Phase Quantization from Observable Regularity: Reducing the Wallström Objection to a Standard Physical Condition

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Abstract

Wallström showed that the Madelung hydrodynamic equations are an incomplete formulation of quantum mechanics: they admit solutions with non-integer phase circulation for which no single-valued wave function exists. Previous attempts to complete the formulation postulate either single-valuedness of the wave function or the quantization condition directly, supplying a quantum ingredient and rendering the derivation circular.

Working within the Onsager-Machlup stochastic variational framework, we show that the incompleteness is remedied without postulating quantization. The probability current $\mathbf{j} = \rho\nabla S/m$ is a physical observable, and as such must be a smooth field throughout physical space—including at nodal zeros, which are not sources and where probability is conserved. This is the same class of condition that every physical theory imposes on its observable fields in the absence of sources. We prove that this smoothness, combined with the Hamilton-Jacobi constraint that the variational principle itself imposes at nodal zeros, forces integer phase circulation, excluding the explicit non-quantized solutions constructed by Reddiger and Poirier. Quantization is not postulated: it emerges from the combination of the dynamics with a standard property of observables.

Three results confirm that the condition is not equivalent to postulating quantization. First, smoothness of $\mathbf{j}$ decoupled from the dynamics is compatible with arbitrary circulation. Second, the Hamilton-Jacobi dynamics alone admits the non-quantized Reddiger-Poirier solutions. Third, C^∞ is the unique regularity class that produces exact quantization: for any finite differentiability C^k, there exist non-integer solutions satisfying it, so the condition cannot be weakened.

With phase circulation quantized, the Madelung transformation recovers the Schrödinger equation without postulating the existence or single-valuedness of the wave function. The Wallström objection correctly identified an incompleteness; we show that what was missing was not a quantum postulate but a condition on observables that was always implicit in the physical structure of the problem.

1 Introduction

The derivation of quantum mechanics from more fundamental principles has been a persistent goal in theoretical physics. Among the most promising approaches are those based on stochastic mechanics, initiated by Nelson [1, 2], and on the Onsager-Machlup variational principle [3]. These approaches derive the hydrodynamic equations of quantum mechanics—the continuity equation and a modified Hamilton-Jacobi equation containing a quantum potential—from stochastic or variational principles.

In 1994, Wallström [4] identified a critical gap in all such derivations. The Madelung hydrodynamic equations, obtained by substituting $\psi = \sqrt{\rho}\, e^{iS/\hbar}$ into the Schrödinger equation, are *necessary* consequences of quantum mechanics. But they are not *sufficient*: the hydrodynamic equations admit solutions where the phase circulation

$$\oint_C \nabla S \cdot d\mathbf{l} = \alpha\hbar, \quad \alpha \notin \mathbb{Z} \tag{1}$$

takes non-integer values, and for such solutions no single-valued wave function ψ exists. The Madelung equations are therefore an *incomplete* formulation of quantum mechanics. Any derivation that arrives only at these equations has not derived quantum mechanics.

This objection is correct. It has not been resolved in the three decades since it was posed. The gap was made concrete in 2023 by Reddiger and Poirier [12], who explicitly constructed non-quantized strong solutions of the Madelung equations in two dimensions, demonstrating that the problem is not merely formal but produces actual pathological solutions.

Previous attempts to close the gap fall into two categories. The first postulates single-valuedness of ψ—but this presupposes the existence of the wave function, which is what the derivation aims to establish. The second postulates the quantization condition $\oint \nabla S\cdot d\mathbf{l} \in 2\pi\hbar\mathbb{Z}$ directly—but this is the quantum mechanical result itself, making the derivation circular. In either case, the additional ingredient is quantum mechanical in character, and the derivation reduces to assuming what it intends to derive.

1.1 The present work

We show that the ingredient missing from the Madelung equations is not quantum mechanical. It is a condition on the probability current $\mathbf{j} = \rho\nabla S/m$

that belongs to the same class of regularity assumptions that every physical theory makes about its observable fields.

The argument has the following structure. Working within the Onsager-Machlup stochastic variational framework, we derive the coupled hydrodynamic equations for the density ρ and phase S. The Hamilton-Jacobi equation—one of these two equations—imposes a constraint at nodal zeros that links the vanishing order of ρ to the local winding number of S. This constraint is not an additional assumption: it is a consequence of the variational principle.

The single ingredient we identify as missing is: the probability current $\mathbf{j} = \rho \nabla S/m$ must be a smooth (C^∞) vector field on all of physical space, including at nodal zeros. This is a condition on a classical observable. It does not mention ψ, it does not mention $\hbar$ (except through the dynamical equations), and by itself it does not imply quantization.

We prove that the combination of this regularity condition with the Hamilton-Jacobi constraint forces integer phase circulation:

$$\oint_C \nabla S \cdot d\mathbf{l} = 2\pi n\hbar, \quad n \in \mathbb{Z} \tag{2}$$

The Madelung transformation then yields the Schrödinger equation without postulating the existence or single-valuedness of ψ.

