Phase Quantization from Observable Regularity: Reducing the Wallström Objection to a Standard Physical Condition

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Abstract

Wallström (1989, 1994) showed that the Madelung hydrodynamic equations admit solutions with non-integer phase circulation, for which no single-valued wave function exists. Previous completions of the Madelung system postulate either single-valuedness or the quantization condition directly.

In this paper we consider the regularity of the probability current $\mathbf{j} = \rho \nabla S/m$ at nodal zeros within the Onsager-Machlup stochastic variational framework. We find that requiring $\mathbf{j} \in C^\infty$, combined with the Hamilton-Jacobi constraint at zeros of ρ, implies integer phase circulation. Neither condition alone has this consequence: smooth currents with arbitrary circulation exist when the dynamics is absent, and the Hamilton-Jacobi constraint alone admits the non-quantized solutions constructed by Reddiger and Poirier (2023). We also find that C^∞ is the only regularity class with this property: for any finite k, non-integer solutions satisfying C^k can be constructed.

When the framework is applied with initial data satisfying $\rho_0 > 0$, the phase is single-valued by simple connectivity, the Schrödinger equation follows from the variational principle, and any nodes formed under subsequent evolution carry integer winding numbers. The variational principle degenerates at zeros of ρ, leaving the winding parameter undetermined—a feature that holds for any variational functional of the form $\int \rho \, G \, d^n x$, not only the Onsager-Machlup action.

The non-quantized solutions correspond to multivalued sections of a non-trivial line bundle and do not arise within the natural domain of the stochastic framework.

Keywords: phase quantization, Wallström objection, Madelung equations, Onsager-Machlup variational principle, stochastic mechanics, probability current regularity
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1 Introduction

The derivation of quantum mechanics from more fundamental principles has been a persistent goal in theoretical physics. Among the approaches based on stochastic methods are those initiated by Fényes [2] and Nelson [8, 9], the stochastic variational principles of [11, 12, 10], and the exact uncertainty relations of [35]. These approaches derive the hydrodynamic equations of quantum mechanics—the continuity equation and a modified Hamilton-Jacobi equation containing a quantum potential—from stochastic or variational principles.

In 1994, Wallström [15, 16] identified a critical gap in all such derivations. The Madelung hydrodynamic equations, obtained by substituting $\psi = \sqrt{\rho}\, e^{iS/\hbar}$ into the Schrödinger equation [1], are *necessary* consequences of quantum mechanics. But they are not *sufficient*: the hydrodynamic equations admit solutions where the phase circulation

$$\oint_C \nabla S \cdot d\mathbf{l} = \alpha\hbar, \quad \alpha \notin \mathbb{Z} \tag{1}$$

takes non-integer values, and for such solutions no single-valued wave function ψ exists. The Madelung equations are therefore an *incomplete* formulation of quantum mechanics.

This objection is correct. The need for a quantization condition in the hydrodynamic picture was already recognized by Takabayasi [3, 4], who formulated the condition that the phase circulation must be an integer multiple of $2\pi\hbar$. The gap was made concrete in 2023 by Reddiger and Poirier [23], who explicitly constructed non-quantized strong solutions of the Madelung equations in two dimensions and showed that Takabayasi's condition holds for C^1 wave functions but may not extend to distributional solutions.

Previous attempts to close the gap fall into several categories. The first postulates single-valuedness of ψ—but this presupposes the existence of the wave function. The second postulates the quantization condition directly—but this is the quantum mechanical result itself. In either case, the derivation is circular. More recently, Schmelzer [19] proposed a regularity condition on the Laplacian of the density at nodal zeros, and Derakhshani [21, 22] proposed a zitterbewegung mechanism that introduces an additional physical hypothesis. Neither approach has been published in a peer-reviewed journal.

1.1 The present work

We present two independent results that address the Wallström objection from complementary directions.

Result 1 (Static characterization). Working with the Madelung equations directly, we identify a condition that completes them: the probability current $\mathbf{j} = \rho\,\nabla S/m$ must be a smooth (C^∞) vector field on all of physical space. This is a condition on a classical observable that does not mention ψ or $\hbar$ and does not by itself imply quantization. Combined with the Hamilton-Jacobi constraint derived from the Onsager-Machlup variational principle, it implies integer phase circulation. We find that C^∞ is the only regularity class with this property.

Result 2 (Dynamic derivation). The Wallström objection does not arise when the Onsager-Machlup framework is applied within its natural domain of validity. For initial data with $\rho_0 > 0$—the regime where the stochastic variational principle is non-degenerate—the Schrödinger equation follows as a theorem without any topological postulate. Any nodal zeros that form under subsequent evolution carry integer winding numbers by the topology of continuous complex-valued functions, and are protected from degradation by the divergent osmotic drift acting as a Bessel-type entrance boundary. The non-quantized Reddiger-Poirier solutions exist only in the Madelung variables (ρ, S) applied outside their domain of validity.

Result 3 (Structural explanation). The Onsager-Machlup action functional is algebraically identical to the Dirichlet energy of a complex field $z = \sqrt{\rho}\, e^{iS/\hbar}$. The non-quantized solutions correspond to multivalued sections of a non-trivial line bundle—they are not functions on Ω but a consequence of decomposing a naturally complex variational problem into amplitude and phase variables that degenerate at $\rho = 0$.

1.2 Why the regularity condition was not identified earlier

Three factors explain why the smoothness of $\mathbf{j}$ was not previously identified as the relevant condition. First, the objection was formulated in the language of ψ, and all responses remained in that language. Second, the non-quantized solutions were abstract until the explicit Reddiger-Poirier construction [23] made it possible to examine concretely what distinguishes them from the physical solutions in the language of observables. Third, the connection between current regularity and quantization requires the intermediate Hamilton-Jacobi constraint linking the vanishing order to the winding number—an argument with two steps, each innocuous in isolation.

1.3 Outline

Section 2 establishes the Onsager-Machlup framework and derives the hydrodynamic equations. Section 3 presents the Madelung transformation and states the Wallström objection. Section 4 contains Result 1: phase quantization from current regularity. Section 5 contains Result 2: the dynamic derivation. Section 6 contains Result 3: the structural explanation. Section 7 discusses the regularity condition in physical context. Section 8 compares with previous approaches. Section 9 concludes.

2 Mathematical Framework

2.1 The Onsager-Machlup Functional

Definition 1 (Admissible Paths)**.** Let M be a smooth n-dimensional Riemannian manifold. The space of admissible paths is

$$\mathcal{P} = \left\{ x \in H^1([0,T], M) : x(0) = x_0,\, x(T) = x_T \right\} \tag{2}$$

where H^1 denotes the Sobolev space of absolutely continuous paths with square-integrable derivatives.

Consider a diffusion process governed by the Itô SDE:

$$dX_t = f(X_t)\, dt + \sigma(X_t)\, dW_t \tag{3}$$

where $f : M \to TM$ is the drift, $\sigma : M \to \mathrm{End}(TM)$ is the diffusion coefficient, and W_t is standard Brownian motion.

Theorem 2 (Onsager-Machlup Action)**.** *The probability density functional for paths of Eq.* (3) *is*

$$P[x(\cdot)] = \mathcal{N} \exp(-S_{\mathrm{OM}}[x]) \tag{4}$$

with action

$$S_{\mathrm{OM}}[x] = \int_0^T L_{\mathrm{OM}}(x, \dot{x})\, dt \tag{5}$$

and Lagrangian density

$$L_{\mathrm{OM}} = \tfrac{1}{4}(\dot{x} - f)^T D^{-1}(\dot{x} - f) + \tfrac{1}{2}\nabla \cdot f \tag{6}$$

where $D = \frac{1}{2}\sigma\sigma^T$ *is the diffusion tensor.*

Proof. Standard discretization of the transition kernel and passage to the continuum limit. See Refs. [10, 34]. □

2.2 Hydrodynamic Equations

Theorem 3 (Coupled Hydrodynamic Equations)**.** *For a conservative system with scalar diffusivity* D *and velocity field* $v = \nabla S/m$*, the stationary point of the ensemble-averaged Onsager-Machlup action yields:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m}\right) = 0 \tag{7}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U(x) + Q(\rho) = 0 \tag{8}$$

where the quantum potential is

$$Q(\rho) = -2mD^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \tag{9}$$

Proof. The ensemble-averaged Onsager-Machlup action is

$$\bar{A}[\rho, S] = \int_0^T \int_{\mathbb{R}^n} \rho\, L_{\mathrm{OM}}\, d^n x\, dt.$$

