Almost Reflection Positivity for Gradient-Flow Observables via Gaussian Localization in Lattice Yang–Mills Theory

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Abstract

Let ω_L^t be the continuum limit state constructed in [1] for Wilson-gradient-flow observables in $SU(N)$ lattice Yang–Mills theory on the four-torus $\mathbb{T}_L^4$. Although the gradient flow is parabolic and propagates information globally [9], we prove a quantitative *Gaussian localization* statement: for observables supported in $\{x_0 > \delta\}$, the influence of links in $\{x_0 < 0\}$ on flowed observables decays as $\exp(-c\,\delta^2/t)$.

This yields an $L^2(\mu_k)$ approximation of flowed observables by half-space-supported conditional expectations, and hence a quantitative *almost*-reflection-positivity estimate for ω_L^t: for observables separated from the reflection plane by a buffer $\delta_0 > \sqrt{8t}$ [9], the RP defect is bounded by $C\exp(-c(\delta_0 - \sqrt{8t})^2/t)$. We record the standard Osterwalder–Schrader reconstruction as a conditional statement: exact reflection positivity on a positive-time algebra implies a Hilbert space, a vacuum, and a non-negative Hamiltonian.

1 Introduction

2 Setup and Notation

We adopt the setup of [1] throughout. In particular:

- $\mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$ is the d-dimensional torus of side L, with $d = 4$.
- Λ_k is the lattice (vertex set) on $\mathbb{T}_L^4$ with spacing $a_k > 0$, and Λ_k^1 denotes its set of *positively oriented* nearest-neighbour links $\ell = (x, \mu)$, $\mu \in \{0, 1, 2, 3\}$.
- $G = \mathrm{SU}(N)$ and gauge configurations are maps $U : \Lambda_k^1 \to G$.
- We extend U to negatively oriented links by the convention $U_{(x,-\mu)} := U^{-1}_{(x-a_k\hat{\mu},\mu)}$, so that link reversal corresponds to group inversion.
- μ_k is the Wilson lattice measure with action $S_W(U) = g_0^{-2} \sum_p \mathrm{Re}\,\mathrm{tr}(\mathbb{1} - U(p))$.
- $\mathcal{W}_{\tau_k}$ is the lattice Wilson flow at time $\tau_k = t/a_k^2$ [9], and $\overline{\mathfrak{A}}^t$ is the C^*-closure of the flow observable algebra ([1, Definition 2.1]).
- The limit state ω_L^t on $\overline{\mathfrak{A}}^t$ exists by [1, Theorem 1.1].
- For each k, we write $G_k := F_k \circ \mathcal{W}_{\tau_k}$ and define the prelimit functional $\omega_k^t(F) := \int G_k(U)\, d\mu_k(U)$.

Remark 2.1 (Time coordinate convention on $\mathbb{T}_L^4$)**.** *Whenever we compare time coordinates to a threshold (e.g. $x_0(\ell) > 0$ or $x_0(\ell) > \delta$), we fix the representative $x_0 \in (-L/2, L/2]$. In particular, we implicitly assume $\delta < L/2$ so that the regions $\{x_0 > 0\}$ and $\{x_0 > \delta\}$ are separated subsets of the chosen fundamental domain.*

Definition 2.2 (Link graph and graph distance)**.** *Fix k and consider the undirected graph $\mathcal{G}_k = (V_k, E_k)$ with vertex set $V_k := \Lambda_k^1$ (positively oriented links). Two vertices $\ell, \ell' \in \Lambda_k^1$ are adjacent (i.e. $\{\ell, \ell'\} \in E_k$) if the corresponding links share at least one endpoint in Λ_k. We write $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)$ for the associated graph distance.*

Definition 2.3 (Single-link derivatives and Jacobian columns)**.** *Fix a bi-invariant Riemannian metric on $G = \mathrm{SU}(N)$ and let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denote the induced inner product on $\mathfrak{g} := \mathrm{Lie}(G)$ under the standard identification $T_{\mathbb{1}} G \simeq \mathfrak{g}$.*

Let $H : G^{\Lambda_k^1} \to G$ be a differentiable map. For a link $\ell_0 \in \Lambda_k^1$, define the single-link directional derivative *at U in direction $\xi \in \mathfrak{g}$ by*

$$(D_{\ell_0} H)(U)[\xi] := \frac{d}{ds}\bigg|_{s=0} H\big(U^{(\ell_0,s)}\big)\, H(U)^{-1} \ \in \ \mathfrak{g},$$

where $U^{(\ell_0,s)}$ agrees with U on all links except ℓ_0, and $U^{(\ell_0,s)}_{\ell_0} := \exp(s\xi)\, U_{\ell_0}$. Thus $(D_{\ell_0} H)(U)$ is a linear map $\mathfrak{g} \to \mathfrak{g}$ after left-trivializing the tangent at $H(U)$.

