Quantum Mechanics from Stochastic Coherence: Resolving the Wallstrom Objection via Topological Stability

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Abstract

We derive the Schrödinger equation from the Onsager-Machlup stochastic variational principle and address the Wallstrom objection within this framework. Wallstrom showed that the Madelung hydrodynamic equations do not enforce quantization of phase circulation unless single-valuedness of the wave function is imposed as an additional condition.

We prove that quantization of phase circulation follows from intrinsic requirements of the stochastic formulation. Reformulating the phase gradient as a flat U(1) connection on the punctured domain where the density is positive, we show that smooth removability of isolated singularities forces trivial holonomy, yielding the quantization condition

$$\oint \nabla S \cdot dl = 2\pi n\hbar$$

For configurations with nodal zeros, quantization emerges from the combination of the Hamilton-Jacobi constraint at nodal zeros and the regularity required for the probability current to be a well-defined observable on physical space. These two conditions, neither of which alone implies quantization, together exclude non-integer winding numbers including the recently constructed non-quantized strong solutions of the Madelung equations.

Under the identification $D = \hbar/(2m)$, the Madelung transformation then recovers the Schrödinger equation without postulating wave function single-valuedness. Quantization emerges as a geometric and regularity consequence of the Onsager-Machlup variational structure.

1 Introduction

The derivation of quantum mechanics from stochastic foundations has been attempted through various approaches, notably Nelson's stochastic mechanics [1] and applications of the Onsager-Machlup variational principle [3]. However, Wallstrom [4] identified a critical gap: the hydrodynamic equations obtained via the Madelung transformation

$$\psi(x,t) = \sqrt{\rho(x,t)}\, e^{iS(x,t)/\hbar} \tag{1}$$

are not equivalent to the Schrödinger equation unless the phase circulation satisfies

$$\oint_C \nabla S \cdot dl = 2\pi n\hbar, \quad n \in \mathbb{Z} \tag{2}$$

for all closed loops C in the domain. Without this topological constraint, the hydrodynamic equations admit solutions where no corresponding single-valued wave function exists.

We resolve this objection by proving that Eq. (2) emerges as a necessary consequence of requiring finite Onsager-Machlup action and topological stability. Our derivation proceeds in three steps:

(I) We establish the Onsager-Machlup action functional for diffusion processes and derive the coupled nonlinear equations for (ρ, S).

(II) We introduce the calibration $D = \hbar/(2m)$ and show that the Madelung transformation linearizes the system to yield the Schrödinger equation *formally*.

(III) We prove that topological stability under continuous deformations enforces Eq. (2), completing the derivation.

2 Mathematical Framework

2.1 The Onsager-Machlup Functional

Definition 1 (Admissible Paths)**.** Let $\mathcal{M}$ be a smooth n-dimensional Riemannian manifold. The space of admissible paths is

$$\mathcal{P} = \left\{x \in H^1([0,T],\mathcal{M}) : x(0) = x_0, x(T) = x_T\right\} \tag{3}$$

where H^1 denotes the Sobolev space of absolutely continuous paths with square-integrable derivatives.

Consider a diffusion process governed by the Itô SDE:

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t \tag{4}$$

where $f : \mathcal{M} \to T\mathcal{M}$ is the drift, $\sigma : \mathcal{M} \to \mathrm{End}(T\mathcal{M})$ is the diffusion coefficient, and W_t is standard Brownian motion on $\mathbb{R}^m$.

Theorem 1 (Onsager-Machlup Action)**.** *The probability density functional for paths of Eq. (4) is given by*

$$\mathcal{P}[x(\cdot)] = \mathcal{N} \exp\left(-S_{OM}[x]\right) \tag{5}$$

where $\mathcal{N}$ is a normalization constant and

$$S_{OM}[x] = \int_0^T L_{OM}(x,\dot{x})dt \tag{6}$$

with Lagrangian density

$$L_{OM}(x, \dot{x}) = \frac{1}{4}(\dot{x} - f)^T \mathbb{D}^{-1}(\dot{x} - f) + \frac{1}{2}\nabla \cdot f \quad (7)$$

where $\mathbb{D} = \frac{1}{2}\sigma\sigma^T$ *is the diffusion tensor.*

Proof. The transition probability for a small time step Δt is given by the Gaussian kernel

$$p(x_{k+1}|x_k) = \frac{1}{(4\pi\Delta t)^{n/2}\sqrt{\det \mathbb{D}}} \times \exp\left[-\frac{|x_{k+1} - x_k - f\Delta t|^2_{\mathbb{D}^{-1}}}{4\Delta t}\right] \quad (8)$$

where $|\cdot|^2_{\mathbb{D}^{-1}} = (\cdot)^T\mathbb{D}^{-1}(\cdot)$. For a trajectory discretized into N steps,

$$\mathcal{P}[x(\cdot)] = \prod_{k=0}^{N-1} p(x_{k+1}|x_k) \quad (9)$$

Taking $-\log$ and applying the Riemann sum limit as $\Delta t \to 0$ yields Eq. (6). The divergence term arises from the Jacobian determinant in the continuum limit. For details, see Ref. [3]. □

2.2 Conservative Drift and Classical Limit

Definition 2 (Fluctuation-Dissipation Relation)**.** We say that the system satisfies the fluctuation-dissipation relation if there exists a mass tensor $M(x)$ and thermal parameter $\beta = (k_B T)^{-1}$ such that

$$\mathbb{D}(x) = \frac{1}{2\beta}M(x)^{-1} \quad (10)$$

The scalar diffusivity is $D(x) = \mathrm{Tr}[\mathbb{D}(x)]$.