For a conservative system with $v = \nabla S/m$ and scalar diffusivity D, the kinetic part of L_{OM} decomposes into current and osmotic contributions. The drift of the diffusion process (3) decomposes into a current velocity $v = \nabla S/m$ and an osmotic velocity $u = D\nabla \ln \rho$, where ρ is the probability density of the process. Substituting $f = v + u$ into the Onsager-Machlup Lagrangian (6) and averaging over the ensemble with measure $\rho\, d^n x$, the kinetic term yields, after integration over fluctuations, a contribution from the mean current velocity $|\nabla S|^2/(2m)$ and a contribution from the osmotic velocity $|u|^2 = D^2|\nabla \ln \rho|^2$. The latter can be rewritten as $4D^2|\nabla\sqrt{\rho}|^2/\rho$ using $\nabla \ln \rho = 2\nabla\sqrt{\rho}/\sqrt{\rho}$. See [12, 11, 9] for detailed derivations. The averaged action takes the form

$$\bar{A}[\rho, S] = \int_0^T \int_{\mathbb{R}^n} \rho \left[\partial_t S + \frac{|\nabla S|^2}{2m} + U + 2mD^2 \frac{|\nabla\sqrt{\rho}|^2}{\rho}\right] d^n x\, dt$$

plus boundary terms that vanish for ρ decaying at infinity.

Variation with respect to S*:* Under $S \to S + \epsilon\, \delta S$ with ρ fixed:

$$\delta_S \bar{A} = \int_0^T \int \rho \left[\partial_t(\delta S) + \frac{\nabla S \cdot \nabla(\delta S)}{m}\right] d^n x\, dt.$$

Integrating by parts in t (first term) and in x (second term):

$$\delta_S \bar{A} = -\int_0^T \int \left[\partial_t \rho + \nabla \cdot \left(\rho \frac{\nabla S}{m}\right)\right] \delta S\, d^n x\, dt.$$

Since δS is arbitrary, the Euler-Lagrange equation is the continuity equation (7).

Variation with respect to ρ: Under $\rho \to \rho + \epsilon\,\delta\rho$ with S fixed and subject to $\int \rho\, d^n x = 1$: the osmotic term satisfies

$$\frac{\delta}{\delta\rho}\int 2mD^2 |\nabla\sqrt{\rho}|^2\, d^n x = -2mD^2 \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

(using $|\nabla\sqrt{\rho}|^2 = |\nabla\rho|^2/(4\rho)$ and integrating by parts twice). Identifying $Q(\rho) = -2mD^2\,\nabla^2\sqrt{\rho}/\sqrt{\rho}$, the Euler-Lagrange equation is the Hamilton-Jacobi equation (8). □

Definition 4 (Quantum Calibration)**.** The diffusivity is fixed by $D = \hbar/(2m)$, yielding

$$Q(\rho) = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{10}$$

Remark 5. The Wallström objection is topological in character and independent of the value of D. The calibration $D = \hbar/(2m)$ determines the physical scale but not the quantization structure. All results in this paper hold for any $D > 0$; the specific value is required only for the identification with quantum mechanics.

3 The Madelung Transformation and the Wallström Objection

Theorem 6 (Formal Schrödinger Equation)**.** *If (ρ, S) satisfy Eqs. (7)–(8) with $D = \hbar/(2m)$, and if $\psi = \sqrt{\rho}\, e^{iS/\hbar}$ is single-valued, then*

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U(x)\psi \tag{11}$$

Proof. Define $\psi = \sqrt{\rho}\, e^{iS/\hbar}$. Computing $i\hbar\,\partial_t\psi$ using Eqs. (7)–(8), and $-(\hbar^2/2m)\nabla^2\psi + U\psi$ using the identity $\nabla^2\psi = [\nabla^2\sqrt{\rho} + (2i/\hbar)\nabla\sqrt{\rho}\cdot\nabla S + (i/\hbar)\sqrt{\rho}\,\nabla^2 S - \sqrt{\rho}\,|\nabla S|^2/\hbar^2]\, e^{iS/\hbar}$, the imaginary parts of both sides are identical (both reduce to the continuity equation). The real parts yield $Q(\rho) = -(\hbar^2/2m)\nabla^2\sqrt{\rho}/\sqrt{\rho}$, which is Eq. (10). See [1, 31]. □

Proposition 7 (Wallström [16])**.** *The system Eqs. (7)–(8) admits solutions where S is multivalued. For such solutions, no single-valued ψ exists.*

The Madelung equations are therefore incomplete. The question is: what is the nature of the additional ingredient needed to recover equivalence with the Schrödinger equation?

4 Result 1: Phase Quantization from Observable Regularity

We show that the smoothness of the probability current completes the Madelung equations. The argument proceeds in two stages: first for configurations without nodal zeros, then for configurations with isolated nodes.

4.1 Node-Free Configurations

When $\rho > 0$ throughout Ω, the phase S is well-defined on $\Omega^* = \Omega$. If Ω is simply connected, every closed 1-form is exact, so S is single-valued and all circulations vanish: $\oint_C \nabla S \cdot d\mathbf{l} = 0 \in 2\pi\hbar\mathbb{Z}$. No regularity condition is needed; the result is purely topological. The non-trivial case arises when ρ has zeros.

4.2 Phase Quantization at Nodal Zeros

Let $Z = \{x \in \Omega : \rho(x) = 0\}$ denote the nodal set and $\Omega^* = \Omega \setminus Z$ the punctured domain. On Ω^* the phase gradient ∇S is well-defined. The phase circulation $\Gamma_\gamma = \oint_\gamma \nabla S \cdot d\mathbf{l}$ around a closed loop $\gamma \subset \Omega^*$ is a homotopy invariant (since the 1-form ∇S is closed on Ω^*). The question is whether Γ_γ must be quantized.

Lemma 8 (Hamilton-Jacobi Constraint). *Let x_0 be an isolated zero of ρ in $n = 2$. In polar coordinates (r, θ) centered at x_0, suppose the leading asymptotic behavior is*

$$\rho \sim r^{2\beta} g(\theta), \quad S \sim \alpha\hbar\theta + S_{\text{reg}}(r, \theta) \tag{12}$$

where $\beta > 0$, $\alpha \in \mathbb{R}$, $g(\theta) > 0$, and S_{reg} is bounded as $r \to 0$. Then the Hamilton-Jacobi equation (8), which holds on Ω^*, requires

$$|\alpha| = \beta \tag{13}$$

as a condition of asymptotic consistency.

Proof. The Hamilton-Jacobi equation (8) holds on $\Omega^* = \{x : \rho(x) > 0\}$. Substituting the asymptotic forms into Eq. (8), the dominant singular contributions as $r \to 0$ are:

$$\frac{(\nabla S)^2}{2m} \sim \frac{\alpha^2\hbar^2}{2mr^2} \tag{14}$$

$$Q(\rho) = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{\hbar^2}{2mr^2}\left(\beta^2 + \frac{w''(\theta)}{w(\theta)}\right) \tag{15}$$

where $w = \sqrt{g}$ and primes denote derivatives with respect to θ. Both the kinetic and quantum-potential contributions are of order r^{-2}. The remaining terms $\partial_t S$ and $U(x)$ are bounded as $r \to 0$. The Hamilton-Jacobi equation on Ω^* therefore requires the coefficient of r^{-2} to vanish for all θ:

$$\alpha^2 = \beta^2 + \frac{w''(\theta)}{w(\theta)}. \tag{16}$$

Since the left-hand side is independent of θ, we require $w''/w = c$ for some constant $c \in \mathbb{R}$. The function $w(\theta) = \sqrt{g(\theta)}$ is positive, smooth, and 2π-periodic. The equation $w'' = cw$ with these constraints admits only the constant solution: for $c > 0$, the solutions are exponential and not periodic; for $c < 0$, the solutions are $w = A\cos(\sqrt{-c}\,\theta + \varphi)$, which is not everywhere positive unless $\sqrt{-c} = 0$. Therefore $c = 0$, w is constant, and $\alpha^2 = \beta^2$, giving $|\alpha| = \beta$.

The non-quantized solutions of Reddiger and Poirier [23] verify this relation explicitly, with $\alpha = \beta = 1/3$. □

Remark 9. This constraint does not force quantization. $\alpha = 1/3$, $\beta = 1/3$, $\rho \sim r^{2/3}$ satisfies Lemma 8 and the full Madelung equations. Nothing in the dynamics alone excludes non-integer winding.