For the Wilson flow $\mathcal{W}_\tau(U) \in G^{\Lambda_k^1}$, define the Jacobian column

$$X_\tau(\ell, \ell_0)(U) \ := \ (D_{\ell_0}(\mathcal{W}_\tau(\cdot)_\ell))(U) \ \in \ \mathcal{L}(\mathfrak{g}, \mathfrak{g}).$$

We write $\|X_\tau(\ell, \ell_0)\|_{\mathrm{op}}$ for the operator norm induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Remark 2.4 (Notation)**.** *In the main text we keep the shorthand $X_\tau(\ell, \ell_0) = \partial_{U_{\ell_0}} \mathcal{W}_\tau(U)_\ell$ for the Jacobian column of Definition 2.3.*

Definition 2.5 (Time reflection)**.** *The reflection Θ acts on gauge configurations by $(\Theta U)_{(x,\mu)} := U_{(\theta x, \mu)}$ for $\mu \in \{1, 2, 3\}$, while for the time direction $\mu = 0$ we set*

$$(\Theta U)_{(x,0)} := U^{-1}_{(\theta x - a_k \hat{0},\, 0)}.$$

Here $\theta(x_0, \vec{x}) = (-x_0, \vec{x})$, and the shift by $-a_k \hat{0}$ is forced because the reflected time-like link is oppositely oriented; the inverse converts it back to a positively oriented link using the convention $U_{(x,-0)} = U^{-1}_{(x - a_k \hat{0},\, 0)}$.

Definition 2.6 (Half-space support at physical distance δ)**.** *Fix scale k with lattice spacing $a_k > 0$. Let $x_0(\cdot)$ denote the* physical *time-coordinate of a lattice site, i.e. $x_0 \in a_k \mathbb{Z}/L\mathbb{Z}$, and whenever we compare to thresholds we fix the representative $x_0(x) \in (-L/2,\, L/2]$ as in Remark 2.1. For a link $\ell = (x, \mu) \in \Lambda_k^1$ set $x_0(\ell) := \min\{x_0(x),\, x_0(x + a_k \hat{\mu})\}$.*

An observable $F_k : G^{\Lambda_k^1} \to \mathbb{C}$ is supported *in $\{x_0 > \delta\}$ (with $\delta > 0$ in physical units) if*

$$F_k(U) = F_k(U') \quad \text{whenever} \quad U_\ell = U'_\ell \ \text{ for all links } \ell \text{ with } x_0(\ell) \le \delta.$$

Equivalently, F_k depends only on the link variables $\{U_\ell : x_0(\ell) > \delta\}$.

Definition 2.7 (Oscillation and normalized Lipschitz seminorm)**.** *Fix a bi-invariant Riemannian metric on* $G = \mathrm{SU}(N)$ *with induced geodesic distance* d_G*. Equip* $G^{\Lambda_k^1}$ *with the* normalized product metric

$$d_k(U, U')^2 := \frac{1}{|\Lambda_k^1|} \sum_{\ell \in \Lambda_k^1} d_G\big(U_\ell, U'_\ell\big)^2.$$

For $F_k : G^{\Lambda_k^1} \to \mathbb{C}$ *and a link* $\ell_0 \in \Lambda_k^1$*, define the* single-link oscillation

$$\mathrm{osc}_{\ell_0}(F_k) := \sup\big\{|F_k(U) - F_k(U')| : \; U_\ell = U'_\ell \; \forall\, \ell \neq \ell_0\big\}.$$

The normalized Lipschitz constant *is*

$$\mathrm{Lip}_k(F_k) := \sup_{U \neq U'} \frac{|F_k(U) - F_k(U')|}{d_k(U, U')}.$$

3 Gaussian Localization of the Gradient Flow

3.1 Heat kernel domination

Lemma 3.1 (Heat kernel domination – Paper 83, Lemma 3.6)**.** *Let* $\mathcal{W}_\tau$ *denote the lattice Wilson flow at time* τ*. For any link* $\ell \in \Lambda_k^1$*, define the Jacobian column* $X_\tau(\ell, \ell_0) = \partial_{U_{\ell_0}} \mathcal{W}_\tau(U)_\ell$*. Then for all* $\tau > 0$*:*

$$\big\|X_\tau(\ell, \ell_0)\big\|_{\mathrm{op}} \leq p_\tau(\ell, \ell_0),$$

where p_τ *is the heat kernel on the link graph* $\mathcal{G}_k$ *(Definition 2.2):*

$$p_\tau(\ell, \ell_0) \leq \frac{C}{(\tau+1)^{d/2}} \exp\Big(-c\, \frac{\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)^2}{\tau + 1}\Big).$$

Remark 3.2 (Diffusion length and the scale $\sqrt{8t}$)**.** *At leading order in the continuum, the (modified) gradient flow is governed by the heat equation and acts by convolution with the heat kernel* $K_t(z) = (4\pi t)^{-d/2} e^{-|z|^2/(4t)}$*; in* $d = 4$ *the mean-square smoothing radius is* $\sqrt{8t}$ *[9, eqs. (2.11)–(2.12) and the discussion below]. Lemma 3.1 is the discrete analogue used in our estimates. The specific constant 8 is not essential for our bounds—all that matters is the Gaussian decay* $\exp(-c\,\delta^2/t)$ *with some* $c > 0$*—but* $\sqrt{8t}$ *serves as a convenient physical reference scale throughout.*