Definition 3 (Conservative System)**.** The drift is conservative if there exists a potential $U : \mathcal{M} \to \mathbb{R}$ such that

$$f(x) = -\mathbb{D}(x)\nabla U(x) \quad (11)$$

Theorem 2 (Classical Limit)**.** *Suppose M is constant, $\mathbb{D}$ satisfies Eq. (10), and f satisfies Eq. (11). Then the Onsager-Machlup action reduces to*

$$S_{OM}[x] = \beta \int_0^T \left[\frac{1}{2}M\dot{x}^T\dot{x} - V(x)\right] dt + \textit{boundary terms} \quad (12)$$

where $V(x) = U(x) + O(\beta^{-1})$.

Proof. Substituting Eq. (10) into Eq. (7):

$$L_{OM} = \frac{\beta}{2}(\dot{x} - f)^T M(\dot{x} - f) + \frac{1}{2}\nabla \cdot f \quad (13)$$

$$= \frac{\beta}{2}\left[\dot{x}^T M\dot{x} - 2\dot{x}^T Mf + f^T Mf\right] + \frac{1}{2}\nabla \cdot f \quad (14)$$

Using $f = -(2\beta)^{-1}M^{-1}\nabla U$, we have $Mf = -(2\beta)^{-1}\nabla U$. Thus:

$$-2\dot{x}^T Mf = \frac{1}{\beta}\dot{x}^T\nabla U \quad (15)$$

$$f^T Mf = \frac{1}{4\beta^2}(\nabla U)^T M^{-1}(\nabla U) \quad (16)$$

$$\frac{1}{2}\nabla \cdot f = -\frac{1}{4\beta}\mathrm{Tr}[M^{-1}\nabla^2 U] \quad (17)$$

Collecting terms and absorbing higher-order contributions into an effective potential yields Eq. (12). □

3 Quantum Regime: Hydrodynamic Equations

3.1 Variational Derivation

We now derive the equations governing the probability density $\rho(x, t)$ and phase $S(x, t)$ by applying the variational principle to the ensemble average of the Onsager-Machlup action.

Lemma 3 (Ensemble Action)**.** *The expected action over the ensemble of paths with endpoints distributed according to $\rho(x, t)$ is*

$$\bar{S} = \int \mathcal{D}x\, \rho[x]\, S_{OM}[x] \quad (18)$$

where $\mathcal{D}x$ denotes the path integral measure.

For a conservative system with scalar diffusivity D, the velocity field decomposes as

$$v(x, t) = \frac{\nabla S(x, t)}{m} \quad (19)$$

where $S(x, t)$ is the action phase.

Theorem 4 (Coupled Hydrodynamic Equations)**.** *The stationary point of $\bar{S}$ with respect to variations in ρ and S yields:*

(i) Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\frac{\nabla S}{m}\right) = 0 \quad (20)$$

(ii) Stochastic Hamilton-Jacobi Equation:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U(x) + Q(\rho) = 0 \quad (21)$$

where the quantum potential is

$$Q(\rho) = -2mD\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \quad (22)$$

Proof. The variational principle $\delta\bar{S} = 0$ subject to probability conservation $\int \rho\, d^n x = 1$ yields the Euler-Lagrange equations.

For the density ρ: Using $\delta\rho$ variations and integrating by parts, the conservation law Eq. (20) follows from requiring the functional derivative to vanish.

For the phase S: The kinetic term in the action contributes

$$\frac{1}{4D}\int(\dot{x}-f)^2dt = \frac{1}{4D}\int\left(\frac{\nabla S}{m}\right)^2 dt \tag{23}$$

when evaluated along trajectories satisfying Eq. (19). Variation with respect to S and applying $\delta S/\delta x = -\partial S/\partial t - H$ where H is the Hamiltonian yields Eq. (21).

The quantum potential arises from the diffusive term. To see this explicitly, note that the optimal drift for a given ρ is

$$f_{opt} = -D\nabla \log\rho = -\frac{D}{\rho}\nabla\rho \tag{24}$$

Substituting into the action and performing integration by parts generates the second-derivative term in $Q(\rho)$. □

3.2 The Quantum Calibration

We now transition from the classical regime to quantum mechanics by specifying the diffusion coefficient.

Definition 4 (Quantum Diffusivity)**.** For quantum systems, the scalar diffusivity is fixed by the relation

$$D = \frac{\hbar}{2m} \tag{25}$$

where $\hbar$ is Planck's reduced constant and m is the inertial mass. This is the only free parameter in the framework.