Remark 10 (Role of the regular part of S)**.** The ansatz $S \sim \alpha\hbar\theta + S_{\text{reg}}(r,\theta)$ with S_{reg} bounded as $r \to 0$ implies $\nabla S_{\text{reg}} = O(1)$ and therefore $(\nabla S_{\text{reg}})^2 = O(1)$ as $r \to 0$. The cross term $2(\nabla(\alpha\hbar\theta)) \cdot (\nabla S_{\text{reg}}) \sim (2\alpha\hbar/r)\,\partial_\theta S_{\text{reg}}/r = O(r^{-1})$ is subleading with respect to the r^{-2} terms in the Hamilton-Jacobi equation. The asymptotic balance at order r^{-2} is therefore determined entirely by the singular parts of $(\nabla S)^2$ and $Q(\rho)$, and the regular part S_{reg} does not affect the conclusion $|\alpha| = \beta$.

Lemma 11 (Smoothness of Radial Powers)**.** *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = (x^2+y^2)^s$ belongs to $C^\infty(\mathbb{R}^2)$ if and only if $s \in \mathbb{N}_0 = \{0,1,2,\ldots\}$.*

Proof. ($\Leftarrow$) If $s = k \in \mathbb{N}_0$, then $(x^2+y^2)^k$ is a polynomial, hence C^∞.

($\Rightarrow$) Write $s = k + \varepsilon$ with $k \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. Restricting to $y = 0$: $h(x) = |x|^{2s} = |x|^{2k+2\varepsilon}$. The $(2k+1)$-th derivative behaves as $|x|^{2\varepsilon-1} \to \infty$ as $x \to 0$. Therefore $f \notin C^{2k+1}$ and in particular $f \notin C^\infty$. □

Proposition 12 (Integer Winding from Current Regularity)**.** *Let x_0 be an isolated zero of ρ in $n = 2$, with $\rho \sim r^{2|\alpha|}$ (by Lemma 8) and $S \sim \alpha\hbar\theta$. If $\mathbf{j} = \rho\nabla S/m \in C^\infty(\mathbb{R}^2)$ including at x_0, then $\alpha \in \mathbb{Z}$.*

Proof. In Cartesian coordinates, $\hat{e}_\theta = (-y/r, x/r)$. Near x_0:

$$\mathbf{j} \sim \frac{\alpha\hbar}{m}(x^2+y^2)^{|\alpha|-1}(-y,\, x) \tag{17}$$

The factor $(-y,x)$ is a smooth (linear) vector field. The regularity of $\mathbf{j}$ at the origin reduces to the smoothness of the scalar prefactor $(x^2+y^2)^{|\alpha|-1}$. By Lemma 11, this is C^∞ if and only if $|\alpha|-1 \in \mathbb{N}_0$, i.e., $|\alpha| \in \{1,2,3,\ldots\}$. The case $\alpha = 0$ gives trivial circulation. Therefore $\alpha \in \mathbb{Z}$. □

Theorem 13 (Phase Quantization from Current Regularity)**.** *Let (ρ, S) satisfy the Madelung equations* (7)–(8) *with $D = \hbar/(2m)$ on $\Omega \subset \mathbb{R}^n$, with $\rho \geq 0$ and isolated nodal zeros. If $\mathbf{j} = \rho\nabla S/m \in C^\infty(\Omega)$, then*

$$\oint_C \nabla S \cdot d\mathbf{l} = 2\pi n\hbar, \quad n \in \mathbb{Z} \tag{18}$$

for every closed loop C in $\Omega^* = \{x : \rho(x) > 0\}$.

Proof. For loops contractible within Ω^*: the 1-form ∇S is closed (by commutativity of mixed partials), so $\Gamma_C = 0 \in 2\pi\hbar\mathbb{Z}$ by Stokes' theorem.

For loops encircling isolated zeros of ρ: Lemma 8 gives $|\alpha_i| = \beta_i$ at each zero (from the dynamics). Proposition 12 gives $\alpha_i \in \mathbb{Z}$ (from current regularity). The total circulation decomposes as $\Gamma_C = \sum_i 2\pi\alpha_i\hbar \in 2\pi\hbar\mathbb{Z}$, where the sum runs over zeros enclosed by C. □

Example 14 (Phase quantization in the 2D harmonic oscillator)**.** Consider the two-dimensional isotropic harmonic oscillator with eigenstates $\psi_{n,l}(r,\theta) = R_{n,l}(r)\,e^{il\theta}$, where $n \in \mathbb{N}_0$, $l \in \mathbb{Z}$, and $R_{n,l}(r) \sim r^{|l|}$ as $r \to 0$. The Madelung variables are $\rho = R^2_{n,l}(r)$ and $S = l\hbar\theta$. Near $r = 0$:

$$\rho \sim r^{2|l|}, \qquad S = l\hbar\theta.$$

The probability current in polar coordinates is $\mathbf{j} = (\rho\,\nabla S)/m = (l\hbar/m)\,R^2_{n,l}(r)/r\ \hat{e}_\theta \sim (l\hbar/m)\,r^{2|l|-1}\,\hat{e}_\theta$. Converting to Cartesian coordinates using $r^{-1}\hat{e}_\theta = r^{-2}(-y,x)$:

$$\mathbf{j} \sim \frac{l\hbar}{m}\,(x^2+y^2)^{|l|-1}\,(-y,\, x).$$

Since $|l| \in \mathbb{N}$, the exponent $|l| - 1 \in \mathbb{N}_0$, so $(x^2 + y^2)^{|l|-1}$ is a polynomial and $\mathbf{j} \in C^\infty(\mathbb{R}^2)$.

For the Reddiger-Poirier solution with $\alpha = 1/3$: $\rho \sim r^{2/3}$, $S = (1/3)\hbar\theta$, giving $\mathbf{j} \sim (\hbar/3m)\,(x^2 + y^2)^{-2/3}\,(-y,\, x)$. The prefactor $(x^2 + y^2)^{-2/3}$ diverges as $r \to 0$: $\mathbf{j}$ is not continuous at the origin.

4.3 Extension to Three Dimensions

In $n = 3$, the zeros of smooth solutions of second-order elliptic equations generically form curves (codimension-2 submanifolds) rather than isolated points [28, 27]. For a smooth nodal line $L \subset \mathbb{R}^3$, the fundamental group of the complement satisfies $\pi_1(\mathbb{R}^3 \setminus L) \cong \mathbb{Z}$, generated by a meridional loop encircling L.

Near a regular point $p \in L$, there exist local coordinates $(r_\perp, \theta, s)$ where s parametrizes L and $(r_\perp, \theta)$ are polar coordinates in the transverse plane. In these coordinates, the Laplacian decomposes as $\nabla^2 = \nabla_\perp^2 + \partial_s^2 +$ curvature terms. Since ψ vanishes on L, its leading behavior in the transverse direction is $\psi \sim r_\perp^k e^{ik\theta} \cdot \chi(s)$ for some integer $k \geq 1$ and smooth function $\chi(s) \neq 0$. The curvature terms and the s-dependence contribute at subleading order in $r_\perp$. At leading order $r_\perp^{-2}$, the asymptotic balance reduces to the two-dimensional case: the density behaves as $\rho \sim r_\perp^{2|\alpha|}$ and the transverse current as $\mathbf{j}_\perp \sim r_\perp^{2|\alpha|-1}\hat{e}_\theta$.

Since $\mathbf{j} \in C^\infty(\mathbb{R}^3)$ implies smoothness in every transverse plane, the argument of Proposition 12 gives $\alpha \in \mathbb{Z}$ for each nodal line. The phase circulation around any loop linking L is therefore quantized.

4.4 General Nodal Sets

The results above apply to isolated zeros (in 2D) and smooth nodal lines (in 3D). For completeness, we address the structure of generic nodal sets.

Proposition 15 (Nodal set structure)**.** *Let $z : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ be a non-trivial C^∞ function satisfying a second-order elliptic equation with smooth coefficients. Let $Z = \{x : z(x) = 0\}$ be its zero set. Then:*

(i) Z is a smooth manifold of codimension 2 outside a singular subset Z_{sing} of codimension at least 3 (isolated points in $\mathbb{R}^3$).

(ii) On each connected component of $Z \setminus Z_{\text{sing}}$, the vanishing order $k \in \mathbb{N}$ is constant.

(iii) The phase circulation around any loop linking $Z \setminus Z_{\text{sing}}$ is quantized by the transverse argument of Proposition 12.

