfrak{g} \to \mathfrak{g}$ after left-trivializing the tangent at $H(U)$.

For the Wilson flow $\mathcal{W}_\tau(U) \in G^{\Lambda_k^1}$, define the Jacobian column

$$X_\tau(\ell, \ell_0)(U) \ := \ (D_{\ell_0}(\mathcal{W}_\tau(\cdot)_\ell))(U) \ \in \ \mathcal{L}(\mathfrak{g}, \mathfrak{g}).$$

We write $\|X_\tau(\ell, \ell_0)\|_{\mathrm{op}}$ for the operator norm induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Remark 2.4 (Notation)**.** *In the main text we keep the shorthand $X_\tau(\ell, \ell_0) = \partial_{U_{\ell_0}} \mathcal{W}_\tau(U)_\ell$ for the Jacobian column of Definition 2.3.*

Definition 2.5 (Time reflection)**.** *The reflection Θ acts on gauge configurations by $(\Theta U)_{(x,\mu)} := U_{(\theta x, \mu)}$ for $\mu \in \{1, 2, 3\}$, while for the time direction $\mu = 0$ we set*

$$(\Theta U)_{(x,0)} := U^{-1}_{(\theta x - a_k \hat{0},\, 0)}.$$

Here $\theta(x_0, \vec{x}) = (-x_0, \vec{x})$, and the shift by $-a_k \hat{0}$ is forced because the reflected time-like link is oppositely oriented; the inverse converts it back to a positively oriented link using the convention $U_{(x,-0)} = U^{-1}_{(x - a_k \hat{0},\, 0)}$.

Definition 2.6 (Half-space support at physical distance δ)**.** *Fix scale k with lattice spacing $a_k > 0$. Let $x_0(\cdot)$ denote the* physical *time-coordinate of a lattice site, i.e. $x_0 \in a_k\mathbb{Z}/L\mathbb{Z}$, and whenever we compare to thresholds we fix the representative $x_0(x) \in (-L/2,\, L/2]$ as in Remark 2.1. For a link $\ell = (x, \mu) \in \Lambda_k^1$ set $x_0(\ell) := \min\{x_0(x),\, x_0(x + a_k \hat{\mu})\}$.*

An observable $F_k : G^{\Lambda_k^1} \to \mathbb{C}$ is supported *in $\{x_0 > \delta\}$ (with $\delta > 0$ in physical units) if*

$$F_k(U) = F_k(U') \quad \textit{whenever} \quad U_\ell = U'_\ell \ \textit{ for all links } \ell \textit{ with } x_0(\ell) \le \delta.$$

Equivalently, F_k depends only on the link variables $\{U_\ell : x_0(\ell) > \delta\}$.

Definition 2.7 (Oscillation and normalized Lipschitz seminorm)**.** *Fix a bi-invariant Riemannian metric on $G = \mathrm{SU}(N)$ with induced geodesic distance d_G. Equip $G^{\Lambda_k^1}$ with the* normalized product metric

$$d_k(U, U')^2 := \frac{1}{|\Lambda_k^1|} \sum_{\ell \in \Lambda_k^1} d_G\big(U_\ell, U'_\ell\big)^2.$$

For $F_k : G^{\Lambda_k^1} \to \mathbb{C}$ and a link $\ell_0 \in \Lambda_k^1$, define the single-link oscillation

$$\mathrm{osc}_{\ell_0}(F_k) := \sup\big\{|F_k(U) - F_k(U')| : \ U_\ell = U'_\ell \ \forall\, \ell \neq \ell_0\big\}.$$

The normalized Lipschitz constant *is*

$$\mathrm{Lip}_k(F_k) := \sup_{U \neq U'} \frac{|F_k(U) - F_k(U')|}{d_k(U, U')}.$$

3 Gaussian Localization of the Gradient Flow

3.1 Heat kernel domination

Lemma 3.1 (Heat kernel domination – Paper 83, Lemma 3.6)**.** *Let $\mathcal{W}_\tau$ denote the lattice Wilson flow at time τ. For any link $\ell \in \Lambda_k^1$, define the Jacobian column $X_\tau(\ell, \ell_0) = \partial_{U_{\ell_0}} \mathcal{W}_\tau(U)_\ell$. Then for all $\tau > 0$:*

$$\big\|X_\tau(\ell, \ell_0)\big\|_{\mathrm{op}} \leq p_\tau(\ell, \ell_0),$$

where p_τ is the heat kernel on the link graph $\mathcal{G}_k$ (Definition 2.2):

$$p_\tau(\ell, \ell_0) \leq \frac{C}{(\tau + 1)^{d/2}} \exp\Big(-c\, \frac{\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)^2}{\tau + 1}\Big).$$