To establish that the derivation is not circular, we prove three independent results:

(i) The regularity condition alone, decoupled from the Hamilton-Jacobi dynamics, is compatible with arbitrary (non-integer) winding numbers. Therefore it is not equivalent to postulating quantization.

(ii) The Hamilton-Jacobi dynamics alone admits non-quantized solutions—these are precisely the Reddiger-Poirier solutions. Therefore the dynamics alone does not imply quantization.

(iii) C^∞ is the unique regularity class that produces quantization: for any finite k, there exist non-integer winding numbers whose currents are C^k. No finite differentiability suffices.

1.2 Why this condition was not identified earlier

Three factors explain why the regularity of $\mathbf{j}$ was not previously recognized as the missing ingredient.

First, the Wallström objection was formulated in the language of the wave function ψ, and all responses remained in that language. The natural question became "why is ψ single-valued?" and the natural answers were topological or axiomatic. Reformulating the question as "what happens to $\mathbf{j}$ at nodal zeros?" requires treating the hydrodynamic formulation as fundamental rather than derived—a perspective that the objection itself seemed to discourage.

Second, the non-quantized solutions were abstract until 2023. Their existence was known, but no one had constructed them explicitly. The Reddiger-Poirier construction [12] made it possible to examine *concretely* what is pathological about these solutions in the language of observables: their probability currents are singular at nodal zeros. This observation opens the door to our argument.

Third, the connection between current regularity and quantization requires the intermediate step of the Hamilton-Jacobi constraint linking the vanishing order β to the winding number $|\alpha|$. Without this link, regularity of $\mathbf{j}$ says nothing about quantization. The argument has two steps, each of which appears innocuous in isolation, and their combination is not obvious.

1.3 Outline

Section 2 establishes the Onsager-Machlup action functional. Section 3 derives the hydrodynamic equations. Section 4 presents the Madelung transformation and states the Wallström objection precisely. Section 5 contains the main results: phase quantization from current regularity, including the proof that C^∞ is the unique regularity class. Section 6 discusses the regularity condition in the context of physical theory. Section 7 compares with previous approaches. Section 8 concludes.

2 Mathematical Framework

2.1 The Onsager-Machlup Functional

Definition 1 (Admissible Paths)**.** Let M be a smooth n-dimensional Riemannian manifold. The space of admissible paths is

$$\mathcal{P} = \left\{ x \in H^1([0,T], M) : x(0) = x_0,\, x(T) = x_T \right\} \tag{3}$$

where H^1 denotes the Sobolev space of absolutely continuous paths with square-integrable derivatives.

Consider a diffusion process governed by the Itô SDE:

$$dX_t = f(X_t)\, dt + \sigma(X_t)\, dW_t \tag{4}$$

where $f : M \to TM$ is the drift, $\sigma : M \to \mathrm{End}(TM)$ is the diffusion coefficient, and W_t is standard Brownian motion on $\mathbb{R}^m$.

Theorem 2 (Onsager-Machlup Action)**.** *The probability density functional for paths of Eq.* (4) *is given by*

$$P[x(\cdot)] = \mathcal{N} \exp(-S_{\mathrm{OM}}[x]) \tag{5}$$

where $\mathcal{N}$ is a normalization constant and

$$S_{\mathrm{OM}}[x] = \int_0^T L_{\mathrm{OM}}(x, \dot{x})\, dt \tag{6}$$

with Lagrangian density

$$L_{\mathrm{OM}}(x, \dot{x}) = \frac{1}{4}(\dot{x} - f)^T D^{-1}(\dot{x} - f) + \frac{1}{2}\nabla \cdot f \tag{7}$$

where $D = \frac{1}{2}\sigma\sigma^T$ is the diffusion tensor.

Proof. The transition probability for a small time step Δt is given by the Gaussian kernel

$$p(x_{k+1}|x_k) = \frac{1}{(4\pi\Delta t)^{n/2}\sqrt{\det D}} \times \exp\left(-\frac{|x_{k+1} - x_k - f\Delta t|^2_{D^{-1}}}{4\Delta t}\right) \quad (8)$$

where $|\cdot|^2_{D^{-1}} = (\cdot)^T D^{-1}(\cdot)$. For a trajectory discretized into N steps,

$$P[x(\cdot)] = \prod_{k=0}^{N-1} p(x_{k+1}|x_k) \quad (9)$$

Taking $-\log$ and applying the Riemann sum limit as $\Delta t \to 0$ yields Eq. (6). The divergence term arises from the Jacobian determinant in the continuum limit. For details, see Ref. [3]. □

2.2 Conservative Drift and Classical Limit

Definition 3 (Fluctuation-Dissipation Relation)**.** The system satisfies the fluctuation-dissipation relation if there exists a mass tensor $M(x)$ and thermal parameter $\beta = (k_B T)^{-1}$ such that

$$D(x) = \frac{1}{2\beta} M(x)^{-1} \quad (10)$$

Definition 4 (Conservative System)**.** The drift is conservative if there exists a potential $U : M \to \mathbb{R}$ such that

$$f(x) = -D(x)\nabla U(x) \quad (11)$$

Theorem 5 (Classical Limit)**.** *Suppose M is constant, D satisfies Definition 3, and f satisfies Definition 4. Then the Onsager-Machlup action reduces to*

$$S_{\mathrm{OM}}[x] = \beta \int_0^T \left(\tfrac{1}{2} M \dot{x}^T \dot{x} - V(x)\right) dt + \textit{boundary terms} \quad (12)$$

where $V(x) = U(x) + O(\beta^{-1})$.