(iv) The singular points Z_{sing} do not affect quantization: any loop around Z can be deformed by homotopy to avoid Z_{sing}, and the circulation is a homotopy invariant.

In particular, if (ρ, S) satisfy the Madelung equations with $\mathbf{j} \in C^\infty(\Omega)$ and the phase circulation is quantized (Theorem 13), then $z = \sqrt{\rho}\, e^{iS/\hbar}$ is a well-defined C^∞ function whose zero set has this structure.

Proof. (i) $Z = \{\operatorname{Re} z = 0\} \cap \{\operatorname{Im} z = 0\}$ is the intersection of two real hypersurfaces. By the unique continuation principle for elliptic operators [26], z cannot vanish to infinite order, so Z has the structure of a real-analytic variety. In $\mathbb{R}^3$, such a variety is a smooth 1-manifold outside a discrete set of singular points.

(ii) The vanishing order k is the degree of the leading homogeneous harmonic polynomial in the Taylor expansion of z near a zero. This degree is a positive integer that is locally constant along $Z \setminus Z_{\text{sing}}$, since it can change only where the leading polynomial degenerates.

(iii) At each regular point of Z, the transverse plane analysis reduces to the 2-dimensional case with $|z|^2 \sim r_\perp^{2k}$, and Proposition 12 gives $k \in \mathbb{N}$.

(iv) Z_{sing} has codimension ≥ 3 in $\mathbb{R}^3$ (i.e., it consists of isolated points). Any loop in $\mathbb{R}^n \setminus Z$ can be continuously deformed to avoid any finite set of points without changing its homotopy class relative to Z. Since phase circulation is a homotopy invariant (the 1-form ∇S is closed on Ω^*), the quantization at regular points extends to all loops. □

4.5 Non-Circularity

Proposition 16. *If the vanishing order β and the winding number α are not constrained by the Hamilton-Jacobi equation, smooth currents exist for non-integer α.*

Proof. Take $\rho = r^{2k}$ with $k \in \mathbb{N}$ and $S = \alpha\hbar\theta$ with $\alpha \in \mathbb{R}$ arbitrary. Then $\mathbf{j} = (\alpha\hbar/m)(x^2+y^2)^{k-1}(-y, x)$, a polynomial vector field, hence C^∞, for any α. The regularity of $\mathbf{j}$ imposes no constraint on α when β is free. □

Proposition 17. *The Hamilton-Jacobi constraint alone does not imply quantization.*

Proof. Reddiger and Poirier [23] construct explicit strong solutions of the Madelung equations in two dimensions with $\alpha = \beta = 1/3$, satisfying Lemma 8 with $\alpha \notin \mathbb{Z}$. See [23], Theorem 3.1. □

Quantization arises only from the combination: the Hamilton-Jacobi equation binds $\beta = |\alpha|$ (dynamics); current regularity gives $|\alpha| \in \mathbb{N}$ (observables). Neither condition alone is equivalent to quantization, so the derivation is not circular.

4.6 C^∞ as the Unique Regularity Class

Proposition 18. *For any finite k, there exist non-integer α with $\mathbf{j} \in C^k$. Only C^∞ implies $\alpha \in \mathbb{Z}$.*

Proof. The regularity of $\mathbf{j}$ at a nodal zero with $|\alpha| = \beta$ reduces to the smoothness of $(x^2 + y^2)^{|\alpha|-1}$ (Proposition 12). For $\alpha = k/2 + 3/2 \notin \mathbb{Z}$, the exponent is $s = k/2 + 1/2$, and Lemma 11 gives $(x^2+y^2)^s \in C^k \setminus C^{k+1}$. For any finite k, a sufficiently large half-integer α passes the C^k test. Only C^∞—which requires $s \in \mathbb{N}_0$ for *all* k simultaneously—gives $|\alpha| \in \mathbb{N}$. □

Example 19 (Explicit finite-regularity counterexample)**.** We illustrate that arbitrarily high finite regularity does not imply quantization. Take $\alpha = 5/2$ and $\beta = |\alpha| = 5/2$, so $\rho \sim r^5$ and $S = (5/2)\hbar\theta$. The probability current is $\mathbf{j} \sim (5\hbar/2m)\,(x^2 + y^2)^{3/2}\,(-y, x)$. The regularity of the scalar prefactor $f(x, y) = (x^2 + y^2)^{3/2}$ is determined by restricting to $y = 0$: $h(x) = |x|^3$. Computing derivatives: $h'(x) = 3x|x|$ (continuous), $h''(x) = 6|x|$ (continuous), $h'''(x) = 6\,\text{sgn}(x)$ (discontinuous at $x = 0$). Therefore $f \in C^2(\mathbb{R}^2) \setminus C^3(\mathbb{R}^2)$ and $\mathbf{j} \in C^2 \setminus C^3$.

In general, for any $k \in \mathbb{N}$, taking $\alpha = (2k + 1)/2$ gives the prefactor $(x^2 + y^2)^{(2k-1)/2}$. Restricting to $y = 0$: $h(x) = |x|^{2k-1}$, whose $(2k - 1)$-th derivative is proportional to $\text{sgn}(x)$, discontinuous at $x = 0$. Thus $\mathbf{j} \in C^{2k-2} \setminus C^{2k-1}$, with non-integer α. Only $\alpha \in \mathbb{Z}$ gives $\mathbf{j} \in C^\infty$.

Remark 20. C^∞ is not an arbitrary choice. It is the only regularity class equivalent to phase quantization given the Hamilton-Jacobi constraint. No weaker condition suffices; no stronger condition is needed. In particular, membership in Sobolev spaces $W^{k,p}$ for any finite k does not force quantization, since the non-integer solutions that pass the C^k test *a fortiori* belong to $W^{k,p}_{\text{loc}}$.

Remark 21 (Physical interpretation of the regularity ladder)**.** The regularity classes C^0, C^1, C^k, C^∞ correspond to successively stronger physical conditions on the probability current at nodal zeros. At C^0 (continuity), the current does not diverge at nodes, but its derivative—needed for $\nabla \cdot \mathbf{j}$ in the continuity equation—may not exist classically; the Reddiger-Poirier solutions with $\alpha = 1/3$ fail even this condition. At C^1, the divergence $\nabla \cdot \mathbf{j}$ exists and the continuity equation holds pointwise at nodes, but half-integer values of α can still satisfy this (e.g., $\alpha = 3/2$). At C^k for any finite k, Example 19 shows that non-integer α remains possible. Only C^∞ produces quantization (Proposition 18). This is also the natural regularity class for solutions of elliptic equations with smooth coefficients, and therefore the class to which all eigenstates of Schrödinger operators with smooth potentials belong.

5 Result 2: Dynamic Phase Quantization

We now show that the Wallström objection does not arise when the Onsager-Machlup framework is applied within its natural domain. This provides a derivation of the Schrödinger equation with automatic phase quantization, requiring no postulate beyond the variational principle and the quantum calibration.

Result 2 is logically independent of Result 1. Result 1 provides a static characterization: given any solution of the Madelung equations (with or without nodal zeros), the condition $\mathbf{j} \in C^\infty$ is necessary and sufficient for phase quantization. Result 2 provides a dynamic derivation: starting from the natural domain $\rho_0 > 0$, quantization follows without any regularity postulate on $\mathbf{j}$. The two results address different aspects of the Wallström objection: Result 1 identifies what condition completes the Madelung equations; Result 2 shows that this condition is automatically satisfied within the stochastic variational framework.

5.1 The Natural Domain: $\rho_0 > 0$

The Onsager-Machlup variational principle is formulated over diffusion processes governed by the SDE (3). The drift decomposes as $b = v + u$ where $v = \nabla S/m$ is the current velocity and $u = D\nabla \ln \rho$ is the osmotic velocity. The osmotic term $\nabla \ln \rho$ diverges wherever $\rho = 0$, rendering the variational principle degenerate at nodal zeros.

The natural domain of the framework is therefore initial data with $\rho_0 > 0$ on Ω. This is not an exotic restriction: it is the regime where the stochastic process and its variational principle are well-defined. Gaussian states, coherent states, thermal states, and the ground state of any confining potential all satisfy $\rho_0 > 0$.

This restriction does not exclude physically relevant states. States with nodal structure—such as excited eigenstates—arise dynamically from regular initial configurations under Schrödinger evolution. They are not excluded from the theory; they are not admissible as initial data for the variational principle. The condition $\rho_0 > 0$ is a domain of applicability of the stochastic variational principle, not a restriction on the physical state space. This is analogous to the requirement of regular initial data in classical field theories: singular configurations may arise dynamically, but are not taken as initial conditions for the formulation of the variational principle. Moreover, states with $\rho > 0$ everywhere are dense in $L^2(\Omega)$: any quantum state can be approximated arbitrarily well by a node-free state. The domain $\rho_0 > 0$ is therefore variationally complete.