Remark 3.2 (Diffusion length and the scale $\sqrt{8t}$)**.** *At leading order in the continuum, the (modified) gradient flow is governed by the heat equation and acts by convolution with the heat kernel $K_t(z) = (4\pi t)^{-d/2} e^{-|z|^2/(4t)}$; in $d = 4$ the mean-square smoothing radius is $\sqrt{8t}$ [9, eqs. (2.11)–(2.12) and the discussion below]. Lemma 3.1 is the discrete analogue used in our estimates. The specific constant 8 is not essential for our bounds—all that matters is the Gaussian decay $\exp(-c\,\delta^2/t)$ with some $c > 0$—but $\sqrt{8t}$ serves as a convenient physical reference scale throughout.*

3.2 Trans-plane influence decay

Lemma 3.3 (Graph distance controls time separation)**.** *Let $\mathcal{G}_k$ be the link graph from Definition 2.2. Then for adjacent links $\ell \sim \ell'$ one has*

$$|x_0(\ell) - x_0(\ell')| \le a_k.$$

Consequently, for any $\ell, \ell_0 \in \Lambda_k^1$,

$$\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \;\ge\; \frac{|x_0(\ell) - x_0(\ell_0)|}{a_k}.$$

In particular, if $x_0(\ell) > \delta$ and $x_0(\ell_0) < 0$, then $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge \delta/a_k$*.*

Proof. If $\ell \sim \ell'$, then the underlying links share a lattice endpoint v. Each link has endpoints of the form v and $v \pm a_k\hat{\mu}$ for some μ, so the time coordinates of its endpoints differ by at most a_k; hence the minimum time coordinate $x_0(\cdot)$ of the two links differs by at most a_k.

For a shortest path $\ell = \ell^{(0)} \sim \ell^{(1)} \sim \cdots \sim \ell^{(m)} = \ell_0$ with $m = \mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)$, the previous bound and the triangle inequality give $|x_0(\ell) - x_0(\ell_0)| \le \sum_{i=0}^{m-1} |x_0(\ell^{(i)}) - x_0(\ell^{(i+1)})| \le m\, a_k$, proving $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge |x_0(\ell) - x_0(\ell_0)|/a_k$. The final claim follows since $|x_0(\ell) - x_0(\ell_0)| > \delta$. $\square$

Lemma 3.4 (Trans-plane influence decay)**.** *Let F_k be gauge-invariant and supported in $\{x_0 > \delta\}$ for some $\delta > 0$ (physical units). Let ℓ_0 be any link with $x_0(\ell_0) < 0$. Then*

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \le C\,\frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}}\ \exp\!\Big(-c\,\frac{\delta^2}{t}\Big),$$

with constants $C, c > 0$ independent of k (for fixed $L < \infty$), where $\mathrm{diam}(G)$ *is the diameter of (G, d_G).*

Lemma 3.5 (Chain rule for oscillation)**.** *Let F_k be supported in $\{x_0 > \delta\}$ with normalized Lipschitz constant* $\mathrm{Lip}_k(F_k)$ *(Definition 2.7). Then for any link $\ell_0 \in \Lambda_k^1$,*

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \le \frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}} \sum_{\substack{\ell \in \Lambda_k^1 \\ x_0(\ell) > \delta}} \big\|X_{\tau_k}(\ell, \ell_0)\big\|_{\mathrm{op}}.$$

Proof. Fix ℓ_0 and let U, U' differ only at ℓ_0. Since F_k is supported in $\{x_0 > \delta\}$:

$$|F_k(\mathcal{W}_{\tau_k}(U)) - F_k(\mathcal{W}_{\tau_k}(U'))| \le \mathrm{Lip}_k(F_k)\, d_k\big(\mathcal{W}_{\tau_k}(U),\, \mathcal{W}_{\tau_k}(U')\big).$$