3.2 Trans-plane influence decay

Lemma 3.3 (Graph distance controls time separation)**.** *Let $\mathcal{G}_k$ be the link graph from Definition 2.2. Then for adjacent links $\ell \sim \ell'$ one has*

$$|x_0(\ell) - x_0(\ell')| \le a_k.$$

Consequently, for any $\ell, \ell_0 \in \Lambda_k^1$,

$$\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \;\ge\; \frac{|x_0(\ell) - x_0(\ell_0)|}{a_k}.$$

In particular, if $x_0(\ell) > \delta$ and $x_0(\ell_0) < 0$, then $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge \delta / a_k$.

Proof. If $\ell \sim \ell'$, then the underlying links share a lattice endpoint v. Each link has endpoints of the form v and $v \pm a_k \hat{\mu}$ for some μ, so the time coordinates of its endpoints differ by at most a_k; hence the minimum time coordinate $x_0(\cdot)$ of the two links differs by at most a_k.

For a shortest path $\ell = \ell^{(0)} \sim \ell^{(1)} \sim \cdots \sim \ell^{(m)} = \ell_0$ with $m = \mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0)$, the previous bound and the triangle inequality give $|x_0(\ell) - x_0(\ell_0)| \le \sum_{i=0}^{m-1} |x_0(\ell^{(i)}) - x_0(\ell^{(i+1)})| \le m\, a_k$, proving $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \ge |x_0(\ell) - x_0(\ell_0)|/a_k$. The final claim follows since $|x_0(\ell) - x_0(\ell_0)| > \delta$. □

Lemma 3.4 (Trans-plane influence decay)**.** *Let F_k be gauge-invariant and supported in $\{x_0 > \delta\}$ for some $\delta > 0$ (physical units). Let ℓ_0 be any link with $x_0(\ell_0) < 0$. Then*

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \le C\, \frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}}\ \exp\!\Big(-c\, \frac{\delta^2}{t}\Big),$$

with constants $C, c > 0$ independent of k (for fixed $L < \infty$), where $\mathrm{diam}(G)$ *is the diameter of (G, d_G).*

Lemma 3.5 (Chain rule for oscillation)**.** *Let F_k be supported in $\{x_0 > \delta\}$ with normalized Lipschitz constant* $\mathrm{Lip}_k(F_k)$ *(Definition 2.7). Then for any link $\ell_0 \in \Lambda_k^1$,*

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \le \frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}} \sum_{\substack{\ell \in \Lambda_k^1 \\ x_0(\ell) > \delta}} \big\| X_{\tau_k}(\ell, \ell_0) \big\|_{\mathrm{op}}.$$

Proof. Fix ℓ_0 and let U, U' differ only at ℓ_0. Since F_k is supported in $\{x_0 > \delta\}$:

$$|F_k(\mathcal{W}_{\tau_k}(U)) - F_k(\mathcal{W}_{\tau_k}(U'))| \leq \mathrm{Lip}_k(F_k)\, d_k\big(\mathcal{W}_{\tau_k}(U),\, \mathcal{W}_{\tau_k}(U')\big).$$

By the definition of d_k and the support condition,

$$\begin{aligned} d_k(\mathcal{W}_{\tau_k}(U), \mathcal{W}_{\tau_k}(U')) &\leq \frac{1}{|\Lambda_k^1|^{1/2}} \Big(\sum_{\ell:\, x_0(\ell)>\delta} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}^2\, d_G(U_{\ell_0}, U'_{\ell_0})^2 \Big)^{1/2} \\ &\leq \frac{d_G(U_{\ell_0}, U'_{\ell_0})}{|\Lambda_k^1|^{1/2}} \sum_{\ell:\, x_0(\ell)>\delta} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}, \end{aligned}$$

where the last step uses $(\sum a_i^2)^{1/2} \leq \sum |a_i|$. Since $G = \mathrm{SU}(N)$ is compact, $d_G(U_{\ell_0}, U'_{\ell_0}) \leq \mathrm{diam}(G)$. Taking the supremum over $U_{\ell_0} \neq U'_{\ell_0}$ yields the claim (the factor $\mathrm{diam}(G)$ is absorbed into the constant C in subsequent applications). ☐

Lemma 3.6 (Gaussian tail on the link graph)**.** *Let Λ_k^1 have link graph with polynomial volume growth: $|\{\ell : \mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) = r\}| \leq C_d\, r^{d-1}$ for $r \leq c\, L/a_k$. Then for $R \geq 1$ and $\tau > 0$,*

$$\sum_{\substack{\ell \\ \mathrm{dist}_{\mathrm{gr}}(\ell,\ell_0)\geq R}} p_\tau(\ell, \ell_0) \leq C'_d \exp\Big(-c\, \frac{R^2}{\tau+1}\Big).$$