We emphasize that the restriction to $\rho_0 > 0$ is specific to Result 2 (the dynamic derivation). Result 1 applies to any solution of the Madelung equations—with or without nodal zeros in the initial data—and requires only that the probability current $\mathbf{j}$ be smooth on all of space. In particular, Result 1 covers excited eigenstates with nodal structure directly, without approximation by node-free configurations.

The following result shows that $\rho_0 > 0$ is not merely a convenient choice but a *necessary* condition for the variational principle to fully determine the dynamics.

Proposition 22 (Variational degeneracy at zeros of ρ)**.** *Let $A[\rho, S]$ be the Onsager-Machlup action and let (ρ, S) be a stationary point with $\rho(x_0) = 0$. Then:*

(i) The first variation $\delta_S A$ vanishes at x_0 for all δS, independently of $\nabla S(x_0)$. The phase S is not determined by the variational principle at zeros of ρ.

(ii) The first variation $\delta_\rho A$ at x_0 satisfies an inequality (Karush-Kuhn-Tucker condition for $\rho \geq 0$), not an equality. The Hamilton-Jacobi equation does not hold as an equality at x_0.

(iii) The winding parameter α of S around x_0 is a free parameter of the variational problem—it is not fixed by the stationarity of the action.

Proof. (i) The phase enters the action through $\int \rho\,(\partial_\mu S)(\partial^\mu S)\,d^n x$. Under $S \to S + \epsilon\,\delta S$, the first variation is $\delta_S A = -\int 2\,\partial_\mu(\rho\,\partial^\mu S)\,\delta S\,d^n x$ (after integration by parts). At x_0 with $\rho(x_0) = 0$, the original integrand $\rho\,(\partial_\mu S)(\partial^\mu \delta S)$ vanishes before integration by parts, regardless of δS. The Euler-Lagrange equation $\partial_\mu(\rho\,\partial^\mu S) = 0$ is derived only where δS is truly arbitrary in the sense that its contribution to the variation is nonzero—that is, in the interior of $\{\rho > 0\}$.

(ii) The variation $\delta_\rho A$ subject to $\rho \geq 0$ yields the Hamilton-Jacobi equation as an equality where $\rho > 0$ (the constraint is inactive). At points where $\rho = 0$, only $\delta\rho \geq 0$ is admissible, and the stationarity condition reduces to an inequality.

(iii) The parameter α enters through ∇S near x_0. By (i), the variational principle does not constrain ∇S at x_0. The asymptotic relation $|\alpha| = \beta$ (Lemma 8) follows from the Hamilton-Jacobi equation on $\{\rho > 0\}$ near x_0, but this only links α to β—it does not select integer values. Any $\alpha > 0$ is consistent with the stationarity of the action. □

Remark 23. This result clarifies the distinct roles of the two main results. When $\rho_0 > 0$, the variational principle determines S at every point, S is single-valued (by simple connectivity), and the Schrödinger equation follows without ambiguity (Result 2). When ρ has zeros, the variational principle leaves α undetermined, and an additional condition—the regularity of $\mathbf{j}$—is needed to select the quantized values (Result 1). The condition $\rho_0 > 0$ is therefore the necessary and sufficient condition for the Onsager-Machlup variational principle to determine the dynamics without supplementary input.

The degeneracy identified in Proposition 22 is not specific to the Onsager-Machlup functional. It is an intrinsic feature of the amplitude-phase parametrization, unavoidable for two independent reasons.

First, *probabilistically*: any variational principle over a statistical ensemble defines its action as an expectation value of a Lagrangian density G:

$$\mathcal{F}[\rho, S] = \mathbb{E}[G] = \int \rho(x)\, G(\rho, \nabla\rho, S, \nabla S)\, d^n x. \tag{19}$$

This is the definition of the ensemble average—not a choice of formulation.

Second, *algebraically*: the polar decomposition of the Dirichlet energy for a complex field $z = \sqrt{\rho}\, e^{iS/\hbar}$ gives $|\nabla z|^2 = |\nabla\sqrt{\rho}|^2 + \rho\, |\nabla S/\hbar|^2$ (Proposition 32). The phase gradient ∇S enters multiplied by ρ, not as an independent term. This is a consequence of the chain rule applied to the polar decomposition, not a modeling choice.

Both reasons lead to the same conclusion.

Proposition 24 (Structural degeneracy of amplitude-phase variational principles)**.** *Let $\mathcal{F}[\rho, S] = \int \rho\, G(\rho, \nabla\rho, S, \nabla S)\, d^n x$ be any variational functional with finite action formulated in amplitude-phase variables. Then $\delta_S \mathcal{F} = 0$ identically at every zero of ρ, for all smooth δS. The winding parameter α of S around any nodal zero is undetermined by the stationarity of $\mathcal{F}$. This holds for any G and is independent of the specific dynamics.*

Proof. The first variation with respect to S yields an integrand proportional to $\rho\, (\partial G/\partial(\nabla S)) \cdot \nabla(\delta S)$. At any point x^* where $\rho(x^*) = 0$, this vanishes before integration by parts, independently of δS. The stationarity condition is trivially satisfied at nodal zeros and imposes no constraint on ∇S. □

Remark 25. This structural degeneracy explains why all previous stochastic and hydrodynamic derivations of quantum mechanics—including Nelson [8], Yasue [11], Guerra and Morato [12], and Fritsche and Haugk [18]—encounter the Wallström objection. The obstruction is not a defect of any particular dynamics but is intrinsic to the amplitude-phase parametrization at nodal zeros.

Theorem 26 (Automatic Phase Quantization)**.** *Let (ρ_0, S_0) be initial data for the system Eqs.* (7)–(8) *with $D = \hbar/(2m)$, $\rho_0 > 0$ on $\Omega = \mathbb{R}^n$, and $U \in C^\infty$. Then:*

- *(i) S_0 is single-valued on Ω.*
- *(ii) $\psi_0 = \sqrt{\rho_0}\, e^{iS_0/\hbar}$ is a well-defined, single-valued function satisfying the Schrödinger equation* (11).
- *(iii) For all $t > 0$, if $\psi(\cdot, t)$ develops isolated zeros, these carry integer winding numbers.*
- *(iv) Phase circulation satisfies $\oint \nabla S \cdot d\mathbf{l} \in 2\pi\hbar\mathbb{Z}$ for all $t \geq 0$.*

The proof of (iii) requires two standard results from elliptic PDE theory, which we state for completeness.

Lemma 27 (Unique Continuation [26, 30])**.** *Let $\Omega \subset \mathbb{R}^n$ be connected and let $L = -\sum_{i,j} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i + c$ be a second-order elliptic operator with smooth coefficients. If $u \in H^2_{\text{loc}}(\Omega)$ satisfies $Lu = 0$ and u vanishes to infinite order at some $x_0 \in \Omega$, then $u \equiv 0$ on Ω.*

In our application, ψ satisfies $(-\hbar^2/(2m))\nabla^2\psi + (U - E)\psi = 0$, which is elliptic with smooth coefficients when $U \in C^\infty$. By the lemma, every zero of a non-trivial ψ has finite vanishing order.

Lemma 28 (Local structure of zeros)**.** *Let $\psi \in C^\infty(\Omega)$ satisfy $\nabla^2\psi + V(x)\psi = 0$ with $V \in C^\infty$, and let x_0 be a zero of finite vanishing order $k \geq 1$. Then $\psi(x) = P_k(x - x_0) + O(|x - x_0|^{k+1})$, where P_k is a non-trivial homogeneous harmonic polynomial of degree k: $\nabla^2 P_k = 0$.*

Proof. Substituting the Taylor expansion of ψ at x_0 into $\nabla^2\psi + V\psi = 0$ and collecting terms of degree $k - 2$: the leading contribution from $\nabla^2 P_k$ has degree $k - 2$, while $V\psi$ contributes at degree $\geq k$. Balancing gives $\nabla^2 P_k = 0$. In $\mathbb{R}^2$, every homogeneous harmonic polynomial of degree k takes the form $P_k = \mathrm{Re}(c\, z^k)$ (up to rotation), with winding number $\pm k \in \mathbb{Z}$. See Hardt and Simon [28] and Cheng [27] for the general theory. □

Proof of Theorem 26. (i) Since $\rho_0 > 0$, the nodal set $Z = \emptyset$ and $\Omega = \mathbb{R}^n$, which is simply connected: $\pi_1(\mathbb{R}^n) = 0$. The 1-form ∇S_0 is closed on Ω (since $\partial_i\partial_j S_0 = \partial_j\partial_i S_0$ for any C^2 function). On a simply connected domain, every closed 1-form is exact: there exists a single-valued function S_0 with the given gradient. Note that in the Onsager-Machlup variational framework, S enters as a functional variable defined on Ω—the variational principle is formulated over functions, not over sections of non-trivial bundles. The simple connectivity of Ω guarantees that no multivalued S can arise as a variational extremum.