By the definition of d_k and the support condition,

$$\begin{aligned} d_k(\mathcal{W}_{\tau_k}(U), \mathcal{W}_{\tau_k}(U')) &\le \frac{1}{|\Lambda_k^1|^{1/2}} \Big(\sum_{\ell:\, x_0(\ell) > \delta} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}^2\, d_G(U_{\ell_0}, U'_{\ell_0})^2 \Big)^{1/2} \\ &\le \frac{d_G(U_{\ell_0}, U'_{\ell_0})}{|\Lambda_k^1|^{1/2}} \sum_{\ell:\, x_0(\ell) > \delta} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}, \end{aligned}$$

where the last step uses $(\sum a_i^2)^{1/2} \le \sum |a_i|$. Since $G = \mathrm{SU}(N)$ is compact, $d_G(U_{\ell_0}, U'_{\ell_0}) \le \mathrm{diam}(G)$. Taking the supremum over $U_{\ell_0} \ne U'_{\ell_0}$ yields the claim (the factor $\mathrm{diam}(G)$ is absorbed into the constant C in subsequent applications). $\square$

Lemma 3.6 (Gaussian tail on the link graph)**.** *Let Λ_k^1 have link graph with polynomial volume growth: $|\{\ell : \mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) = r\}| \le C_d\, r^{d-1}$ for $r \le c\, L/a_k$. Then for $R \ge 1$ and $\tau > 0$,*

$$\sum_{\substack{\ell \\ \mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge R}} p_\tau(\ell, \ell_0) \le C'_d \exp\!\Big(-c\, \frac{R^2}{\tau + 1}\Big).$$

Proof. Split the sum into shells $\{r \le \mathrm{dist}_{\mathrm{gr}} < r+1\}$ for $r \ge R$. Each shell has $O(r^{d-1})$ links and the heat kernel bound gives $p_\tau \le C(\tau+1)^{-d/2} e^{-cr^2/(\tau+1)}$. The sum $\sum_{r \ge R} r^{d-1} e^{-cr^2/(\tau+1)}$ is dominated by $C' e^{-c' R^2/(\tau+1)}$ via standard Gaussian integral comparison. For $r > c\, L/a_k$ (comparable to the torus diameter), the volume growth saturates, but the total number of links is at most $|\Lambda_k^1|$ and the Gaussian factor provides more than sufficient decay, so the bound remains valid with adjusted constants. $\square$

Proof of Lemma 3.4. By Lemma 3.5,

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \le \frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}} \sum_{\substack{\ell \\ x_0(\ell) > \delta}} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}.$$

For ℓ with $x_0(\ell) > \delta$ and ℓ_0 with $x_0(\ell_0) < 0$, Lemma 3.3 gives $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge \delta/a_k$. By Lemma 3.1, $\|X_{\tau_k}(\ell, \ell_0)\| \le p_{\tau_k}(\ell, \ell_0)$ with $\tau_k = t/a_k^2$. By Lemma 3.6

with $R = \delta/a_k$ and $\tau = \tau_k$:

$$\sum_{\mathrm{dist}_{\mathrm{gr}} \geq \delta/a_k} p_{\tau_k}(\ell, \ell_0) \leq C'_d \exp\Big(-c\,\frac{(\delta/a_k)^2}{\tau_k + 1}\Big) = C'_d \exp\Big(-c\,\frac{\delta^2}{t + a_k^2}\Big).$$

To make the bound uniform in k, fix k_0 so that $a_k^2 \leq t$ for all $k \geq k_0$. Then for $k \geq k_0$ we have $t + a_k^2 \leq 2t$ and hence $\exp\big(-c\,\delta^2/(t+a_k^2)\big) \leq \exp\big(-\frac{c}{2}\,\delta^2/t\big)$. For the finitely many $k < k_0$, we absorb the maximum of the corresponding prefactors into the constant C. This yields the stated bound for all k with constants independent of k (for fixed $L < \infty$). □

Remark 3.7 (Buffer form of Lemma 3.4)**.** *In applications we often take $\delta = \sqrt{8t} + \varepsilon$ with $\varepsilon > 0$. Since $\delta^2 \geq 8t + \varepsilon^2$, the bound in Lemma 3.4 implies $\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \leq C(F, t, L) \exp(-c\,\varepsilon^2/t)$ after absorbing the factor e^{-8c} into the prefactor (see Remark 3.2 for the origin of $\sqrt{8t}$).*