Proof. Substituting Definition 3 into Eq. (7):

$$L_{\mathrm{OM}} = \frac{\beta}{2}(\dot{x} - f)^T M(\dot{x} - f) + \frac{1}{2}\nabla \cdot f \quad (13)$$

Using $f = -(2\beta)^{-1} M^{-1}\nabla U$, we have $Mf = -(2\beta)^{-1}\nabla U$. Expanding the quadratic form:

$$-2\dot{x}^T M f = \frac{1}{\beta}\dot{x}^T \nabla U \quad (14)$$

$$f^T M f = \frac{1}{4\beta^2}(\nabla U)^T M^{-1}(\nabla U) \quad (15)$$

$$\frac{1}{2}\nabla \cdot f = -\frac{1}{4\beta}\mathrm{Tr}[M^{-1}\nabla^2 U] \quad (16)$$

Collecting terms and absorbing higher-order contributions into an effective potential yields the result. □

3 Quantum Regime: Hydrodynamic Equations

3.1 Variational Derivation

Lemma 6 (Ensemble Action)**.** *The expected action over the ensemble of paths with endpoints distributed according to $\rho(x,t)$ is*

$$\bar{S} = \int \mathcal{D}x\, \rho[x]\, S_{\mathrm{OM}}[x] \quad (17)$$

For a conservative system with scalar diffusivity D, the velocity field decomposes as $v(x,t) = \nabla S(x,t)/m$.

Theorem 7 (Coupled Hydrodynamic Equations)**.** *The stationary point of $\bar{S}$ with respect to variations in ρ and S yields:*

(i) Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m}\right) = 0 \quad (18)$$

(ii) Stochastic Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U(x) + Q(\rho) = 0 \quad (19)$$

where the quantum potential is

$$Q(\rho) = -2mD\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \quad (20)$$

Proof. The variational principle $\delta\bar{S} = 0$ subject to $\int \rho\, d^n x = 1$ yields the Euler-Lagrange equations. Variation with respect to ρ yields the Hamilton-Jacobi equation (19). Variation with respect to S yields the continuity equation (18). The quantum potential arises from the diffusive term: the optimal drift for a given ρ is $f_{\mathrm{opt}} = -D\nabla \log \rho$, and substituting into the action generates the second-derivative term. □

3.2 The Quantum Calibration

Definition 8 (Quantum Diffusivity)**.** For quantum systems, the scalar diffusivity is fixed by

$$D = \frac{\hbar}{2m} \quad (21)$$

where $\hbar$ is Planck's reduced constant and m is the inertial mass.

Remark 9. The calibration $D = \hbar/(2m)$ replaces the thermal energy scale $k_B T$ with the quantum scale $\hbar$. It is the unique value reproducing the quantum mechanical spectrum and plays a role analogous to fixing the speed of light in special relativity: it is not derived within the framework but determines its physical content.

Under Eq. (21), the quantum potential becomes

$$Q(\rho) = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \quad (22)$$

4 The Madelung Transformation

4.1 Formal Linearization

Definition 10 (Wave Function)**.** Define the complex-valued function

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp\left(\frac{iS(x,t)}{\hbar}\right) \tag{23}$$

Theorem 11 (Formal Schrödinger Equation)**.** *If (ρ, S) satisfy Eqs.* (18)–(19) *with $D = \hbar/(2m)$, and if ψ defined by Eq.* (23) *is single-valued, then ψ satisfies*

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U(x)\psi \tag{24}$$

Proof. Computing the time derivative of ψ:

$$\frac{\partial \psi}{\partial t} = \left(\frac{1}{2\sqrt{\rho}}\frac{\partial \rho}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\frac{\partial S}{\partial t}\right)e^{iS/\hbar} \tag{25}$$

From Eq. (18):

$$\frac{\partial \rho}{\partial t} = -\frac{1}{m}\left(\nabla\rho\cdot\nabla S + \rho\nabla^2 S\right) \tag{26}$$

The spatial derivatives give:

$$\nabla\psi = \left(\frac{\nabla\rho}{2\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\,\nabla S\right)e^{iS/\hbar} \tag{27}$$

$$\nabla^2\psi = \left(\nabla^2\sqrt{\rho} + \frac{i}{\hbar}\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\,\nabla^2 S - \frac{\sqrt{\rho}(\nabla S)^2}{\hbar^2}\right)e^{iS/\hbar} \tag{28}$$

using the identity $\nabla^2\sqrt{\rho} = \nabla^2\rho/(2\sqrt{\rho}) - |\nabla\rho|^2/(4\rho^{3/2})$.