Remark 1. In thermal equilibrium, the classical relation $\mathbb{D} = (2\beta)^{-1}M^{-1}$ with $\beta = (k_BT)^{-1}$ yields $D = (2\beta m)^{-1} = k_BT/(2m)$ for a scalar system. The quantum calibration Eq. (25) replaces the thermal energy scale k_BT with the quantum scale $\hbar$, reflecting the fact that quantum fluctuations are not temperature-dependent but intrinsic to the vacuum substrate. The calibration $D = \hbar/(2m)$ is the unique value reproducing the quantum mechanical spectrum. It plays a role analogous to fixing the speed of light in special relativity: it is not derived within the framework but determines its physical content.

Under Eq. (25), the quantum potential becomes

$$Q(\rho) = -2mD\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{26}$$

4 The Madelung Transformation

4.1 Formal Linearization

Definition 5 (Wave Function)**.** Define the complex-valued function

$$\psi(x,t) = \sqrt{\rho(x,t)}\,\exp\left(\frac{iS(x,t)}{\hbar}\right) \tag{27}$$

Theorem 5 (Formal Schrödinger Equation)**.** *If (ρ, S) satisfy Eqs. (20)–(21) with $D = \hbar/(2m)$, and if ψ defined by Eq. (27) is single-valued, then ψ satisfies*

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U(x)\psi \tag{28}$$

Proof. We compute derivatives of ψ explicitly.

Time derivative:

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \frac{\partial\sqrt{\rho}}{\partial t}e^{iS/\hbar} + \sqrt{\rho}\,\frac{i}{\hbar}\frac{\partial S}{\partial t}e^{iS/\hbar} \\ &= \left[\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\frac{\partial S}{\partial t}\right]e^{iS/\hbar}\end{aligned} \tag{29}$$

From Eq. (20):

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\left(\rho\frac{\nabla S}{m}\right) = -\frac{1}{m}\left[\nabla\rho\cdot\nabla S + \rho\nabla^2 S\right] \tag{30}$$

Thus:

$$\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t} = -\frac{1}{2m}\left[\frac{\nabla\rho}{\sqrt{\rho}}\cdot\nabla S + \sqrt{\rho}\nabla^2 S\right] \tag{31}$$

From Eq. (21):

$$\frac{\partial S}{\partial t} = -\frac{(\nabla S)^2}{2m} - U(x) - Q(\rho) \tag{32}$$

Spatial derivatives:

$$\nabla\psi = \left[\frac{\nabla\rho}{2\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\nabla S\right]e^{iS/\hbar} \tag{33}$$

Computing the Laplacian:

$$\begin{aligned}\nabla^2\psi &= \nabla\cdot\left[\left(\frac{\nabla\rho}{2\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\nabla S\right)e^{iS/\hbar}\right] \\ &= \left[\frac{\nabla^2\rho}{2\sqrt{\rho}} - \frac{|\nabla\rho|^2}{4\rho^{3/2}} + \frac{i}{\hbar}\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}}\right. \\ &\quad \left. + \frac{i}{\hbar}\sqrt{\rho}\nabla^2 S - \frac{(\nabla S)^2}{\hbar^2}\sqrt{\rho}\right]e^{iS/\hbar}\end{aligned} \tag{34}$$

Using the identity

$$\nabla^2\sqrt{\rho} = \frac{\nabla^2\rho}{2\sqrt{\rho}} - \frac{|\nabla\rho|^2}{4\rho^{3/2}} \tag{35}$$

we recognize that the first two terms in Eq. (34) are precisely $\nabla^2\sqrt{\rho}$. Thus Eq. (34) becomes:

$$\begin{aligned}\nabla^2\psi = \left[\nabla^2\sqrt{\rho} + \frac{i}{\hbar}\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\nabla^2 S\right. \\ \left. - \frac{\sqrt{\rho}(\nabla S)^2}{\hbar^2}\right]e^{iS/\hbar}\end{aligned} \tag{36}$$

Now we compute $i\hbar\partial_t\psi$ from Eq. (29):

$$\begin{aligned}i\hbar\frac{\partial\psi}{\partial t} &= i\hbar\left[\frac{1}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\frac{\partial S}{\partial t}\right]e^{iS/\hbar} \\ &= \left[\frac{i\hbar}{2\sqrt{\rho}}\frac{\partial\rho}{\partial t} - \sqrt{\rho}\frac{\partial S}{\partial t}\right]e^{iS/\hbar}\end{aligned} \tag{37}$$

Substituting Eqs. (31) and (32):

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{i\hbar}{2m}\left(\frac{\nabla\rho}{\sqrt{\rho}}\cdot\nabla S + \sqrt{\rho}\nabla^2 S\right) + \sqrt{\rho}\left(\frac{(\nabla S)^2}{2m} + U + Q(\rho)\right)\right]e^{iS/\hbar} \tag{38}$$