3.3 Construction of half-space approximants

Proposition 3.8 (Half-space approximation)**.** *Assume the hypotheses of Lemma 3.4, fix $\delta = \sqrt{8t} + \varepsilon$ with $\varepsilon > 0$, and assume $\sup_k \mathrm{Lip}_k(F_k) \leq M < \infty$. Then there exists a gauge-invariant function $A_{k,\varepsilon} : G^{\Lambda_k^1} \to \mathbb{C}$ depending only on links in $\{x_0 > 0\}$ such that*

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|_{L^2(\mu_k)} \leq C(F, t, L)\, \exp\Big(-c\,\frac{\varepsilon^2}{t}\Big),$$

with constants $C(F, t, L), c > 0$ independent of k (for fixed $L < \infty$).

Proof. Define the approximant as the *conditional expectation*

$$A_{k,\varepsilon}(U_+) := \mathbb{E}_{\mu_k}\big[\, F_k \circ \mathcal{W}_{\tau_k}(U) \,\big|\, \sigma\big(U_\ell : x_0(\ell) > 0\big)\big](U_+), \tag{1}$$

where U_+ denotes the configuration restricted to $\{x_0 > 0\}$.

Dependence on half-space. By construction, $A_{k,\varepsilon}$ is $\sigma(U_\ell : x_0(\ell) > 0)$-measurable.

Gauge invariance. The Wilson measure μ_k is gauge-invariant, the flow $\mathcal{W}_{\tau_k}$ is gauge-equivariant, and F_k is gauge-invariant. Therefore $F_k \circ \mathcal{W}_{\tau_k}$ is gauge-invariant, and the conditional expectation onto a gauge-invariant sub-σ-algebra inherits gauge invariance.

L^2 **error bound.** By the variance–oscillation inequality (Lemma A.1),

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|^2_{L^2(\mu_k)} \le \frac{1}{4} \sum_{\substack{\ell_0 \in \Lambda_k^1 \\ x_0(\ell_0) \le 0}} \operatorname{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k})^2.$$

By Lemma 3.4, each summand is bounded by $C^2 \operatorname{diam}(G)^2 \operatorname{Lip}_k(F_k)^2 |\Lambda_k^1|^{-1} e^{-2c\delta^2/t}$. Since there are at most $|\Lambda_k^1|$ links with $x_0(\ell_0) \le 0$, the factor $|\Lambda_k^1|$ from the sum cancels the $|\Lambda_k^1|^{-1}$ from each oscillation bound:

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|^2_{L^2(\mu_k)} \le \tfrac{1}{4} C^2 \operatorname{diam}(G)^2 \operatorname{Lip}_k(F_k)^2 e^{-2c\delta^2/t}.$$

Thus, under the hypothesis $\sup_k \operatorname{Lip}_k(F_k) \le M$ (see Remark 3.9 below),

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|_{L^2(\mu_k)} \le C(F,t,L)\, e^{-c\varepsilon^2/t}$$

after substituting $\delta = \sqrt{8t} + \varepsilon$, with $C(F,t,L)$ independent of k (for fixed $L < \infty$). □

Remark 3.9 (Uniform Lipschitz condition)**.** *The hypothesis* $\sup_k \operatorname{Lip}_k(F_k) \le M$ *is a regularity condition on the family* $(F_k)_{k\ge1}$ *that must be verified for each class of observables. For observables depending on only* $O(1)$ *links in physical units (e.g. a single plaquette), one typically has* $\operatorname{osc}_{\ell_0}(F_k) = O(1)$ *but, because the metric* d_k *is normalized by* $|\Lambda_k^1|^{-1/2}$*, the Lipschitz constant usually scales as* $\operatorname{Lip}_k(F_k) \sim |\Lambda_k^1|^{1/2}$ *and is* not *uniformly bounded. Uniform boundedness of* $\operatorname{Lip}_k(F_k)$ *is instead satisfied by suitably normalized or spatially averaged observables (e.g. with an explicit* $|\Lambda_k^1|^{-1/2}$ *or* $|\Lambda_k^1|^{-1}$ *prefactor, depending on the normalization). For Wilson loops of growing physical perimeter, the condition requires separate verification.*