Proof. Split the sum into shells $\{r \leq \mathrm{dist}_{\mathrm{gr}} < r+1\}$ for $r \geq R$. Each shell has $O(r^{d-1})$ links and the heat kernel bound gives $p_\tau \leq C(\tau+1)^{-d/2} e^{-cr^2/(\tau+1)}$. The sum $\sum_{r\geq R} r^{d-1} e^{-cr^2/(\tau+1)}$ is dominated by $C' e^{-c' R^2/(\tau+1)}$ via standard Gaussian integral comparison. For $r > c\, L/a_k$ (comparable to the torus diameter), the volume growth saturates, but the total number of links is at most $|\Lambda_k^1|$ and the Gaussian factor provides more than sufficient decay, so the bound remains valid with adjusted constants. ☐

Proof of Lemma 3.4. By Lemma 3.5,

$$\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \leq \frac{\mathrm{diam}(G)\ \mathrm{Lip}_k(F_k)}{|\Lambda_k^1|^{1/2}} \sum_{\substack{\ell \\ x_0(\ell)>\delta}} \|X_{\tau_k}(\ell, \ell_0)\|_{\mathrm{op}}.$$

For ℓ with $x_0(\ell) > \delta$ and ℓ_0 with $x_0(\ell_0) < 0$, Lemma 3.3 gives $\mathrm{dist}_{\mathrm{gr}}(\ell, \ell_0) \geq \delta/a_k$. By Lemma 3.1, $\|X_{\tau_k}(\ell, \ell_0)\| \leq p_{\tau_k}(\ell, \ell_0)$ with $\tau_k = t/a_k^2$. By Lemma 3.6

with $R = \delta/a_k$ and $\tau = \tau_k$:

$$\sum_{\mathrm{dist}_{\mathrm{gr}} \geq \delta/a_k} p_{\tau_k}(\ell, \ell_0) \leq C'_d \exp\Bigl(-c\, \frac{(\delta/a_k)^2}{\tau_k + 1}\Bigr) = C'_d \exp\Bigl(-c\, \frac{\delta^2}{t + a_k^2}\Bigr).$$

To make the bound uniform in k, fix k_0 so that $a_k^2 \leq t$ for all $k \geq k_0$. Then for $k \geq k_0$ we have $t + a_k^2 \leq 2t$ and hence $\exp\bigl(-c\, \delta^2/(t + a_k^2)\bigr) \leq \exp\bigl(-\frac{c}{2}\, \delta^2/t\bigr)$. For the finitely many $k < k_0$, we absorb the maximum of the corresponding prefactors into the constant C. This yields the stated bound for all k with constants independent of k (for fixed $L < \infty$). □

Remark 3.7 (Buffer form of Lemma 3.4)**.** *In applications we often take* $\delta = \sqrt{8t} + \varepsilon$ *with* $\varepsilon > 0$*. Since* $\delta^2 \geq 8t + \varepsilon^2$*, the bound in Lemma 3.4 implies* $\mathrm{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k}) \leq C(F, t, L) \exp(-c\, \varepsilon^2/t)$ *after absorbing the factor* e^{-8c} *into the prefactor (see Remark 3.2 for the origin of* $\sqrt{8t}$*).*

3.3 Construction of half-space approximants

Proposition 3.8 (Half-space approximation)**.** *Assume the hypotheses of Lemma 3.4, fix* $\delta = \sqrt{8t} + \varepsilon$ *with* $\varepsilon > 0$*, and assume* $\sup_k \mathrm{Lip}_k(F_k) \leq M < \infty$*. Then there exists a gauge-invariant function* $A_{k,\varepsilon} : G^{\Lambda_k^1} \to \mathbb{C}$ *depending only on links in* $\{x_0 > 0\}$ *such that*

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|_{L^2(\mu_k)} \leq C(F, t, L)\, \exp\Bigl(-c\, \frac{\varepsilon^2}{t}\Bigr),$$

with constants $C(F, t, L), c > 0$ *independent of* k *(for fixed* $L < \infty$*).*

Proof. Define the approximant as the *conditional expectation*

$$A_{k,\varepsilon}(U_+) := \mathbb{E}_{\mu_k}\bigl[\, F_k \circ \mathcal{W}_{\tau_k}(U) \bigm| \sigma\bigl(U_\ell : x_0(\ell) > 0\bigr)\bigr](U_+), \tag{1}$$

where U_+ denotes the configuration restricted to $\{x_0 > 0\}$.

Dependence on half-space. By construction, $A_{k,\varepsilon}$ is $\sigma(U_\ell : x_0(\ell) > 0)$-measurable.

Gauge invariance. The Wilson measure μ_k is gauge-invariant, the flow $\mathcal{W}_{\tau_k}$ is gauge-equivariant, and F_k is gauge-invariant. Therefore $F_k \circ \mathcal{W}_{\tau_k}$ is gauge-invariant, and the conditional expectation onto a gauge-invariant sub-σ-algebra inherits gauge invariance.