For the right-hand side, compute $-(\hbar^2/(2m))\nabla^2\psi + U\psi$:

$$\begin{aligned}-\frac{\hbar^2}{2m}\nabla^2\psi &= -\frac{\hbar^2}{2m}\left[\nabla^2\sqrt{\rho} + \frac{i}{\hbar}\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}} + \frac{i}{\hbar}\sqrt{\rho}\nabla^2 S - \frac{\sqrt{\rho}(\nabla S)^2}{\hbar^2}\right]e^{iS/\hbar} \\ &= \left[-\frac{\hbar^2}{2m}\nabla^2\sqrt{\rho} - \frac{i\hbar}{2m}\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}} - \frac{i\hbar}{2m}\sqrt{\rho}\nabla^2 S + \frac{(\nabla S)^2}{2m}\sqrt{\rho}\right]e^{iS/\hbar}\end{aligned} \tag{39}$$

Therefore:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = \left[-\frac{\hbar^2}{2m}\nabla^2\sqrt{\rho} - \frac{i\hbar}{2m}\left(\frac{\nabla\rho\cdot\nabla S}{\sqrt{\rho}} + \sqrt{\rho}\nabla^2 S\right) + \frac{(\nabla S)^2}{2m}\sqrt{\rho} + U\sqrt{\rho}\right]e^{iS/\hbar} \tag{40}$$

Comparing Eqs. (38) and (40), the imaginary parts are identical. The real parts yield:

$$\sqrt{\rho}\cdot Q(\rho) = -\frac{\hbar^2}{2m}\cdot\nabla^2\sqrt{\rho} \tag{41}$$

Dividing by $\sqrt{\rho}$:

$$Q(\rho) = -\frac{\hbar^2}{2m}\cdot\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{42}$$

which is precisely Eq. (26). Thus the Schrödinger equation holds. □

4.2 Wallstrom's Objection

Theorem 5 establishes that *if* ψ is single-valued, then it satisfies the Schrödinger equation. However, the hydrodynamic equations (20)–(21) do not by themselves guarantee single-valuedness. Specifically:

Proposition 6 (Wallstrom [4])**.** *The coupled system Eqs. (20)–(21) admits solutions where S is multivalued (i.e., ∇S is defined but S cannot be written as a single-valued function). For such solutions, no corresponding wave function ψ exists.*

This is the Wallstrom objection: stochastic mechanics is incomplete without an additional constraint ensuring single-valuedness of ψ.

5 Topological Quantization via Flat U(1) Connections

We now prove that the quantization condition Eq. (2) emerges as a necessary consequence of requiring smooth extension of physical observables over the domain, including at nodal zeros. The key insight is to reformulate the problem in the language of principal U(1) bundles and flat connections, where quantization arises from a removable singularity theorem rather than heuristic physical arguments.

5.1 Geometric Reformulation

Definition 6 (Configuration Space)**.** Let $\Omega\subset\mathbb{R}^n$ be a domain and $Z\subset\Omega$ the nodal set where $\rho = 0$. Define the punctured domain

$$\Omega^* = \Omega\setminus Z \tag{43}$$

We work on Ω^* where $\rho > 0$ and the phase S is well-defined.

Definition 7 (U(1) Connection One-Form)**.** The phase gradient ∇S defines a connection one-form ω on the principal U(1) bundle over Ω^*:

$$\omega = \frac{1}{\hbar}\nabla S\cdot dx = \frac{1}{\hbar}\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}dx_i \tag{44}$$

Remark 2. The connection ω is intrinsic to the hydrodynamic formulation: it appears in the current $\mathbf{j} = \rho\nabla S/m = (m\hbar\rho)\omega$, independent of any wave function ψ. The Madelung transformation $\psi = \sqrt{\rho}e^{iS/\hbar}$ would then correspond to a section of the associated complex line bundle, but we do not assume its existence.

Lemma 7 (Flatness of the Connection)**.** *The connection ω is flat, i.e., its curvature vanishes: $d\omega = 0$ on Ω^*.*

Proof. The curvature two-form is

$$F = d\omega = \frac{1}{\hbar}\sum_{i<j}\left(\frac{\partial^2 S}{\partial x_i\partial x_j} - \frac{\partial^2 S}{\partial x_j\partial x_i}\right)dx_i\wedge dx_j \tag{45}$$

Since S is a scalar function with C^2 regularity (required by the hydrodynamic equations), mixed partial derivatives commute: $\partial_i\partial_j S = \partial_j\partial_i S$. Therefore $F = 0$. □

Lemma 8 (Finite Action and L^2 Regularity)**.** *If $S_{OM}[\rho, S] < \infty$, then $\nabla S\in L^2_{\mathrm{loc}}(\Omega^*)$ and the probability current $\mathbf{j} = \rho\nabla S/m$ satisfies*

$$\int_{\Omega^*}\frac{|\mathbf{j}|^2}{\rho}d^n x < \infty \tag{46}$$

Proof. The kinetic term in the Onsager-Machlup action is

$$\begin{aligned}&\frac{1}{4D}\int_0^T\int_{\Omega^*}\rho\left(\frac{\nabla S}{m}\right)^2 d^n x\,dt \\ &= \frac{1}{4Dm^2}\int_0^T\int_{\Omega^*}\frac{|\mathbf{j}|^2}{\rho}d^n x\,dt\end{aligned} \tag{47}$$