4 Reflection Positivity

4.1 Lattice RP for Wilson measure

Theorem 4.1 (Lattice reflection positivity and transfer matrix [8, Thm. 2.1 and §2.3])**.** *Let* μ_k *be the* $\mathrm{SU}(N)$ *Wilson lattice gauge measure on* $\Lambda_k^1 \subset \mathbb{T}_L^d$*. Let* $\mathfrak{E}_{k,+}$ *denote the algebra of gauge-invariant functions depending only on link variables in the positive-time half-space* $\{x_0 > 0\}$*. Then for all* $A \in \mathfrak{E}_{k,+}$*:*

$$\int (\Theta A)^* \cdot A \, d\mu_k \ge 0.$$

Consequently, the associated lattice transfer operator T_k is positive and self-adjoint with $\|T_k\| \leq 1$. By spectral calculus, there exists a (possibly unbounded) self-adjoint operator $H_k \geq 0$ on the corresponding lattice Hilbert space and a time step $\Delta t_k \in \{a_k, 2a_k\}$ (depending on the chosen time-slice convention; see Remark 4.2) such that

$$T_k = e^{-\Delta t_k \, H_k}.$$

Remark 4.2 (Time-slice convention in [8]). *Osterwalder–Seiler implement reflection positivity by choosing the lattice so that no sites lie on the reflection plane, and the transfer operator shifts by two time steps [8, §2.3]. This is equivalent to working with a shifted reflection plane (e.g. $x_0 = a_k/2$) or with a two-step transfer operator in our periodic torus setup. Throughout this paper we keep the notation T_k for the transfer operator arising from the chosen convention, and encode the corresponding step size in Δt_k so that $T_k = e^{-\Delta t_k \, H_k}$.*

Remark 4.3 (Two levels of RP: lattice base vs. flow observables). *It is important to distinguish two levels of reflection positivity in our framework:*

1. ***Exact RP (lattice base).*** *Theorem 4.1 provides exact RP for the Wilson lattice measure μ_k, restricted to the algebra $\mathfrak{E}_{k,+}$ of gauge-invariant functions of half-space links. This produces a lattice Hilbert space, transfer matrix, and Hamiltonian $H_k \geq 0$* without *reference to the gradient flow.*

2. ***Almost-RP (flow observables).*** *For flowed observables $F_k \circ \mathcal{W}_{\tau_k}$, which depend on* all *links, the Gaussian localization of §3 provides half-space approximants with exponentially small error. This yields the almost-RP estimate of Theorem 4.4 below, but not exact RP on the full flow algebra $\overline{\mathfrak{A}}^t$.*

In the companion paper [2], the Hamiltonian H_k is reconstructed from the lattice base (level 1), and the gradient flow enters only at the level of observables, not in the positivity structure.

4.2 RP for flowed observables

Theorem 4.4 (Quantitative RP for flowed observables). *Let $F = (F_k)_{k\geq 1}$ be a family of gauge-invariant observables in $\overline{\mathfrak{A}}^t$, with each F_k supported in*

$\{x_0 > \delta_0\}$ *for some fixed* $\delta_0 > \sqrt{8t}$*. Set* $\varepsilon_0 := \delta_0 - \sqrt{8t} > 0$*. Assume moreover the uniform regularity bound*

$$\sup_{k\geq 1} \mathrm{Lip}_k(F_k) < \infty.$$

Then

$$\omega_L^t\big((\Theta F)^* \cdot F\big) \geq -C(F,t,L)\, \exp\Big(-c\,\frac{\varepsilon_0^2}{t}\Big), \tag{2}$$

where $C(F,t,L), c > 0$ *are independent of* k *(with* $L < \infty$ *fixed).*

Proof. Set $\varepsilon = \varepsilon_0$ and $\delta = \sqrt{8t} + \varepsilon_0 = \delta_0$.

Step 1 (Approximant). By Proposition 3.8, for each k there exists a gauge-invariant $A_k : G^{\Lambda_k^1} \to \mathbb{C}$ depending only on links in $\{x_0 > 0\}$ such that

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_k\|_{L^2(\mu_k)} \leq \eta := C(F,t,L)\, e^{-c\,\varepsilon_0^2/t},$$

with η independent of k.