(ii) With S_0 single-valued and $\rho_0 > 0$, the function $\psi_0 = \sqrt{\rho_0}\, e^{iS_0/\hbar}$ is well-defined and single-valued on Ω. By Theorem 6, ψ_0 satisfies the Schrödinger equation. This is a theorem of the Onsager-Machlup framework, not a postulate.

(iii) The time-dependent Schrödinger equation with $U \in C^\infty$ and smooth initial data preserves regularity: $\psi(\cdot, t) \in C^\infty(\mathbb{R}^n)$ for all $t > 0$ (see Reed and Simon [29], Theorem X.71). If $\psi(\cdot, t)$ develops a zero at x^*, the zero has finite vanishing order by Lemma 27, and its leading behavior is a homogeneous harmonic polynomial P_k of degree $k \geq 1$ by Lemma 28. Since ψ is continuous and single-valued, its winding number around x^* is $\pm k \in \mathbb{Z} \setminus \{0\}$. In $\mathbb{R}^3$, zeros generically form codimension-2 curves, and the same argument applies in the transverse plane.

(iv) Phase circulation at $t = 0$ is zero (by (i), S_0 is single-valued). Circulation is conserved between topological transitions: away from nodes, $\partial_t \oint \nabla S \cdot d\mathbf{l} = -\oint \nabla(U + Q) \cdot d\mathbf{l} = 0$ by Stokes' theorem. At the instant a node forms, the circulation changes by $2\pi k\hbar$ with $k \in \mathbb{Z}$ determined by the local structure of the zero (by (iii)). Circulation is thereafter conserved until the next topological transition. □

Remark 29 (No additional postulate)**.** Theorem 26 requires only the Onsager-Machlup action, the calibration $D = \hbar/(2m)$, and initial data in the natural domain $\rho_0 > 0$. No regularity condition on $\mathbf{j}$, no single-valuedness of ψ, and no quantization condition are postulated. All three emerge as theorems.

5.2 Protection by Osmotic Drift

Once a quantized node forms, it is protected from topological degradation by the stochastic dynamics.

Proposition 30 (Bessel Protection)**.** *Near a nodal zero with $\rho \sim r^{2n}$ ($n \in \mathbb{N}$), the radial component of the SDE reduces to a Bessel process of effective dimension $d > 2$. The origin is an entrance boundary: trajectories starting at $r > 0$ almost surely never reach $r = 0$.*

Proof. The osmotic drift $u = D\nabla \ln \rho$ contributes a radially repulsive term $\sim 2nD/r$. Including the Itô correction from d-dimensional polar coordinates (where d is the spatial dimension), the radial SDE is

$$dR_t = \frac{(d-1+2n)D}{R_t}\, dt + \sqrt{2D}\, dW_t \tag{20}$$

This is a Bessel process with effective topological dimension $d_{\text{eff}} = d + 2n$. For $d \geq 2$ and $n \geq 1$, $d_{\text{eff}} \geq 4 > 2$. By the Feller boundary classification, the origin is an entrance boundary for $d_{\text{eff}} \geq 2$, so the process is repelled from the origin with probability one. □

Remark 31. Bessel repulsion holds for any $\alpha > 0$, integer or not. But only integer winding numbers can arise from the formation mechanism in Theorem 26(iii). The osmotic drift preserves whatever winding exists; the dynamic derivation ensures that only integer winding is created.

At the level of individual stochastic trajectories, the Bessel repulsion means that a particle following the diffusion process (3) with drift $b = v + u$ almost surely never reaches a nodal zero. The winding number of S around a nodal line, measured along any stochastic trajectory, is therefore a dynamical invariant: it is fixed at the moment the node forms and cannot change thereafter. This provides a trajectory-level explanation for the conservation of circulation established in Theorem 26(iv) at the ensemble level.

5.3 The Wallström Objection Revisited

The Wallström objection asks: what prevents non-integer phase circulation in the Madelung equations? Within the natural domain of the Onsager-Machlup framework ($\rho_0 > 0$), the answer is: nothing prevents it because it never arises.

When $\rho > 0$ everywhere, the domain is simply connected, the phase is single-valued, and circulation is zero. The Schrödinger equation—derived, not postulated—governs subsequent evolution. Any nodes that form carry integer winding by continuity and single-valuedness of ψ. The Reddiger-Poirier solutions require $\rho = 0$ as an *initial condition*, which lies outside the domain where the stochastic variational principle is non-degenerate.

The objection is correct as a statement about the Madelung equations. It does not apply to the Onsager-Machlup framework operating in its natural regime.

6 Result 3: Structural Explanation

The Onsager-Machlup action has an algebraic structure that explains why the Madelung decomposition introduces non-physical solutions.

Proposition 32 (Dirichlet Energy Identity). *Define $z = \sqrt{\rho}\, e^{iS/\hbar}$. Then*

$$|\nabla z|^2 = |\nabla \sqrt{\rho}|^2 + \rho \left| \frac{\nabla S}{\hbar} \right|^2 \tag{21}$$

identically, with no cross terms and no boundary contributions.

Proof. Compute $\nabla z = (\nabla\sqrt{\rho} + i\sqrt{\rho}\,\nabla S/\hbar)\, e^{iS/\hbar}$ and $\overline{\nabla z} = (\nabla\sqrt{\rho} - i\sqrt{\rho}\,\nabla S/\hbar)\, e^{-iS/\hbar}$. Then:

$$\begin{aligned}|\nabla z|^2 &= \nabla z \cdot \overline{\nabla z}\\ &= |\nabla\sqrt{\rho}|^2 + \frac{i\sqrt{\rho}\,\nabla\sqrt{\rho}\cdot\nabla S}{\hbar} - \frac{i\sqrt{\rho}\,\nabla\sqrt{\rho}\cdot\nabla S}{\hbar} + \frac{\rho\,|\nabla S|^2}{\hbar^2}\\ &= |\nabla\sqrt{\rho}|^2 + \rho\left|\frac{\nabla S}{\hbar}\right|^2. \end{aligned} \tag{22}$$

The cross terms cancel exactly. The identity is pointwise and algebraic; no integration or approximation is involved. ☐

The kinetic part of the Onsager-Machlup action is therefore a Dirichlet energy $\int |\nabla z|^2\, d^n x$ for the complex field z. The Madelung decomposition $z \mapsto (\rho, S)$ splits this naturally complex variational problem into real amplitude and phase variables that degenerate at $\rho = 0$.

Remark 33 (Nature of the Reddiger-Poirier solutions)**.** For $\alpha = 1/3$, the corresponding $z = r^{1/3} e^{i\theta/3}$ is not a function on $\mathbb{R}^2$: upon traversing a closed loop around the origin, $z \to z \cdot e^{i2\pi/3} \neq z$. It is a section of a non-trivial complex line bundle, or equivalently, a function on a 3-fold covering space. The Madelung variables (ρ, S) do not detect this because only ∇S (which is single-valued) and ρ (which is single-valued) enter the equations. The decomposition strips away the global topological information carried by z.

Remark 34. One might argue that z must be a function (not a section), forcing quantization. However, this reasoning is circular: assuming the trivial bundle is equivalent to assuming single-valuedness of ψ. The non-circular arguments are provided by Result 1 (regularity of $\mathbf{j}$) and Result 2 (dynamic derivation from $\rho_0 > 0$). Result 3 explains *why* the Madelung equations admit non-physical solutions; Results 1 and 2 show *how* to exclude them without circularity.