Finiteness of S_{OM} implies square-integrability of the integrand. □

5.2 Holonomy and Phase Circulation

Definition 8 (Holonomy)**.** For a closed loop $\gamma : [0,1] \to \Omega^*$ with $\gamma(0) = \gamma(1) = x_0$, the holonomy of the connection ω is

$$\mathrm{hol}_\gamma(\omega) = \exp\left(i \oint_\gamma \omega\right) = \exp\left(\frac{i}{\hbar}\oint_\gamma \nabla S \cdot dl\right) \quad (48)$$

This is a well-defined element of $U(1)$ even if S is multivalued, since only ∇S enters.

Definition 9 (Phase Circulation)**.** The phase circulation along γ is

$$\Gamma_\gamma = \oint_\gamma \nabla S \cdot dl \quad (49)$$

The holonomy relates to circulation by $\mathrm{hol}_\gamma(\omega) = e^{i\Gamma_\gamma/\hbar}$.

Proposition 9 (Flatness Implies Homotopy Invariance)**.** *Since ω is flat, the holonomy $\mathrm{hol}_\gamma(\omega)$ depends only on the homotopy class of γ in $\pi_1(\Omega^*, x_0)$.*

Proof. For a flat connection, the parallel transport around a contractible loop is trivial. If γ_0 and γ_1 are homotopic, their holonomies coincide: $\mathrm{hol}_{\gamma_0}(\omega) = \mathrm{hol}_{\gamma_1}(\omega)$. □

5.3 Main Theorem: Smooth Extension Forces Trivial Holonomy

We now establish the central mathematical result: smooth extension of the observable current over nodal zeros forces quantization of the holonomy.

Lemma 10 (Removable Singularity for Flat U(1) Connections)**.** *Let ω be a flat U(1) connection on the punctured domain $\Omega^* = \Omega \setminus Z$, where Z consists of isolated points (in $n = 2, 3$). The connection ω extends smoothly over Z if and only if its holonomy around small loops encircling each point of Z is trivial.*

Proof. This is a standard result in gauge theory. A flat connection on a punctured manifold extends to a smooth connection on the full manifold precisely when the monodromy around each puncture is trivial. For U(1) bundles over contractible domains, smooth extension over isolated singularities is controlled entirely by local holonomy.

Explicitly, near an isolated zero $z_0 \in Z$, we can work in polar coordinates (r, θ) centered at z_0. If the connection has non-trivial holonomy $\alpha \neq 0 \pmod{2\pi}$ around z_0, it locally behaves as

$$\omega \sim \frac{\alpha}{2\pi}\frac{d\theta}{\hbar} \quad (50)$$

This has a non-removable singularity at $r = 0$. Conversely, if the holonomy is trivial ($\alpha = 2\pi n$ for $n \in \mathbb{Z}$), the connection is gauge-equivalent to the trivial connection in a neighborhood of z_0 and extends smoothly. □

Theorem 11 (Smooth Current Extension Implies Phase Quantization)**.** *Let (ρ, S) satisfy the hydrodynamic equations (20)–(21) with $D = \hbar/(2m)$ on domain $\Omega \subset \mathbb{R}^n$. Let $Z = \{x : \rho(x) = 0\}$ be the nodal set. Suppose:*

(i) $\rho \geq 0$ on Ω, with $\rho > 0$ on $\Omega^* = \Omega \setminus Z$

(ii) $S_{OM}[\rho, S] < \infty$

(iii) The probability current $\mathbf{j} = \rho\nabla S/m$ extends as a smooth (C^∞) vector field on all of Ω, including at points in Z

Then for any closed loop γ in Ω^*,

$$\Gamma_\gamma = \oint_\gamma \nabla S \cdot dl = 2\pi n\hbar, \quad n \in \mathbb{Z} \quad (51)$$

Proof. The proof proceeds in two steps: first, we show that smooth extension of $\mathbf{j}$ implies smooth extension of the connection ω; second, we apply Lemma 10 to conclude quantization.

Step 1: Current smoothness implies connection smoothness.

The current and connection are related by

$$\mathbf{j} = m\hbar\rho\omega \quad (52)$$

where we identify the one-form ω with the vector $\nabla S/\hbar$ via the metric.

Near an isolated zero $z_0 \in Z$, suppose $\rho \sim r^{2\beta}$ with $\beta > 0$ (the exponent is determined by the Hamilton-Jacobi equation, as shown in Section 5.4). Then

$$\omega = \frac{\mathbf{j}}{m\hbar\rho} \sim \frac{\mathbf{j}_{\text{smooth}}}{r^{2\beta}} \quad (53)$$

For $\mathbf{j}$ to be C^∞ at z_0 (assumption (iii)), we require $\mathbf{j}(z_0) = 0$. Moreover, the derivatives of $\mathbf{j}$ must vanish to sufficiently high order to compensate for the $r^{-2\beta}$ singularity in the denominator.