L^2 **error bound.** By the variance–oscillation inequality (Lemma A.1),

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|^2_{L^2(\mu_k)} \leq \frac{1}{4} \sum_{\substack{\ell_0 \in \Lambda^1_k \\ x_0(\ell_0) \leq 0}} \operatorname{osc}_{\ell_0}(F_k \circ \mathcal{W}_{\tau_k})^2.$$

By Lemma 3.4, each summand is bounded by $C^2 \operatorname{diam}(G)^2 \operatorname{Lip}_k(F_k)^2 \, |\Lambda^1_k|^{-1} \, e^{-2c\delta^2/t}$. Since there are at most $|\Lambda^1_k|$ links with $x_0(\ell_0) \leq 0$, the factor $|\Lambda^1_k|$ from the sum cancels the $|\Lambda^1_k|^{-1}$ from each oscillation bound:

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|^2_{L^2(\mu_k)} \leq \tfrac{1}{4} \, C^2 \operatorname{diam}(G)^2 \operatorname{Lip}_k(F_k)^2 \, e^{-2c\delta^2/t}.$$

Thus, under the hypothesis $\sup_k \operatorname{Lip}_k(F_k) \leq M$ (see Remark 3.9 below),

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_{k,\varepsilon}\|_{L^2(\mu_k)} \leq C(F, t, L) \, e^{-c\varepsilon^2/t}$$

after substituting $\delta = \sqrt{8t} + \varepsilon$, with $C(F, t, L)$ independent of k (for fixed $L < \infty$). □

Remark 3.9 (Uniform Lipschitz condition)**.** *The hypothesis* $\sup_k \operatorname{Lip}_k(F_k) \leq M$ *is a regularity condition on the family* $(F_k)_{k \geq 1}$ *that must be verified for each class of observables. For observables depending on only* $O(1)$ *links in physical units (e.g. a single plaquette), one typically has* $\operatorname{osc}_{\ell_0}(F_k) = O(1)$ *but, because the metric* d_k *is normalized by* $|\Lambda^1_k|^{-1/2}$*, the Lipschitz constant usually scales as* $\operatorname{Lip}_k(F_k) \sim |\Lambda^1_k|^{1/2}$ *and is* not *uniformly bounded. Uniform boundedness of* $\operatorname{Lip}_k(F_k)$ *is instead satisfied by suitably normalized or spatially averaged observables (e.g. with an explicit* $|\Lambda^1_k|^{-1/2}$ *or* $|\Lambda^1_k|^{-1}$ *prefactor, depending on the normalization). For Wilson loops of growing physical perimeter, the condition requires separate verification.*

4 Reflection Positivity

4.1 Lattice RP for Wilson measure

Theorem 4.1 (Lattice reflection positivity and transfer matrix [8, Thm. 2.1 and §2.3])**.** *Let* μ_k *be the* $\mathrm{SU}(N)$ *Wilson lattice gauge measure on* $\Lambda^1_k \subset \mathbb{T}^d_L$*. Let* $\mathfrak{E}_{k,+}$ *denote the algebra of gauge-invariant functions depending only on link variables in the positive-time half-space* $\{x_0 > 0\}$*. Then for all* $A \in \mathfrak{E}_{k,+}$*:*

$$\int (\Theta A)^* \cdot A \, d\mu_k \geq 0.$$

Consequently, the associated lattice transfer operator T_k is positive and self-adjoint with $\|T_k\| \leq 1$. By spectral calculus, there exists a (possibly unbounded) self-adjoint operator $H_k \geq 0$ on the corresponding lattice Hilbert space and a time step $\Delta t_k \in \{a_k, 2a_k\}$ (depending on the chosen time-slice convention; see Remark 4.2) such that

$$T_k = e^{-\Delta t_k \, H_k}.$$

Remark 4.2 (Time-slice convention in [8]). *Osterwalder–Seiler implement reflection positivity by choosing the lattice so that no sites lie on the reflection plane, and the transfer operator shifts by two time steps [8, §2.3]. This is equivalent to working with a shifted reflection plane (e.g. $x_0 = a_k/2$) or with a two-step transfer operator in our periodic torus setup. Throughout this paper we keep the notation T_k for the transfer operator arising from the chosen convention, and encode the corresponding step size in Δt_k so that $T_k = e^{-\Delta t_k \, H_k}$.*

Remark 4.3 (Two levels of RP: lattice base vs. flow observables). *It is important to distinguish two levels of reflection positivity in our framework:*

1. ***Exact RP (lattice base).*** *Theorem 4.1 provides exact RP for the Wilson lattice measure μ_k, restricted to the algebra $\mathfrak{E}_{k,+}$ of gauge-invariant functions of half-space links. This produces a lattice Hilbert space, transfer matrix, and Hamiltonian $H_k \geq 0$* without *reference to the gradient flow.*

2. ***Almost-RP (flow observables).*** *For flowed observables $F_k \circ \mathcal{W}_{\tau_k}$, which depend on* all *links, the Gaussian localization of §3 provides half-space approximants with exponentially small error. This yields the almost-RP estimate of Theorem 4.4 below, but not exact RP on the full flow algebra $\overline{\mathfrak{A}}^t$.*

In the companion paper [2], the Hamiltonian H_k is reconstructed from the lattice base (level 1), and the gradient flow enters only at the level of observables, not in the positivity structure.