Corollary 35 (Status of non-quantized solutions)**.** *The non-quantized solutions of Reddiger and Poirier [23] are excluded from the physical sector of the theory by three independent routes:*

- *(i)* Observable regularity *(Result 1): their probability current* $\mathbf{j} \sim r^{-1/3}\hat{e}_\theta$ *is not continuous at the nodal zero, violating* $\mathbf{j} \in C^\infty(\Omega)$*.*
- *(ii)* Dynamic inaccessibility *(Result 2): they cannot arise from Schrödinger evolution of any initial state with* $\rho_0 > 0$*, since such evolution produces only integer winding numbers at newly formed nodes.*
- *(iii)* Absence of wave function *(Result 3): they correspond to multivalued sections* $z = r^{1/3}e^{i\theta/3}$ *of a non-trivial line bundle; no single-valued wave function* ψ *exists for these configurations.*

7 The Regularity Condition in Physical Context

Result 1 identifies $\mathbf{j} \in C^\infty$ as the condition that completes the Madelung equations. Result 2 shows that this condition is automatically satisfied within the natural domain of the framework. For completeness, we discuss why the regularity condition is physically natural independently of Result 2.

7.1 Physical Meaning

The probability current $\mathbf{j} = \rho \nabla S/m$ determines all measurable flow properties. At nodal zeros ($\rho = 0$), where no particles are present, smoothness of $\mathbf{j}$ states that the flow field does not develop singularities where there is no matter to source them. The continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$ requires $\nabla \cdot \mathbf{j}$ to be well-defined throughout Ω. For the Reddiger-Poirier solution with $\alpha = 1/3$, the current diverges as $\mathbf{j} \sim r^{-1/3} \hat{e}_\theta$ at the nodal zero—it is not even continuous at $r = 0$, let alone smooth. More precisely, a divergent probability current at a point where $\rho = 0$ would imply an unbounded flux of probability through a region containing no particles. Since probability is conserved ($\int \rho \, d^n x = 1$ for all t), such a configuration would require the empty region to act as an infinite source or sink of probability flux—a condition incompatible with the conservation law that the continuity equation is meant to express.

7.2 Regularity Assumptions in Physics

Every physical theory assumes smoothness of observable fields in the absence of sources. In electrodynamics, $\mathbf{E}$ and $\mathbf{B}$ are smooth away from charges. In general relativity, $g_{\mu\nu}$ is smooth away from matter. In fluid mechanics, the velocity field is assumed smooth (whether solutions remain smooth is the Clay Millennium problem, but the equations are *formulated* under smoothness). In each case, singularities signal sources, not source-free physics. Nodal zeros are not sources.

Beyond the analogy with other physical theories, the smoothness of $\mathbf{j}$ has a direct functional role. The continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$ requires $\nabla \cdot \mathbf{j}$ to be a well-defined function throughout Ω for the equation to hold as a classical partial differential equation. If $\mathbf{j}$ is not at least C^1, the divergence $\nabla \cdot \mathbf{j}$ is not classically defined, and the continuity equation cannot be interpreted pointwise at nodal zeros. One could pass to a distributional formulation, but as Reddiger and Poirier [23] showed, the distributional formulation introduces additional complications (their concept of "quantum quasi-irrotationality"). The condition $\mathbf{j} \in C^\infty$ ensures that all equations of the Madelung system have a classical (pointwise) interpretation on all of Ω, including at nodal zeros.

Remark 36 (Nodal singularities vs. fluid shocks)**.** One might object that classical fluids develop singularities (shocks, turbulence), so why should the probability fluid be C^∞? The analogy is misleading. Shocks in Euler or Navier-Stokes equations are singularities of the *velocity* field at points where the density *increases* (compression). The singularities of the non-quantized solutions are of a fundamentally different type: the current $\mathbf{j}$ diverges where the density *vanishes*. No classical fluid develops infinite mass flux in a vacuum. Furthermore, Proposition 18 shows that C^∞ is not an arbitrary strengthening of a weaker condition: it is the only regularity class that produces quantization. Any finite C^k admits non-integer solutions. The condition is dictated by the mathematical structure of the problem, not imported from quantum mechanics. Finally, we note that even if one does not accept the C^∞ condition of Result 1, Result 2 derives quantization independently, without any regularity postulate on $\mathbf{j}$.

7.3 Singular Potentials

Result 1 as stated requires $U \in C^\infty$ to guarantee $\mathbf{j} \in C^\infty$ globally. We now show that the Hamilton-Jacobi constraint extends to physically relevant singular potentials.

Proposition 37 (HJ constraint for singular potentials)**.** *Let x_0 be an isolated zero of ρ that coincides with a singularity of the potential satisfying $|U(x)| \leq C|x - x_0|^{-\gamma}$ with $0 < \gamma < 2$. Then the Hamilton-Jacobi constraint $|\alpha| = \beta$ (Lemma 8) continues to hold.*

Proof. The asymptotic analysis of Lemma 8 carries through with one modification. The dominant singular terms as $r \to 0$ are the quantum potential $Q \sim -\hbar^2(\beta^2 + w''/w)/(2mr^2)$ (including the angular contribution, as shown in Lemma 8) and the kinetic term $(\nabla S)^2/(2m) \sim \alpha^2\hbar^2/(2mr^2)$, both of order r^{-2}. The singular potential contributes $U \sim r^{-\gamma}$ with $\gamma < 2$. Since $r^{-\gamma} = o(r^{-2})$ as $r \to 0$, the potential is subleading in the singular balance. The argument that $w''/w = 0$ (from periodicity and positivity of w) is unaffected by the presence of a subleading potential term. The cancellation of the r^{-2} coefficient still requires $\alpha^2 = \beta^2$. □

Remark 38. The condition $\gamma < 2$ covers all physically relevant potentials: Coulomb ($\gamma = 1$), Yukawa ($\gamma = 1$), and any potential less singular than r^{-2}. The inverse-square potential $\gamma = 2$ is the borderline case, which already requires special treatment in standard quantum mechanics (fall to the center).

For nodes that do not coincide with the potential singularity, Result 1 applies without modification, since U is smooth at such points. This covers the vast majority of physical configurations: for example, the radial nodes of hydrogen eigenstates lie at finite $r > 0$, where the Coulomb potential is smooth.

For nodes that do coincide with a potential singularity—such as the $l \geq 1$ states of hydrogen, where $\rho \sim r^{2l}$ at $r = 0$—the current regularity can be verified directly. With $\rho \sim r^{2l}$ and $S \sim m_l\hbar\phi$ (azimuthal phase), the current in Cartesian coordinates becomes $\mathbf{j} \sim (x^2 + y^2)^{l-1}(-y, x, 0) \cdot m_l\hbar/m$, which is a polynomial vector field and therefore C^∞, despite the Coulomb singularity at the origin. The vanishing of ρ absorbs both the singularity of ∇S and that of U.

More generally, the two results are complementary: Result 1 covers smooth potentials (and nodes away from singularities) with a purely static condition; Result 2 covers singular potentials through the dynamics, since the Schrödinger equation derived for $\rho_0 > 0$ produces integer winding at all nodes regardless of the local regularity at the potential singularity.

7.4 Consequences of Relaxing Regularity

Without $\mathbf{j} \in C^\infty$, the Madelung equations admit the non-quantized solutions. Relaxing regularity does not yield a more general theory; it yields one where the probability current is not a well-defined classical field at nodal zeros, the continuity equation has no classical meaning at those points, and no single-valued wave function exists.

8 Comparison with Previous Approaches

8.1 Historical Context

The hydrodynamic formulation of quantum mechanics originates with Madelung [1], who decomposed the Schrödinger equation into continuity and Hamilton-Jacobi equations. Takabayasi [3, 4] developed this formulation extensively and recognized the need for a quantization condition on the phase circulation—the condition now bearing his name. In a later work, Takabayasi [5] established the global condition that quantizes circulation around singular vortex lines in quantum fluids, connecting the phase quantization problem to vortex dynamics. Bohm [6, 7] independently developed a trajectory interpretation based on the same decomposition, starting from the Schrödinger equation; see Holland [31] and Dürr and Teufel [32] for comprehensive treatments. In Bohm's formulation, the velocity field $v = \nabla S/m$ is singular at nodal zeros, but the probability current $\mathbf{j} = \rho v$ is well-defined because ρ vanishes at those points. The regularity of $\mathbf{j}$ is guaranteed by the existence of a C^∞ wave function ψ, from which $\mathbf{j} = (\hbar/m)\,\mathrm{Im}(\bar{\psi}\nabla\psi)$ is manifestly smooth. The present work proceeds in the opposite direction: starting from the Madelung variables (ρ, S) without assuming the existence of ψ, we show that the regularity of $\mathbf{j}$ implies the conditions under which a single-valued ψ can be constructed. Wyatt [33] developed computational methods for quantum dynamics in the hydrodynamic picture.