The key observation is that this is possible *only* if the connection ω itself extends smoothly over z_0. If ω had a residual singularity (such as an angular component $\alpha d\theta/\hbar$ with $\alpha \notin 2\pi\hbar\mathbb{Z}$), then $\mathbf{j} = m\hbar\rho\omega$ would inherit a non-removable singularity from the $r^{2\beta-1}$ factor multiplying the angular part, violating C^∞ regularity.

Therefore, smooth extension of $\mathbf{j}$ over Z forces smooth extension of ω over Z.

Step 2: Quantization from removable singularities.

By Lemma 10, smooth extension of the flat connection ω over each point $z_i \in Z$ requires

$$\mathrm{hol}_{\gamma_i}(\omega) = 1 \quad (54)$$

where γ_i is a small loop encircling z_i. This means

$$\Gamma_{\gamma_i} = \oint_{\gamma_i} \nabla S \cdot dl = 2\pi n_i\hbar, \quad n_i \in \mathbb{Z} \quad (55)$$

For an arbitrary closed loop γ in Ω^*, decompose it using the fundamental group structure:

$$[\gamma] = \sum_i m_i[\gamma_i] \in \pi_1(\Omega^*) \quad (56)$$

where $m_i \in \mathbb{Z}$ are winding numbers around each z_i. By homotopy invariance (flatness of ω):

$$\Gamma_\gamma = \sum_i m_i \Gamma_{\gamma_i} = 2\pi \left(\sum_i m_i n_i \right) \hbar \in 2\pi\hbar\mathbb{Z} \tag{57}$$

This completes the proof. □

Remark 3 (Independence from Wave Function). The proof makes no reference to the existence of ψ. The quantization condition emerges purely from requiring that the *observable* quantity $\mathbf{j} = \rho\nabla S/m$—which appears directly in the continuity equation and determines all measurable flow properties—be a regular vector field on physical space $\mathbb{R}^n$.

Remark 4 (Comparison with Heuristic Arguments). Previous attempts to derive quantization from stochastic mechanics relied on either:

1. Infinite action costs from branch cuts in S (heuristic, since the relevant action is for the *ensemble* of trajectories, not individual paths)

2. Inadmissibility of Riemann surfaces for physical space (ontologically contentious)

Our approach avoids both issues. The quantization arises from a rigorous mathematical theorem: flat connections over punctured domains extend smoothly if and only if their holonomy is quantized. This is a standard result in differential geometry, not a physical postulate.

5.4 Extension to Nodal Configurations

Theorem 11 establishes phase quantization under the assumption $\rho(x, t) > 0$ everywhere. However, physically relevant quantum states—such as excited hydrogen orbitals—possess nodal zeros where $\rho = 0$. At these zeros, the phase S becomes undefined and the velocity field $\nabla S/m$ diverges, creating precisely the configurations where Wallstrom's objection has its strongest force.

We now extend the quantization result to configurations with isolated nodal zeros. The derivation combines two independent physical ingredients, neither of which alone implies quantization, but whose combination forces integer winding numbers.

Remark 5 (Scope). We restrict to configurations where the nodal set Z consists of isolated zeros of finite vanishing order (in $n = 2$) or smooth codimension-2 submanifolds (in $n = 3$). Extension to general nodal sets remains open.

5.4.1 Ingredient 1: Hamilton-Jacobi Constraint at Nodal Zeros

Lemma 12 (Hamilton-Jacobi Constraint)**.** *Let x_0 be an isolated zero of ρ in dimension $n = 2$. In polar coordinates (r, θ) centered at x_0, suppose*

$$\rho \sim r^{2\beta} g(\theta), \qquad S \sim \alpha\hbar\theta + S_{\text{reg}}(r, \theta) \tag{58}$$

where $\beta > 0$, $\alpha \in \mathbb{R}$, $g(\theta) > 0$, and S_{reg} is regular at $r = 0$. Then the stochastic Hamilton-Jacobi equation (21) requires

$$\alpha^2 = \beta^2 \tag{59}$$

Proof. Near x_0, the quantum potential behaves as

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{\hbar^2 \beta^2}{2mr^2} \tag{60}$$

while the kinetic term contributes

$$\frac{(\nabla S)^2}{2m} \sim \frac{\alpha^2 \hbar^2}{2mr^2} \tag{61}$$

The stochastic Hamilton-Jacobi equation (Eq. 21) reads

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U(x) + Q(\rho) = 0 \tag{62}$$

The terms $\partial S/\partial t$ and $U(x)$ are bounded near x_0. The singular contributions therefore yield

$$\frac{(\alpha^2 - \beta^2)\hbar^2}{2mr^2} + \text{bounded terms} = 0 \tag{63}$$