Computing $i\hbar\,\partial_t\psi$ and $-(\hbar^2/2m)\nabla^2\psi + U\psi$ separately, the imaginary parts are identical. The real parts yield

$$Q(\rho) = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{29}$$

which is precisely Eq. (22). The Schrödinger equation holds. □

4.2 The Wallström Incompleteness

Theorem 11 requires that ψ be single-valued. However, the hydrodynamic equations (18)–(19) do not guarantee this.

Proposition 12 (Wallström [4])**.** *The coupled system Eqs.* (18)–(19) *admits solutions where S is multivalued, i.e., ∇S is defined but S cannot be written as a single-valued function. For such solutions, no corresponding wave function ψ exists.*

This is the Wallström objection: the Madelung hydrodynamic equations are an incomplete formulation of quantum mechanics. Something additional is needed to exclude the non-quantized solutions and recover equivalence with the Schrödinger equation.

The question is: what is the nature of this additional ingredient?

5 Phase Quantization from Observable Regularity

We now prove that the additional ingredient needed to complete the Madelung equations is the smoothness of the probability current—a condition on a classical observable, not a quantum postulate.

5.1 Geometric Reformulation

Definition 13 (Configuration Space)**.** Let $\Omega \subset \mathbb{R}^n$ be a domain and $Z \subset \Omega$ the nodal set where $\rho = 0$. Define the punctured domain $\Omega^* = \Omega \setminus Z$.

Definition 14 (U(1) Connection One-Form)**.** The phase gradient defines a connection one-form on the principal $U(1)$ bundle over Ω^*:

$$\omega = \frac{1}{\hbar}\nabla S\cdot d\mathbf{x} = \frac{1}{\hbar}\sum_{i=1}^{n}\frac{\partial S}{\partial x^i}dx^i \tag{30}$$

Remark 15. The connection ω is intrinsic to the hydrodynamic formulation: it appears in the current $\mathbf{j} = \rho\nabla S/m = (m\hbar\rho)\omega$, independent of any wave function ψ.

Lemma 16 (Flatness)**.** *The connection ω is flat: $d\omega = 0$ on Ω^*.*

Proof. Since S has C^2 regularity (required by the hydrodynamic equations), mixed partial derivatives commute: $\partial_i\partial_j S = \partial_j\partial_i S$. Therefore the curvature $F = d\omega = 0$. □

Lemma 17 (Finite Action and L^2 Regularity)**.** *If $S_{\mathrm{OM}}[\rho, S] < \infty$, then $\nabla S \in L^2_{\mathrm{loc}}(\Omega^*)$ and*

$$\int_{\Omega^*}\frac{|\mathbf{j}|^2}{\rho}\,d^n x < \infty \tag{31}$$

Proof. The kinetic term in the Onsager-Machlup action is $(4Dm^2)^{-1}\int |\mathbf{j}|^2/\rho\;d^n x\,dt$. Finiteness of S_{OM} implies square-integrability. □

5.2 Holonomy and Phase Circulation

Definition 18 (Holonomy)**.** For a closed loop $\gamma : [0,1] \to \Omega^*$, the holonomy is

$$\mathrm{hol}_\gamma(\omega) = \exp\left(i\oint_\gamma \omega\right) = \exp\left(\frac{i}{\hbar}\oint_\gamma \nabla S\cdot d\mathbf{l}\right) \tag{32}$$

Definition 19 (Phase Circulation)**.** $\Gamma_\gamma = \oint_\gamma \nabla S\cdot d\mathbf{l}$.

Proposition 20 (Homotopy Invariance)**.** *Since ω is flat, $\mathrm{hol}_\gamma(\omega)$ depends only on the homotopy class of γ in $\pi_1(\Omega^*, x_0)$.*

Proof. For a flat connection, parallel transport around a contractible loop is trivial. Homotopic loops have identical holonomies. □

5.3 Smooth Extension Forces Trivial Holonomy

Lemma 21 (Removable Singularity for Flat $U(1)$ Connections)**.** *Let ω be a flat $U(1)$ connection on $\Omega^* = \Omega \setminus Z$, where Z consists of isolated points (in $n = 2, 3$). The connection ω extends smoothly over Z if and only if its holonomy around small loops encircling each point of Z is trivial.*

Proof. This is a standard result in gauge theory. Near an isolated zero $z_0 \in Z$ in polar coordinates (r, θ), a connection with non-trivial holonomy $\alpha \neq 0 \pmod{2\pi}$ locally behaves as $\omega \sim (\alpha/2\pi)d\theta/\hbar$, which has a non-removable singularity at $r = 0$. Conversely, if the holonomy is trivial ($\alpha = 2\pi n$ for $n \in \mathbb{Z}$), the connection is gauge-equivalent to the trivial connection near z_0 and extends smoothly. $\square$

Theorem 22 (Phase Quantization for $\rho > 0$)**.** *Let (ρ, S) satisfy the hydrodynamic equations (18)–(19) with $D = \hbar/(2m)$ on Ω. Let $Z = \{x : \rho(x) = 0\}$. Suppose:*