8.2 Nelson's Stochastic Mechanics and Variants

Fényes [2] proposed the first probabilistic foundation for quantum mechanics. Nelson [8, 9] developed stochastic mechanics by postulating forward and backward diffusion processes and a stochastic Newton's law. Carlen [13] established the mathematical foundations of conservative diffusions, including existence results for the singular drift fields that arise near nodal zeros—a technical point directly relevant to the present work. Yasue [11] and Guerra and Morato [12] developed stochastic variational principles that yield the Schrödinger equation. Blanchard, Golin and Serva [14] studied repeated measurements in stochastic mechanics and the role of effective collapse. Bacciagaluppi [17] characterized the most general diffusion processes preserving the quantum equilibrium distribution. Fritsche and Haugk [18] proposed a stochastic derivation that assumes single-valuedness as part of the formulation. All of these approaches either assume single-valuedness of ψ or do not address the Wallström objection.

8.3 Direct Attempts to Resolve the Wallström Objection

Three approaches in the literature address the Wallström objection directly.

Schmelzer [19] proposed that the Laplacian of the density $\Delta\rho$ must be finite and positive at zeros of ρ, and showed that this implies the quantization condition. The justification for this postulate is based on a "principle of minimal distortion" from a conjectured subquantum theory, which Schmelzer acknowledges is "speculative in character." Our approach differs in that the condition $\mathbf{j} \in C^\infty$ is placed on an observable quantity (the probability current) rather than on ρ itself, and does not appeal to subquantum physics. The two conditions are not equivalent. Consider the case $\beta = \alpha = 3/2$ (non-integer, satisfying the Hamilton-Jacobi constraint $|\alpha| = \beta$). The density $\rho \sim r^3$ has $\Delta\rho \sim 9r \to 0$ at the origin, so the Laplacian is finite and Schmelzer's condition would be satisfied. However, the probability current $\mathbf{j} \sim (x^2+y^2)^{1/2}(-y,x)$ is not C^∞: the prefactor $(x^2+y^2)^{1/2}$ is not even C^1 (by Lemma 11 with $s = 1/2 \notin \mathbb{N}_0$). More generally, any half-integer $\beta = \alpha \geq 1$ gives finite $\Delta\rho$ but non-smooth $\mathbf{j}$. Our condition on the current detects non-integer winding even when the density's Laplacian is regular at the node. In a later work, Schmelzer [20] proposed accepting the Wallström objection and restricting to $|\psi|^2 > 0$ as a physical condition. Our Result 2 arrives at a similar restriction ($\rho_0 > 0$) but derives it as the natural domain of the variational principle rather than postulating it.

Derakhshani [21, 22] proposed a modification of Nelson-Yasue stochastic mechanics in which the particle undergoes a zitterbewegung oscillation at the Compton frequency. The quantization condition then follows from the periodicity of the oscillation. This approach introduces an explicit physical hypothesis (the zbw oscillation) that goes beyond standard stochastic mechanics. Our approach does not introduce new physics: it uses the regularity of an existing observable.

Reddiger and Poirier [23] did not propose a resolution but provided a thorough mathematical analysis of the problem. They proved that Takabayasi's condition holds for C^1 wave functions, constructed the explicit non-quantized strong solutions in two dimensions, and introduced the concept of quantum quasi-irrotationality. The present work builds directly on their analysis.

Remark 39. The relationship between Takabayasi's condition (as proven by Reddiger and Poirier for C^1 wave functions) and our Result 1 operates at different levels of regularity. If $\psi \in C^1$, then $\mathbf{j} = (\hbar/m)\,\mathrm{Im}(\bar{\psi}\nabla\psi)$ is C^0 but not necessarily C^∞. Conversely, our condition $\mathbf{j} \in C^\infty$ does not presuppose the existence of ψ and is formulated entirely in the Madelung variables (ρ, S). The two conditions are logically independent: Reddiger-Poirier's result derives the quantization condition from the existence of a C^1 wave function; our Result 1 derives it from the regularity of an observable without assuming a wave function exists.

8.4 Geometric and Topological Approaches

Reddiger [24] proposed the Madelung picture as a foundation for geometric quantum theory. Foskett and Tronci [25] developed a geometric approach to quantum hydrodynamics based on gauge connections, endowing the Madelung fluid with intrinsic holonomy and allowing for vortex filament solutions. Their framework provides a natural topological setting in which the phase quantization problem can be formulated.

8.5 Other Approaches

Hall and Reginatto [35] derived the Schrödinger equation from an exact uncertainty principle. Santamato [36] obtained it from classical mechanics in a curved Weyl space. Both approaches arrive at the Madelung equations and are therefore subject to the Wallström objection. Hall, Deckert and Wiseman [37] proposed a many-interacting-worlds approach that recovers quantum phenomena from interactions between classical worlds; this program does not use the Madelung decomposition and is not directly subject to the objection.

8.6 Summary

The approaches reviewed above differ in the nature of the condition used to address the Wallström objection. Nelson, Yasue, Guerra-Morato, and Fritsche-Haugk assume the existence of a single-valued wave function, which presupposes the result. Schmelzer imposes a regularity condition on the density ρ at nodal zeros, motivated by a conjectured subquantum theory. Derakhshani introduces a zitterbewegung hypothesis that modifies the stochastic dynamics. The present work imposes a regularity condition on the probability current $\mathbf{j}$ (Result 1), which is an observable of the existing framework, or restricts to the natural domain $\rho_0 > 0$ of the variational principle (Result 2), which is a property of the mathematical formulation rather than a physical postulate.

9 Conclusion

The Wallström objection correctly identified the Madelung equations as an incomplete formulation of quantum mechanics. We have addressed this incompleteness through two independent routes.

The **static characterization** (Result 1) shows that if the probability current $\mathbf{j} = \rho\nabla S/m$ is smooth throughout space, then the Hamilton-Jacobi dynamics implies integer phase circulation. The derivation is not circular: neither condition alone has this consequence, and C^∞ is the only regularity class with this property. The result extends to three dimensions via the transverse plane argument, to general nodal sets via the structure theorem for elliptic zero sets [28, 27], and to singular potentials with $|U| \leq Cr^{-\gamma}$ ($\gamma < 2$) since the potential is subleading in the singular balance.

The **dynamic derivation** (Result 2) shows that the objection does not arise within the natural domain of the Onsager-Machlup framework. The condition $\rho_0 > 0$ is not merely convenient but necessary: at zeros of ρ, the variational principle degenerates and leaves the winding parameter α undetermined (Proposition 22). For $\rho_0 > 0$, the phase is automatically single-valued, the Schrödinger equation is a theorem, and any nodes that form carry integer winding by continuity of ψ. The non-quantized solutions arise only when the Madelung decomposition is applied in the degenerate regime (Corollary 35).

The two results are complementary. Result 1 answers: *what completes the Madelung equations?* Result 2 answers: *why does the incompleteness not infect the underlying physics?* For smooth potentials, either result suffices. For singular potentials, Result 2 provides the quantization guarantee.

The logical structure of the complete derivation is:

$$\underbrace{\text{OM action}}_{\text{dynamics}}, \quad \underbrace{D = \tfrac{\hbar}{2m}}_{\text{calibration}}, \quad \underbrace{\rho_0 > 0}_{\text{domain}} \implies i\hbar\,\partial_t\psi = \hat{H}\psi \tag{23}$$

with phase quantization at all times. The Wallström objection asked what is missing from the Madelung equations. The answer is: a standard physical condition on observables (Result 1), which is automatically satisfied when the stochastic framework operates in its natural regime (Result 2).

Several extensions remain open. In the many-body case, the configuration space is $\mathbb{R}^{3N}$ and the nodal set of the wave function is a submanifold of codimension 2 in $\mathbb{R}^{3N}$. The topological structure of the complement is richer than in the one-body case, and the fundamental group may be non-abelian for identical particles (cf. the connection with the braid group and anyonic statistics in $d = 2$). The condition $\mathbf{j} \in C^\infty$ on the $3N$-dimensional probability current remains well-defined, but the analysis of the angular structure near higher-codimension nodal sets requires further work. The extension to relativistic field theory raises additional questions about the definition of the Onsager-Machlup functional on infinite-dimensional configuration spaces. Finally, the formal connection between the topology of individual stochastic trajectories and the ensemble-level phase quantization condition remains an open problem; tools from large deviations theory and Malliavin calculus may be relevant (cf. Carlen [13]).

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Conflict of Interest

The author declares no competing interests.

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