Since r^{-2} is not locally integrable in two dimensions ($\int r^{-2} \cdot r\, dr\, d\theta$ diverges logarithmically), this equation cannot hold even distributionally unless the singular coefficient vanishes:

$$\alpha^2 - \beta^2 = 0 \implies |\alpha| = \beta \tag{64}$$

The argument extends to $n = 3$, where nodal zeros are generically lines: in the transverse plane, the same r^{-2} singularity structure applies. □

5.4.2 Ingredient 2: Current Regularity Excludes Non-Integer Winding

Lemma 13 (Smoothness Criterion for Radial Powers)**.** *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2)^s$ is C^∞ at the origin if and only if $s \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.*

Proof. ($\Leftarrow$) If $s = k \in \mathbb{N}_0$, then $(x^2 + y^2)^k$ is a polynomial in (x, y), hence C^∞.

($\Rightarrow$) Suppose $s \notin \mathbb{N}_0$. Write $s = k + \varepsilon$ with $k \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. Then

$$(x^2 + y^2)^s = (x^2 + y^2)^k \cdot (x^2 + y^2)^\varepsilon \tag{65}$$

The first factor is a polynomial. For the second, restrict to the line $y = 0$:

$$h(x) = (x^2)^\varepsilon = |x|^{2\varepsilon} \tag{66}$$

Computing derivatives:

$$h'(x) = 2\varepsilon |x|^{2\varepsilon - 1} \operatorname{sgn}(x) \tag{67}$$

$$h''(x) = 2\varepsilon(2\varepsilon - 1)|x|^{2\varepsilon - 2} \tag{68}$$

For $0 < \varepsilon < 1/2$: $h''(x) \sim |x|^{2\varepsilon - 2} \to \infty$ as $x \to 0$, so $h \notin C^2$. For $1/2 \leq \varepsilon < 1$: the derivative of order $\lceil 2s \rceil$ diverges at the origin by the same mechanism. Therefore $(x^2 + y^2)^s \notin C^\infty$ for $s \notin \mathbb{N}_0$. □

Proposition 14 (Current Regularity Forces Integer Winding)**.** *Let (r, θ) be polar coordinates centered at an isolated zero x_0 of ρ, with $\rho \sim r^{2|\alpha|}$ (by Lemma 12) and $S \sim \alpha\hbar\theta$. If the probability current $\mathbf{j} = \rho\nabla S/m$ is a smooth (C^∞) vector field on physical space $\mathbb{R}^n$ including at x_0, then $\alpha \in \mathbb{Z}$.*

Proof. In Cartesian coordinates, the azimuthal unit vector is $\hat{e}_\theta = (-y/r,\ x/r)$. The probability current near x_0 is

$$\mathbf{j} \sim \frac{\alpha\hbar}{m} r^{2|\alpha|} \cdot \frac{1}{r}\hat{e}_\theta = \frac{\alpha\hbar}{m}(x^2+y^2)^{|\alpha|-1}(-y, x) \tag{69}$$

The factor $(-y, x)$ is a smooth (linear) vector field on $\mathbb{R}^2$. The regularity of $\mathbf{j}$ at the origin therefore reduces to the smoothness of the scalar prefactor

$$f(x, y) = (x^2+y^2)^{|\alpha|-1} \tag{70}$$

By Lemma 13, $f \in C^\infty(\mathbb{R}^2)$ if and only if $|\alpha| - 1 \in \mathbb{N}_0$, i.e., $|\alpha| \in \{1, 2, 3, \ldots\}$.

The case $\alpha = 0$ corresponds to contractible loops, for which $\Gamma_C = 0 \in 2\pi\hbar\mathbb{Z}$ by Stokes' theorem. Therefore $\alpha \in \mathbb{Z}$ for all configurations. □

5.4.3 Combined Result: Quantization from Two Ingredients

Theorem 15 (Phase Quantization at Nodal Zeros)**.** *Let (ρ, S) satisfy the hydrodynamic equations (20)–(21) with $D = \hbar/(2m)$, where $\rho \geq 0$ may have isolated zeros. Suppose:*

(i) $S_{\mathrm{OM}}[\rho, S] < \infty$

(ii) (ρ, S) is a stationary point of the Onsager-Machlup functional

(iii) The probability current $\mathbf{j} = \rho\nabla S/m$ is a C^∞ vector field on all of Ω, including at nodal zeros

Then for any closed loop C that does not pass through a zero of ρ,

$$\oint_C \nabla S \cdot dl = 2\pi n\hbar, \qquad n \in \mathbb{Z} \tag{71}$$

Proof. Consider a closed loop C in $\{x : \rho(x) > 0\}$.

Case 1: C is contractible within $\{x : \rho(x) > 0\}$. Then Theorem 11 applies directly, giving $\Gamma_C \in 2\pi\hbar\mathbb{Z}$.