(i) $\rho \geq 0$ on Ω, with $\rho > 0$ on $\Omega^* = \Omega \setminus Z$

(ii) $S_{\mathrm{OM}}[\rho, S] < \infty$

(iii) The probability current $\mathbf{j} = \rho \nabla S/m$ extends as a C^∞ vector field on all of Ω

Then for any closed loop γ in Ω^*,

$$\Gamma_\gamma = \oint_\gamma \nabla S \cdot d\mathbf{l} = 2\pi n\hbar, \quad n \in \mathbb{Z} \tag{33}$$

Proof. The current and connection are related by $\mathbf{j} = m\hbar\rho\omega$. Near an isolated zero $z_0 \in Z$ with $\rho \sim r^{2\beta}$ ($\beta > 0$), $\omega = \mathbf{j}/(m\hbar\rho) \sim \mathbf{j}_{\mathrm{smooth}}/r^{2\beta}$.

If ω had a residual angular singularity $\alpha\, d\theta/\hbar$ with $\alpha \notin 2\pi\hbar\mathbb{Z}$, then $\mathbf{j} = m\hbar\rho\omega$ would inherit a non-removable singularity from the $r^{2\beta-1}$ factor multiplying the angular part, violating C^∞ regularity. Therefore smooth extension of $\mathbf{j}$ over Z forces smooth extension of ω over Z.

By Lemma 21, smooth extension of the flat connection ω over each $z_i \in Z$ requires trivial holonomy: $\Gamma_{\gamma_i} = 2\pi n_i \hbar$. For arbitrary γ in Ω^*, decompose $[\gamma] = \sum_i m_i[\gamma_i] \in \pi_1(\Omega^*)$ and use homotopy invariance. $\square$

5.4 Extension to Nodal Configurations

Physically relevant quantum states possess nodal zeros where $\rho = 0$. We now prove quantization at these zeros through two independent steps: one dynamical, one observational.

Remark 23 (Scope)**.** We restrict to configurations where Z consists of isolated zeros of finite vanishing order (in $n = 2$) or smooth codimension-2 submanifolds (in $n = 3$). Extension to general nodal sets remains open.

5.4.1 Step 1: The Hamilton-Jacobi Constraint (from the dynamics)

Lemma 24 (Hamilton-Jacobi Constraint)**.** *Let x_0 be an isolated zero of ρ in dimension $n = 2$. In polar coordinates (r, θ) centered at x_0, suppose*

$$\rho \sim r^{2\beta} g(\theta), \quad S \sim \alpha\hbar\theta + S_{\mathrm{reg}}(r, \theta) \tag{34}$$

where $\beta > 0$, $\alpha \in \mathbb{R}$, $g(\theta) > 0$, and S_{reg} is regular at $r = 0$. Then the stochastic Hamilton-Jacobi equation (19) requires

$$\alpha^2 = \beta^2 \tag{35}$$

Proof. Near x_0, the quantum potential behaves as $Q \sim -\hbar^2\beta^2/(2mr^2)$ while the kinetic term contributes $(\nabla S)^2/(2m) \sim \alpha^2\hbar^2/(2mr^2)$. The bounded terms $\partial_t S$ and $U(x)$ cannot cancel the r^{-2} singularity (which is not locally integrable in two dimensions). Therefore the singular coefficient must vanish: $\alpha^2 - \beta^2 = 0$, giving $|\alpha| = \beta$. $\square$

Remark 25. This constraint does not force quantization. The Hamilton-Jacobi equation is perfectly satisfied by $\alpha = 1/3$, $\beta = 1/3$, $\rho \sim r^{2/3}$. These are precisely the Reddiger-Poirier solutions [12]. Nothing in the dynamics alone excludes them.

5.4.2 Step 2: Current Regularity (from the physics of observables)

Lemma 26 (Smoothness Criterion for Radial Powers)**.** *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2)^s$ is C^∞ at the origin if and only if $s \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.*

Proof. ($\Leftarrow$) If $s = k \in \mathbb{N}_0$, then $(x^2 + y^2)^k$ is a polynomial, hence C^∞.

($\Rightarrow$) Suppose $s \notin \mathbb{N}_0$. Write $s = k + \varepsilon$ with $k \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. Then $(x^2+y^2)^s = (x^2+y^2)^k \cdot (x^2+y^2)^\varepsilon$. Restricting to $y = 0$: $h(x) = |x|^{2\varepsilon}$. The derivative of sufficiently high order diverges at the origin. Therefore $(x^2 + y^2)^s \notin C^\infty$ for $s \notin \mathbb{N}_0$. $\square$

Proposition 27 (Current Regularity Forces Integer Winding)**.** *Let (r, θ) be polar coordinates centered at an isolated zero x_0 of ρ, with $\rho \sim r^{2|\alpha|}$ (by Lemma 24) and $S \sim \alpha\hbar\theta$. If $\mathbf{j} = \rho\nabla S/m \in C^\infty(\mathbb{R}^n)$ including at x_0, then $\alpha \in \mathbb{Z}$.*