4.2 RP for flowed observables

Theorem 4.4 (Quantitative RP for flowed observables). *Let $F = (F_k)_{k \geq 1}$ be a family of gauge-invariant observables in $\overline{\mathfrak{A}}^t$, with each F_k supported in*

$\{x_0 > \delta_0\}$ *for some fixed* $\delta_0 > \sqrt{8t}$*. Set* $\varepsilon_0 := \delta_0 - \sqrt{8t} > 0$*. Assume moreover the uniform regularity bound*

$$\sup_{k \geq 1} \mathrm{Lip}_k(F_k) < \infty.$$

Then

$$\omega_L^t\big((\Theta F)^* \cdot F\big) \geq -C(F,t,L)\, \exp\Big(-c\,\frac{\varepsilon_0^2}{t}\Big), \tag{2}$$

where $C(F,t,L), c > 0$ *are independent of* k *(with* $L < \infty$ *fixed).*

Proof. Set $\varepsilon = \varepsilon_0$ and $\delta = \sqrt{8t} + \varepsilon_0 = \delta_0$.

Step 1 (Approximant). By Proposition 3.8, for each k there exists a gauge-invariant $A_k : G^{\Lambda_k^1} \to \mathbb{C}$ depending only on links in $\{x_0 > 0\}$ such that

$$\|F_k \circ \mathcal{W}_{\tau_k} - A_k\|_{L^2(\mu_k)} \leq \eta := C(F,t,L)\, e^{-c\,\varepsilon_0^2/t},$$

with η independent of k.

Step 2 (Lattice RP for approximant). Since A_k depends only on links in $\{x_0 > 0\}$ and is gauge-invariant, Theorem 4.1 gives

$$\int (\Theta A_k)^* \cdot A_k \, d\mu_k \geq 0.$$

Step 3 (Error control). Write $G_k = F_k \circ \mathcal{W}_{\tau_k}$ and decompose

$$\begin{aligned}\int (\Theta G_k)^* G_k \, d\mu_k = &\int (\Theta A_k)^* A_k \, d\mu_k \\ &+ \int (\Theta A_k)^*(G_k - A_k)\, d\mu_k \\ &+ \int (\Theta(G_k - A_k))^* G_k \, d\mu_k. \end{aligned} \tag{3}$$

The first term is ≥ 0 by Step 2. For the cross terms, Cauchy–Schwarz and the L^2-isometry of Θ give

$$\int (\Theta G_k)^* G_k \, d\mu_k \geq -\eta\big(2\|F\|_\infty + \eta\big), \tag{4}$$

where $\|F\|_\infty := \sup_k \|F_k\|_{L^\infty(\mu_k)}$ (finite by the boundedness assumption in $\overline{\mathfrak{A}}^t$). **Step 4 (Limit $k \to \infty$).** By [1, Theorem 1.1], $\omega_k^t((\Theta F)^* F) \to$

$\omega_L^t((\Theta F)^* F)$. The bound (4) is independent of k, so

$$\omega_L^t\big((\Theta F)^* \cdot F\big) \geq -\eta(2\|F\|_\infty + \eta) = -C(F,t,L)\, e^{-c\varepsilon_0^2/t},$$

which is (2). □

Remark 4.5 (Almost-RP vs. exact RP)**.** *The bound* (2) *gives* $\omega_L^t((\Theta F)^* F) \geq -\eta$ *with* η exponentially small *in* ε_0^2/t*, but not identically zero. This is an* almost-*reflection-positivity estimate.*

Exact RP (≥ 0) *for flowed observables would require either:*

(a) observables with compact temporal support, so that the buffer ε_0 *can be sent to* ∞ *and* $\eta \to 0$*; or*

(b) a transfer matrix argument showing that the lattice RP inner product is non-negative for gauge-invariant functions that depend on all *links (not only half-space links). Such an argument requires specifying the boundary Hilbert space with gauge constraints and the integration map* $G_k \mapsto \widehat{G}_k$*; we do not pursue this here.*

As explained in Remark 4.3, the Hamiltonian used in companion papers is reconstructed from exact lattice RP (Theorem 4.1), not from almost-RP on the flow algebra.

5 Osterwalder–Schrader Reconstruction

5.1 Hilbert space and Hamiltonian

Theorem 5.1 (OS reconstruction (conditional))**.** *Let* ω *be a state on a* $*$*-algebra* $\mathfrak{A}$ *equipped with a reflection* Θ *and a positive-time subalgebra* $\mathfrak{E}_+ \subset \mathfrak{A}$*. Assume reflection positivity on* $\mathfrak{E}_+$*:*

$$\omega\big((\Theta F)^* \cdot F\big) \geq 0 \qquad \forall\, F \in \mathfrak{E}_+.$$

Assume moreover that there exists a strongly continuous semigroup $\{\alpha_\tau\}_{\tau\geq 0}$ *of* $*$*-endomorphisms of* $\mathfrak{A}$ *such that:*

(i) $\alpha_\tau(\mathfrak{E}_+) \subset \mathfrak{E}_+$ *for all* $\tau \geq 0$*;*

(ii) $\omega \circ \alpha_\tau = \omega$ *for all* $\tau \geq 0$*;*

(iii) $\Theta \circ \alpha_\tau = \alpha_\tau \circ \Theta$ *for all* $\tau \geq 0$.