Step 2 (Lattice RP for approximant). Since A_k depends only on links in $\{x_0 > 0\}$ and is gauge-invariant, Theorem 4.1 gives

$$\int (\Theta A_k)^* \cdot A_k \; d\mu_k \geq 0.$$

Step 3 (Error control). Write $G_k = F_k \circ \mathcal{W}_{\tau_k}$ and decompose

$$\begin{aligned}\int (\Theta G_k)^* G_k \; d\mu_k = &\int (\Theta A_k)^* A_k \; d\mu_k \\ &+ \int (\Theta A_k)^* (G_k - A_k) \; d\mu_k \\ &+ \int (\Theta(G_k - A_k))^* G_k \; d\mu_k. \end{aligned} \tag{3}$$

The first term is ≥ 0 by Step 2. For the cross terms, Cauchy–Schwarz and the L^2-isometry of Θ give

$$\int (\Theta G_k)^* G_k \; d\mu_k \geq -\eta\big(2\|F\|_\infty + \eta\big), \tag{4}$$

where $\|F\|_\infty := \sup_k \|F_k\|_{L^\infty(\mu_k)}$ (finite by the boundedness assumption in $\overline{\mathfrak{A}}^t$). **Step 4 (Limit $k \to \infty$).** By [1, Theorem 1.1], $\omega_k^t((\Theta F)^* F) \to$

$\omega_L^t((\Theta F)^* F)$. The bound (4) is independent of k, so

$$\omega_L^t\big((\Theta F)^* \cdot F\big) \geq -\eta(2\|F\|_\infty + \eta) = -C(F,t,L)\, e^{-c\varepsilon_0^2/t},$$

which is (2). $\square$

Remark 4.5 (Almost-RP vs. exact RP)**.** *The bound* (2) *gives* $\omega_L^t((\Theta F)^* F) \geq -\eta$ *with* η exponentially small *in* ε_0^2/t*, but not identically zero. This is an* almost-*reflection-positivity estimate.*

Exact RP (≥ 0*) for flowed observables would require either:*

(a) observables with compact temporal support, so that the buffer ε_0 *can be sent to* ∞ *and* $\eta \to 0$*; or*

(b) a transfer matrix argument showing that the lattice RP inner product is non-negative for gauge-invariant functions that depend on all *links (not only half-space links). Such an argument requires specifying the boundary Hilbert space with gauge constraints and the integration map* $G_k \mapsto \widehat{G}_k$*; we do not pursue this here.*

As explained in Remark 4.3, the Hamiltonian used in companion papers is reconstructed from exact lattice RP (Theorem 4.1), not from almost-RP on the flow algebra.

5 Osterwalder–Schrader Reconstruction

5.1 Hilbert space and Hamiltonian

Theorem 5.1 (OS reconstruction (conditional))**.** *Let* ω *be a state on a* $*$*-algebra* $\mathfrak{A}$ *equipped with a reflection* Θ *and a positive-time subalgebra* $\mathfrak{E}_+ \subset \mathfrak{A}$*. Assume reflection positivity on* $\mathfrak{E}_+$*:*

$$\omega\big((\Theta F)^* \cdot F\big) \geq 0 \qquad \forall\, F \in \mathfrak{E}_+.$$

Assume moreover that there exists a strongly continuous semigroup $\{\alpha_\tau\}_{\tau\geq 0}$ *of* $*$*-endomorphisms of* $\mathfrak{A}$ *such that:*

(i) $\alpha_\tau(\mathfrak{E}_+) \subset \mathfrak{E}_+$ *for all* $\tau \geq 0$*;*

(ii) $\omega \circ \alpha_\tau = \omega$ *for all* $\tau \geq 0$*;*

(iii) $\Theta \circ \alpha_\tau = \alpha_\tau \circ \Theta$ *for all* $\tau \geq 0$.

Then the standard Osterwalder–Schrader construction produces:

1. *A separable Hilbert space* $\mathcal{H}$*;*
2. *A unit vector* $\Omega_{\mathrm{vac}} \in \mathcal{H}$ *(the vacuum);*
3. *A self-adjoint operator* $H \geq 0$ *with* $H\Omega_{\mathrm{vac}} = 0$*;*
4. *A continuous unitary representation of the spatial symmetry group (if* ω *is spatially invariant).*

In particular, if exact RP is established for ω_L^t *on* $\overline{\mathfrak{A}}^t$ *(e.g. via a transfer matrix argument as discussed in Remark 4.5(b)), then the above yields* $\mathcal{H}$*,* Ω_{vac}*, and* $H \geq 0$ *for flowed Yang–Mills theory at each* $t > 0$*,* $L < \infty$*. Alternatively, applying this theorem directly to the lattice Wilson measure* μ_k *with the algebra* $\mathfrak{E}_{k,+}$ *of Theorem 4.1 yields the lattice Hilbert space and Hamiltonian* $H_k \geq 0$ *unconditionally.*

Proof. We apply the standard OS reconstruction [6, 7].