Proof. In Cartesian coordinates, $\hat{e}_\theta = (-y/r, x/r)$. The current near x_0 is

$$\mathbf{j} \sim \frac{\alpha\hbar}{m} r^{2|\alpha|} \cdot \frac{1}{r}\hat{e}_\theta = \frac{\alpha\hbar}{m}(x^2 + y^2)^{|\alpha|-1}(-y, x) \tag{36}$$

The factor $(-y, x)$ is smooth. The regularity of $\mathbf{j}$ reduces to the smoothness of $(x^2 + y^2)^{|\alpha|-1}$. By Lemma 26, this is C^∞ if and only if $|\alpha| - 1 \in \mathbb{N}_0$, i.e., $|\alpha| \in \{1, 2, 3, \ldots\}$.

The case $\alpha = 0$ corresponds to contractible loops with $\Gamma_C = 0 \in 2\pi\hbar\mathbb{Z}$. Therefore $\alpha \in \mathbb{Z}$. $\square$

5.4.3 Combined Result

Theorem 28 (Phase Quantization at Nodal Zeros). *Let (ρ, S) satisfy the hydrodynamic equations (18)–(19) with $D = \hbar/(2m)$, where $\rho \geq 0$ may have isolated zeros. Suppose:*

(i) $S_{\mathrm{OM}}[\rho, S] < \infty$

(ii) (ρ, S) *is a stationary point of the Onsager-Machlup functional*

(iii) $\mathbf{j} = \rho \nabla S/m \in C^\infty(\Omega)$

Then for any closed loop C that does not pass through a zero of ρ,

$$\oint_C \nabla S \cdot d\mathbf{l} = 2\pi n\hbar, \quad n \in \mathbb{Z} \tag{37}$$

Proof. For loops contractible within $\{\rho > 0\}$, Theorem 22 applies. For loops enclosing isolated zeros: Lemma 24 gives $|\alpha_i| = \beta_i$ (from the dynamics); Proposition 27 gives $\alpha_i \in \mathbb{Z}$ (from current regularity). The total circulation decomposes as $\Gamma_C = \sum_i 2\pi\alpha_i\hbar \in 2\pi\hbar\mathbb{Z}$. □

5.5 Non-Circularity of the Derivation

The following three results establish that the regularity condition $\mathbf{j} \in C^\infty$ is not equivalent to postulating quantization.

Proposition 29 (Regularity Alone Does Not Imply Quantization). *If the relation between β and α were free (not constrained by the Hamilton-Jacobi equation), smooth currents could exist for non-integer α.*

Proof. Take $\rho \sim r^{2k}$ with $k \in \mathbb{N}$ and α arbitrary. Then $\mathbf{j} \sim r^{2k-1}\hat{e}_\theta$, which is smooth for all $k \geq 1$ regardless of α. The constraint on α arises only through the dynamical link $\beta = |\alpha|$. □

Proposition 30 (Dynamics Alone Does Not Imply Quantization). *The Hamilton-Jacobi equation admits solutions with $\alpha \notin \mathbb{Z}$.*

Proof. These are the Reddiger-Poirier solutions [12], which satisfy $|\alpha| = \beta$ with $\alpha = 1/3$, $\rho \sim r^{2/3}$. □

5.6 C^∞ as the Unique Regularity Class

We now prove that no finite differentiability class suffices to produce quantization.

Proposition 31 (C^∞ Uniqueness). *For any finite $k \in \mathbb{N}$, there exist non-integer values $\alpha \notin \mathbb{Z}$ such that the probability current $\mathbf{j}$ is C^k at the nodal zero. Only C^∞ regularity forces $\alpha \in \mathbb{Z}$.*

Proof. By Proposition 27, the regularity of $\mathbf{j}$ at a nodal zero reduces to the smoothness of

$$f(x, y) = (x^2 + y^2)^{|\alpha|-1} \tag{38}$$

We need to determine when this function is C^k but not C^{k+1}.

Let $s = |\alpha| - 1$. Restricting to $y = 0$: $h(x) = |x|^{2s}$. The m-th derivative of h behaves as $|x|^{2s-m}$ near the origin, which is bounded if and only if $2s - m \geq 0$, i.e., $m \leq 2s$. Therefore $h \in C^k$ if and only if $k \leq \lfloor 2s \rfloor$ when $s \notin \mathbb{N}_0$.

For any finite k, choose $\alpha = k/2 + 3/2$ (which is not an integer for any k). Then $s = |\alpha| - 1 = k/2 + 1/2$, and $\lfloor 2s \rfloor = k$. The current is C^k but not C^{k+1}.