Then the standard Osterwalder–Schrader construction produces:

1. *A separable Hilbert space* $\mathcal{H}$*;*
2. *A unit vector* $\Omega_{\text{vac}} \in \mathcal{H}$ *(the vacuum);*
3. *A self-adjoint operator* $H \geq 0$ *with* $H\Omega_{\text{vac}} = 0$*;*
4. *A continuous unitary representation of the spatial symmetry group (if* ω *is spatially invariant).*

In particular, if exact RP is established for ω_L^t *on* $\overline{\mathfrak{A}}^t$ *(e.g. via a transfer matrix argument as discussed in Remark 4.5(b)), then the above yields* $\mathcal{H}$*,* Ω_{vac}*, and* $H \geq 0$ *for flowed Yang–Mills theory at each* $t > 0$*,* $L < \infty$*. Alternatively, applying this theorem directly to the lattice Wilson measure* μ_k *with the algebra* $\mathfrak{E}_{k,+}$ *of Theorem 4.1 yields the lattice Hilbert space and Hamiltonian* $H_k \geq 0$ *unconditionally.*

Proof. We apply the standard OS reconstruction [6, 7].

Step 1 (Positive form). Define $\mathfrak{E}_+ \subset \mathfrak{A}$ as the positive-time subalgebra. By the hypothesis of reflection positivity, the sesquilinear form

$$\langle F, G\rangle_{\text{OS}} := \omega\big((\Theta F)^* \cdot G\big)$$

is positive semidefinite on $\mathfrak{E}_+$.

Step 2 (Hilbert space). Let $\mathcal{N} = \{F \in \mathfrak{E}_+ : \langle F, F\rangle_{\text{OS}} = 0\}$ be the null space. Define $\mathcal{H}$ as the completion of $\mathfrak{E}_+/\mathcal{N}$ with respect to $\langle \cdot, \cdot \rangle_{\text{OS}}$.

Step 3 (Vacuum). The constant observable $\mathbf{1} \in \mathfrak{E}_+$ defines a vector $\Omega_{\text{vac}} := [\mathbf{1}] \in \mathcal{H}$ with $\|\Omega_{\text{vac}}\|^2 = \omega(\mathbf{1}) = 1$.

Step 4 (Contraction semigroup). For $\tau \geq 0$, define T_τ on $\mathfrak{E}_+/\mathcal{N}$ by $T_\tau[F] := [\alpha_\tau(F)]$. By hypothesis (i), $\alpha_\tau(F) \in \mathfrak{E}_+$, so $T_\tau[F]$ is well-defined. The OS inequality (Cauchy–Schwarz for $\langle \cdot, \cdot \rangle_{\text{OS}}$) gives $\|T_\tau[F]\| \leq \|[F]\|$, so T_τ extends to a contraction on $\mathcal{H}$.

Step 5 (Hamiltonian). The family $\{T_\tau\}_{\tau \geq 0}$ is a strongly continuous contraction semigroup on $\mathcal{H}$ satisfying $T_\tau \Omega_{\text{vac}} = \Omega_{\text{vac}}$. By the Hille–Yosida theorem, $T_\tau = e^{-\tau H}$ with $H \geq 0$ self-adjoint and $H\Omega_{\text{vac}} = 0$.

Step 6 (Spatial symmetries). By [1, Lemma 2.2], ω_L^t is invariant under these symmetries. Hence the spatial action is isometric with respect to $\langle \cdot, \cdot \rangle_{\text{OS}}$ and extends to unitary representations on $\mathcal{H}$ that leave Ω_{vac} fixed. □

Remark 5.2 (Why Theorem 5.1 uses an abstract semigroup)**.** *On the Euclidean torus $\mathbb{T}_L^4$ with the half-space $\{x_0 > 0\}$ defined using the fixed representative $x_0 \in (-L/2, L/2]$ (Remark 2.1), literal time translation by large τ can wrap around and need not preserve the half-space algebra. For the lattice Wilson theory, the correct positivity-preserving evolution is the transfer operator T_k of Theorem 4.1. The abstract semigroup hypothesis in Theorem 5.1 isolates exactly what is needed for OS reconstruction and avoids implicitly using a time-translation action that is not compatible with the chosen half-space on $\mathbb{T}_L^4$.*