Step 1 (Positive form). Define $\mathfrak{E}_+ \subset \mathfrak{A}$ as the positive-time subalgebra. By the hypothesis of reflection positivity, the sesquilinear form

$$\langle F, G \rangle_{\mathrm{OS}} := \omega\big((\Theta F)^* \cdot G\big)$$

is positive semidefinite on $\mathfrak{E}_+$.

Step 2 (Hilbert space). Let $\mathcal{N} = \{F \in \mathfrak{E}_+ : \langle F, F \rangle_{\mathrm{OS}} = 0\}$ be the null space. Define $\mathcal{H}$ as the completion of $\mathfrak{E}_+/\mathcal{N}$ with respect to $\langle \cdot, \cdot \rangle_{\mathrm{OS}}$.

Step 3 (Vacuum). The constant observable $\mathbf{1} \in \mathfrak{E}_+$ defines a vector $\Omega_{\mathrm{vac}} := [\mathbf{1}] \in \mathcal{H}$ with $\|\Omega_{\mathrm{vac}}\|^2 = \omega(\mathbf{1}) = 1$.

Step 4 (Contraction semigroup). For $\tau \geq 0$, define T_τ on $\mathfrak{E}_+/\mathcal{N}$ by $T_\tau[F] := [\alpha_\tau(F)]$. By hypothesis (i), $\alpha_\tau(F) \in \mathfrak{E}_+$, so $T_\tau[F]$ is well-defined. The OS inequality (Cauchy–Schwarz for $\langle \cdot, \cdot \rangle_{\mathrm{OS}}$) gives $\|T_\tau[F]\| \leq \|[F]\|$, so T_τ extends to a contraction on $\mathcal{H}$.

Step 5 (Hamiltonian). The family $\{T_\tau\}_{\tau \geq 0}$ is a strongly continuous contraction semigroup on $\mathcal{H}$ satisfying $T_\tau \Omega_{\mathrm{vac}} = \Omega_{\mathrm{vac}}$. By the Hille–Yosida theorem, $T_\tau = e^{-\tau H}$ with $H \geq 0$ self-adjoint and $H\Omega_{\mathrm{vac}} = 0$.

Step 6 (Spatial symmetries). By [1, Lemma 2.2], ω_L^t is invariant under these symmetries. Hence the spatial action is isometric with respect to $\langle \cdot, \cdot \rangle_{\mathrm{OS}}$ and extends to unitary representations on $\mathcal{H}$ that leave Ω_{vac} fixed. □

Remark 5.2 (Why Theorem 5.1 uses an abstract semigroup). *On the Euclidean torus $\mathbb{T}_L^4$ with the half-space $\{x_0 > 0\}$ defined using the fixed representative $x_0 \in (-L/2, L/2]$ (Remark 2.1), literal time translation by large τ can wrap around and need not preserve the half-space algebra. For the lattice Wilson theory, the correct positivity-preserving evolution is the transfer operator T_k of Theorem 4.1. The abstract semigroup hypothesis in Theorem 5.1 isolates exactly what is needed for OS reconstruction and avoids implicitly using a time-translation action that is not compatible with the chosen half-space on $\mathbb{T}_L^4$.*

5.2 Verification of OS axioms

Proposition 5.3 (OS axioms for ω_L^t). *The state ω_L^t satisfies:*

OS0 *(Temperedness) In the present finite-volume setting on $\mathbb{T}_L^4$, the Schwinger functions are L-periodic distributions on $\mathbb{R}^4$ and hence tempered. Boundedness of flowed observables (guaranteed by continuity of the flow map on the compact configuration space $G^{\Lambda_k^1}$) ensures finiteness of all moments entering the Schwinger functions.*

OS1 *(Euclidean covariance) Inherited from [1], Lemma 2.2.*

OS2 *(Reflection positivity)* Not established unconditionally *for ω_L^t on the full positive-time algebra $\overline{\mathfrak{A}}^t$. Theorem 4.4 provides a quantitative almost-RP estimate (defect exponentially small in the buffer distance). Exact RP holds unconditionally at the lattice level for the Wilson measure (Theorem 4.1); see Remark 4.3.*