Explicitly:

- $\alpha = 3/2$: $s = 1/2$, $\mathbf{j} \sim |x|^1\hat{e}_\theta$, C^0 but not C^1
- $\alpha = 5/2$: $s = 3/2$, $\mathbf{j} \sim |x|^3\hat{e}_\theta$, C^2 but not C^3
- $\alpha = 7/2$: $s = 5/2$, $\mathbf{j} \sim |x|^5\hat{e}_\theta$, C^4 but not C^5
- $\alpha = (2N+1)/2$: C^{2N-2} but not C^{2N-1}

For any given k, a sufficiently large non-integer α passes the C^k test. Only C^∞ requires $s \in \mathbb{N}_0$, i.e., $|\alpha| \in \mathbb{N}$, for all k simultaneously. □

Remark 32. This result shows that C^∞ is not an arbitrary choice of regularity. It is the unique regularity class that is equivalent to phase quantization (given the Hamilton-Jacobi constraint). No weaker condition suffices, and no stronger condition is needed.

5.7 Regularization Argument

We additionally provide a supporting result via regularization.

Proposition 33 (L^2_{loc} Convergence Under Regularization). *Let $\{(\rho_\varepsilon, S_\varepsilon)\}_{\varepsilon>0}$ be a family of configurations satisfying $\rho_\varepsilon > 0$ everywhere, $S_{\mathrm{OM}}[\rho_\varepsilon, S_\varepsilon] \leq C$ uniformly, and $(\rho_\varepsilon, S_\varepsilon) \to (\rho, S)$ pointwise a.e. on $\{\rho > 0\}$. Then for any compact $K \subset \{\rho > 0\}$, $\|\nabla S_\varepsilon - \nabla S\|_{L^2(K)} \to 0$ as $\varepsilon \to 0$. Consequently, $\Gamma_C \in 2\pi\hbar\mathbb{Z}$ for any closed loop $C \subset K$.*

Proof. On K, $\rho \geq \delta_K > 0$. The uniform action bound gives $\int \rho_\varepsilon |\nabla S_\varepsilon|^2 \, d^n x \leq 4Dm^2 C$. For small ε, $\rho_\varepsilon \geq \delta_K/2$ on K, so $\{\nabla S_\varepsilon\}$ is bounded in $L^2(K)$. By Vitali's convergence theorem, $L^2(K)$ convergence follows. Since each $\Gamma_\varepsilon \in 2\pi\hbar\mathbb{Z}$ (by Theorem 22) and $\mathbb{Z}$ is discrete, the limit $\Gamma = \lim \Gamma_\varepsilon \in 2\pi\hbar\mathbb{Z}$. □

5.8 Completeness of the Derivation

Corollary 34 (Schrödinger Equation from Onsager–Machlup). *Under the calibration $D = \hbar/(2m)$ and the condition that $\mathbf{j} = \rho\nabla S/m \in C^\infty(\Omega)$, the Onsager-Machlup variational principle determines quantum mechanical evolution via the Schrödinger equation.*

Proof. Combining Theorem 7 (hydrodynamic equations from the variational principle), Theorem 28 (phase quantization from current regularity), and Theorem 11 (Madelung transformation):

$$\text{OM}+D = \frac{\hbar}{2m} + \mathbf{j} \in C^\infty \implies \exists!\, \psi : i\hbar\partial_t\psi = \hat{H}\psi \tag{39}$$

The wave function exists, is unique (up to global phase), and satisfies the Schrödinger equation. □

5.9 Cohomological Structure

Remark 35 (Continuous vs. Discrete Holonomy)**.** On $\Omega^* = \Omega \setminus Z$ with k isolated punctures, the space of flat $U(1)$ connections modulo gauge gives $H^1(\Omega^*; \mathbb{R}) \cong \mathbb{R}^k$. Each holonomy parameter α_i can take any real value. The discretization $\alpha_i \in 2\pi\hbar\mathbb{Z}$ arises from current regularity: without it, the Madelung equations admit solutions with arbitrary holonomy (the Reddiger-Poirier solutions). Current regularity selects the discrete subset.

6 The Regularity Condition in Physical Context

The central role of the condition $\mathbf{j} \in C^\infty(\Omega)$ in our derivation requires careful discussion. We address three questions: what the condition means physically, why it is the same type of assumption that appears in all physical theories, and what would follow from relaxing it.

6.1 Physical Meaning

The probability current $\mathbf{j} = \rho \nabla S/m$ is the fundamental observable of the hydrodynamic formulation. It determines all measurable flow properties: its integral over surfaces gives particle detection rates, and it appears directly in the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$.

At nodal zeros, where $\rho = 0$, the density vanishes and no particles are present. The condition that $\mathbf{j}$ extends smoothly to these points states that the flow field does not develop singularities where there is no matter to source them. This is an operationally transparent requirement: a singular current at a point where no particle exists would predict infinite flow rates from nothing.

Additionally, the continuity equation requires $\nabla \cdot \mathbf{j}$ to be well-defined throughout Ω, including at nodal zeros. A singular $\mathbf{j}$ would render probability conservation undefined at precisely the points where it is most needed.

The Onsager-Machlup variational principle itself requires well-defined first and second variations of the action for (ρ, S) to constitute a genuine stationary point. Singular fields would undermine the variational structure from which the dynamics are derived.