5.2 Verification of OS axioms

Proposition 5.3 (OS axioms for ω_L^t)**.** *The state ω_L^t satisfies:*

OS0 *(Temperedness) In the present finite-volume setting on $\mathbb{T}_L^4$, the Schwinger functions are L-periodic distributions on $\mathbb{R}^4$ and hence tempered. Boundedness of flowed observables (guaranteed by continuity of the flow map on the compact configuration space $G^{\Lambda_k^1}$) ensures finiteness of all moments entering the Schwinger functions.*

OS1 *(Euclidean covariance) Inherited from [1], Lemma 2.2.*

OS2 *(Reflection positivity)* Not established unconditionally *for ω_L^t on the full positive-time algebra $\overline{\mathfrak{A}}^t$. Theorem 4.4 provides a quantitative almost-RP estimate (defect exponentially small in the buffer distance). Exact RP holds unconditionally at the lattice level for the Wilson measure (Theorem 4.1); see Remark 4.3.*

OS4 *(Symmetry) Gauge invariance of all observables in $\overline{\mathfrak{A}}^t$.*

Remark 5.4 (OS3 – Clustering)**.** *OS3 (uniqueness of the vacuum / cluster property) remains open and requires exponential decay of correlations (mass gap). This will be addressed in the companion paper [2] using a discrete Bakry–Émery approach.*

6 Discussion

A Variance–oscillation bound (no product assumption)

Lemma A.1 (Variance bound via single-edge oscillations)**.** *Let* μ *be any probability measure on* G^E *(not necessarily a product measure) and let* $S \subset E$*. Set* $\mathcal{G}_S = \sigma(U_e : e \in S)$*. Then for any bounded* $\Phi \in L^2(\mu)$*,*

$$\operatorname{Var}_\mu\big(\Phi \,\big|\, \mathcal{G}_S\big) \le \frac{1}{4} \sum_{e \in E \setminus S} \operatorname{osc}_e(\Phi)^2.$$

Remark A.2 (How Lemma A.1 is used in the main text)**.** *In Proposition 3.8 we apply Lemma A.1 with* $E = \Lambda_k^1$*,* $S = \{\ell \in \Lambda_k^1 : x_0(\ell) > 0\}$*, and* $\Phi = F_k \circ \mathcal{W}_{\tau_k}$*.*

Proof. Write $E \setminus S = \{e_1, \dots, e_n\}$ and define filtrations $\mathcal{F}_0 = \mathcal{G}_S$ and $\mathcal{F}_i = \sigma(\mathcal{G}_S, U_{e_1}, \dots, U_{e_i})$ for $i = 1, \dots, n$. Then $\mathcal{F}_n = \sigma(U_e : e \in E)$ and $\Phi - \mathbb{E}[\Phi \mid \mathcal{G}_S] = \sum_{i=1}^n \Delta_i$ where $\Delta_i = \mathbb{E}[\Phi \mid \mathcal{F}_i] - \mathbb{E}[\Phi \mid \mathcal{F}_{i-1}]$ is a martingale difference sequence. By orthogonality,

$$\|\Phi - \mathbb{E}[\Phi \mid \mathcal{G}_S]\|_2^2 = \sum_{i=1}^n \|\Delta_i\|_2^2.$$

Each Δ_i is the conditional expectation (given $\mathcal{F}_{i-1}$) of the difference obtained by varying U_{e_i} only, so $\|\Delta_i\|_{L^\infty} \le \frac{1}{2} \operatorname{osc}_{e_i}(\Phi)$ and hence $\|\Delta_i\|_2^2 \le \frac{1}{4} \operatorname{osc}_{e_i}(\Phi)^2$. Summing gives the result. Note: no independence or product structure on μ is needed; only the tower property of conditional expectation is used. □

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References

[1] L. Eriksson, *Ultraviolet closure for Wilson-flow observables in lattice Yang–Mills theory*, Preprint, 2026.

[2] L. Eriksson, *Spectral gap of the Yang–Mills Hamiltonian at finite volume via log-Sobolev inequalities*, Preprint, 2026.

[3] L. Eriksson, *Blocking renormalisation for lattice gauge observables*, Preprint, 2026.

[4] L. Eriksson, *Thermodynamic limit and volume-uniform mass gap for lattice Yang–Mills theory*, Preprint, 2026.

[5] L. Eriksson, *Yang–Mills existence and mass gap: a complete solution for* SU(N) *gauge theory in four dimensions*, Preprint, 2026.

[6] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions*, Comm. Math. Phys. **31** (1973) 83–112.

[7] K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions II*, Comm. Math. Phys. **42** (1975) 281–305.

[8] K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Ann. Phys. **110** (1978) 440–471.

[9] M. Lüscher, *Properties and uses of the Wilson flow in lattice QCD*, JHEP **1008** (2010) 071.

[10] M. Lüscher and P. Weisz, *Perturbative analysis of the gradient flow in non-abelian gauge theories*, JHEP **1102** (2011) 051.

[11] T. Bałaban, *Averaging operations for lattice gauge theories*, Comm. Math. Phys. **98** (1985) 17–51.

[12] D. Bakry and M. Émery, *Diffusions hypercontractives*, Sém. Prob. XIX, LNM **1123** (1985) 177–206.

[13] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.