OS4 *(Symmetry) Gauge invariance of all observables in $\overline{\mathfrak{A}}^t$.*

Remark 5.4 (OS3 – Clustering). *OS3 (uniqueness of the vacuum / cluster property) remains open and requires exponential decay of correlations (mass gap). This will be addressed in the companion paper [2] using a discrete Bakry–Émery approach.*

6 Discussion

A Variance–oscillation bound (no product assumption)

Lemma A.1 (Variance bound via single-edge oscillations)**.** *Let* μ *be any probability measure on* G^E *(not necessarily a product measure) and let* $S \subset E$*. Set* $\mathcal{G}_S = \sigma(U_e : e \in S)$*. Then for any bounded* $\Phi \in L^2(\mu)$*,*

$$\mathrm{Var}_\mu\big(\Phi \,\big|\, \mathcal{G}_S\big) \le \frac{1}{4} \sum_{e \in E \setminus S} \mathrm{osc}_e(\Phi)^2.$$

Remark A.2 (How Lemma A.1 is used in the main text)**.** *In Proposition 3.8 we apply Lemma A.1 with* $E = \Lambda_k^1$*,* $S = \{\ell \in \Lambda_k^1 : x_0(\ell) > 0\}$*, and* $\Phi = F_k \circ \mathcal{W}_{\tau_k}$*.*

Proof. Write $E \setminus S = \{e_1, \ldots, e_n\}$ and define filtrations $\mathcal{F}_0 = \mathcal{G}_S$ and $\mathcal{F}_i = \sigma(\mathcal{G}_S, U_{e_1}, \ldots, U_{e_i})$ for $i = 1, \ldots, n$. Then $\mathcal{F}_n = \sigma(U_e : e \in E)$ and $\Phi - \mathbb{E}[\Phi \mid \mathcal{G}_S] = \sum_{i=1}^n \Delta_i$ where $\Delta_i = \mathbb{E}[\Phi \mid \mathcal{F}_i] - \mathbb{E}[\Phi \mid \mathcal{F}_{i-1}]$ is a martingale difference sequence. By orthogonality,

$$\|\Phi - \mathbb{E}[\Phi \mid \mathcal{G}_S]\|_2^2 = \sum_{i=1}^n \|\Delta_i\|_2^2.$$

Each Δ_i is the conditional expectation (given $\mathcal{F}_{i-1}$) of the difference obtained by varying U_{e_i} only, so $\|\Delta_i\|_{L^\infty} \le \frac{1}{2}\mathrm{osc}_{e_i}(\Phi)$ and hence $\|\Delta_i\|_2^2 \le \frac{1}{4}\mathrm{osc}_{e_i}(\Phi)^2$. Summing gives the result. Note: no independence or product structure on μ is needed; only the tower property of conditional expectation is used. $\square$

B RP for lattice gauge theories: statement

References

[1] L. Eriksson, *Ultraviolet closure for Wilson-flow observables in lattice Yang–Mills theory*, Preprint, 2026.

[2] L. Eriksson, *Spectral gap of the Yang–Mills Hamiltonian at finite volume via log-Sobolev inequalities*, Preprint, 2026.

[3] L. Eriksson, *Blocking renormalisation for lattice gauge observables*, Preprint, 2026.

[4] L. Eriksson, *Thermodynamic limit and volume-uniform mass gap for lattice Yang–Mills theory*, Preprint, 2026.

[5] L. Eriksson, *Yang–Mills existence and mass gap: a complete solution for* $\mathrm{SU}(N)$ *gauge theory in four dimensions*, Preprint, 2026.

[6] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions*, Comm. Math. Phys. **31** (1973) 83–112.

[7] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions II*, Comm. Math. Phys. **42** (1975) 281–305.

[8] K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Ann. Phys. **110** (1978) 440–471.

[9] M. Lüscher, *Properties and uses of the Wilson flow in lattice QCD*, JHEP **1008** (2010) 071.

[10] M. Lüscher and P. Weisz, *Perturbative analysis of the gradient flow in non-abelian gauge theories*, JHEP **1102** (2011) 051.

[11] T. Bałaban, *Averaging operations for lattice gauge theories*, Comm. Math. Phys. **98** (1985) 17–51.

[12] D. Bakry and M. Émery, *Diffusions hypercontractives*, Sém. Prob. XIX, LNM **1123** (1985) 177–206.

[13] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.