6.2 Regularity Assumptions in Physical Theories

The assumption that observable fields are smooth in the absence of sources is not specific to our framework. It is a constitutive feature of essentially all fundamental physical theories:

In classical electrodynamics, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are assumed smooth away from charge and current sources. Maxwell's equations do not derive this smoothness; it is part of the definition of the theory.

In general relativity, the metric tensor $g_{\mu\nu}$ is assumed smooth (at least C^2) away from matter sources. Einstein's field equations do not derive metric smoothness; they presuppose it.

In fluid mechanics, the velocity field is assumed smooth in the Navier-Stokes equations. Whether solutions remain smooth for all time is an open problem (one of the Clay Millennium problems), but the equations are *formulated* under the assumption of smoothness.

In each case, singularities of observable fields are interpreted as signals of sources, boundaries, or breakdown of the theory—not as physically admissible configurations in source-free regions. The nodal zeros of ρ are not sources: they are points where the density vanishes but probability is conserved. There is no physical reason for $\mathbf{j}$ to be singular at these points, and every reason for it to be regular.

6.3 Consequences of Relaxing the Condition

If one does not require $\mathbf{j} \in C^\infty(\Omega)$, the Madelung equations admit the non-quantized solutions of Reddiger and Poirier [12]. These solutions have probability currents with singularities at nodal zeros. Specifically, for winding number $\alpha = 1/3$, the current behaves as $\mathbf{j} \sim r^{-1/3}\hat{e}_\theta$ near the node—a divergent flow field at a point where no matter is present.

Relaxing the regularity condition does not yield a more general or more fundamental theory. It yields a theory that:

- violates probability conservation at nodal zeros (since $\nabla \cdot \mathbf{j}$ is undefined),
- admits configurations with infinite flow rates from vacuum,
- loses the variational structure at the singular points, and
- does not correspond to quantum mechanics.

The regularity condition does not restrict the physics; it completes it.

7 Comparison with Previous Approaches

7.1 Nelson's Stochastic Mechanics

Nelson [1, 2] postulated forward and backward stochastic processes with decomposition into current velocity $v = \nabla S/m$ and osmotic velocity $u = (D/m)\nabla \log \rho$, then postulated a modified Newton's law from which the Schrödinger equation follows.

In our framework, the osmotic velocity emerges from the quantum potential:

$$u = -\frac{1}{m}\nabla Q(\rho) = \frac{D}{m}\nabla \log \rho = \frac{\hbar}{2m^2}\nabla \log \rho \qquad (40)$$

Nelson's decomposition is a consequence of the Onsager-Machlup variational principle, not a postulate. Nelson did not address the Wallström objection; he assumed single-valuedness of ψ.

7.2 Bohm's Quantum Potential

Bohm [5, 6] applied the Madelung transformation *to* the Schrödinger equation to obtain the quantum potential $Q = -(\hbar^2/2m)\nabla^2 R/R$ where $R = \sqrt{\rho}$. We derive the same quantum potential *from* the stochastic action functional. Bohm's approach is necessarily circular regarding the Wallström objection: it starts from the Schrödinger equation.

7.3 Direct Comparisons on the Wallström Objection

Both Nelson and Bohm fail to address the Wallström incompleteness. Our contribution is to identify the nature of the missing ingredient: it is a regularity condition on the observable $\mathbf{j}$, not a quantum postulate. Theorem 28 proves that this condition, combined with the Hamilton-Jacobi dynamics derived from the variational principle, forces integer phase circulation. The non-circularity is established by Propositions 29 and 30, and the uniqueness of C^∞ by Proposition 31.

8 Conclusion

The Wallström objection correctly identified the Madelung hydrodynamic equations as an incomplete formulation of quantum mechanics. This incompleteness is real: the equations admit solutions with non-integer phase circulation for which no single-valued wave function exists.

We have shown that what completes the formulation is not a quantum postulate but a property of observables. The probability current $\mathbf{j} = \rho\nabla S/m$, as a physical observable, must be smooth throughout physical space—the same requirement that every physical theory imposes on its observable fields in the absence of sources. This condition, combined with the Hamilton-Jacobi dynamics that the Onsager-Machlup variational principle provides, forces integer phase circulation.

The derivation is not circular. We have established this through three independent results: the regularity condition alone does not imply quantization (Proposition 29); the dynamics alone does not imply quantization (Proposition 30); and C^∞ is the unique regularity class that produces exact quantization (Proposition 31).

The logical structure of the complete derivation is:

$$\text{OM functional} + D = \frac{\hbar}{2m} + \mathbf{j} \in C^\infty \implies i\hbar\partial_t\psi = \hat{H}\psi \tag{41}$$

Three ingredients, each with clear physical content: a variational principle (dynamics), a calibration (scale), and a regularity condition (observables). The first two are shared by all stochastic derivations. The third was always implicit in the physics of the problem—it simply had not been made explicit.

The Wallström objection asked: what is missing from the Madelung equations? The answer is: a standard physical condition on observables, not a quantum postulate.

Future work should extend this framework to relativistic field theory (covariant Onsager-Machlup functional), many-body systems (configuration space topology), and the interface with quantum gravity.

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