Case 2: C encloses one or more isolated zeros of ρ. By Lemma 12, the vanishing order of $\sqrt{\rho}$ at each zero equals $|\alpha_i|$ where α_i is the local winding number. By Proposition 14, each $\alpha_i \in \mathbb{Z}$. Since the total circulation decomposes as

$$\Gamma_C = \sum_i 2\pi\alpha_i\hbar \tag{72}$$

where the sum runs over zeros enclosed by C, we obtain $\Gamma_C \in 2\pi\hbar\mathbb{Z}$. □

Remark 6 (Physical Basis for Current Regularity). Assumption (iii) requires justification independent of the wave function ψ. The probability current $\mathbf{j} = \rho\nabla S/m$ is the fundamental observable of the hydrodynamic formulation:

- It determines all measurable flow properties
- It appears directly in the continuity equation $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$
- Its integral over surfaces gives particle detection rates

As an observable field on physical space $\mathbb{R}^n$, $\mathbf{j}$ must admit well-defined measurements at every point, including nodal zeros where no particle is present. This operational requirement translates mathematically to C^∞ regularity.

The Onsager-Machlup variational principle additionally requires well-defined first and second variations of the action for (ρ, S) to be a genuine stationary point, which demands smoothness of all fields entering the functional.

Note that this is *not* an assumption about ψ—it is a requirement on the classical observable $\mathbf{j}$, formulated entirely within the stochastic framework.

Remark 7 (Logical Independence of the Two Ingredients). It is crucial that neither ingredient alone implies quantization:

Ingredient 1 alone does not force quantization. The Hamilton-Jacobi equation is perfectly satisfied by $\alpha = 1/3$, $\beta = 1/3$, $\rho \sim r^{2/3}$. These are precisely the Reddiger-Poirier solutions [12]. Nothing in the dynamics excludes them.

Ingredient 2 alone does not force quantization. If the relation between β and α were free (not constrained by the Hamilton-Jacobi equation), one could have $\rho \sim r^{2k}$ with $k \in \mathbb{N}$ and arbitrary α. Then $\mathbf{j} \sim r^{2k-1}\hat{e}_\theta$, which can be smooth regardless of α. The constraint on α arises only through the dynamical link $\beta = |\alpha|$.

Only the combination forces $\alpha \in \mathbb{Z}$. The Hamilton-Jacobi equation binds $\beta = |\alpha|$ (dynamics). Current regularity requires $|\alpha| \in \mathbb{N}$ (physics of observables). Together: $\alpha \in \mathbb{Z}$.

Neither condition alone is equivalent to quantization, so the derivation is genuinely non-circular.

Remark 8 (Relation to Reddiger-Poirier Solutions). Reddiger and Poirier [12] explicitly constructed non-quantized strong solutions of the Madelung equations in two dimensions with $\alpha \notin \mathbb{Z}$. For such solutions, the winding number satisfies $\alpha = \beta$ (Ingredient 1) but $\alpha \notin \mathbb{Z}$. The probability current acquires the prefactor $(x^2+y^2)^{|\alpha|-1}$ with $|\alpha| - 1 \notin \mathbb{N}_0$. By Lemma 13, this current is not C^∞ at the nodal zero. In our framework, such configurations are excluded because they violate Ingredient 2 (current regularity).

We additionally provide a supporting argument via regularization, which establishes the quantization result through an independent topological route.

Proposition 16 (L^2_{loc} Convergence Under Regularization)**.** *Let $\{(\rho_\varepsilon, S_\varepsilon)\}_{\varepsilon>0}$ be a family of configurations satisfying $\rho_\varepsilon > 0$ everywhere, $S_{\mathrm{OM}}[\rho_\varepsilon, S_\varepsilon] \leq C$ uniformly, and $(\rho_\varepsilon, S_\varepsilon) \to (\rho, S)$ pointwise almost everywhere on $\{\rho > 0\}$. Then for any compact $K \subset \{\rho > 0\}$:*

$$\|\nabla S_\varepsilon - \nabla S\|_{L^2(K)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{73}$$

Consequently, for any closed loop $C \subset K$, the circulation $\Gamma_C \in 2\pi\hbar\mathbb{Z}$.

Proof. On K, compactness and $\rho > 0$ give $\rho \geq \delta_K > 0$ for some δ_K. The uniform action bound provides

$$\int \rho_\varepsilon |\nabla S_\varepsilon|^2 \, d^n x \leq 4Dm^2 C \tag{74}$$

For ε sufficiently small, $\rho_\varepsilon \geq \delta_K/2$ on K (by pointwise convergence), so

$$\int_K |\nabla S_\varepsilon|^2 \, d^n x \leq \frac{2}{\delta_K} \int_K \rho_\varepsilon |\nabla S_\varepsilon|^2 \, d^n x \leq \frac{8Dm^2 C}{\delta_K} \tag{75}$$

Thus $\{\nabla S_\varepsilon\}$ is uniformly bounded in $L^2(K)$. Combined with pointwise convergence $\nabla S_\varepsilon \to \nabla S$ a.e. on K, Vitali's convergence theorem gives $L^2(K)$ convergence.

For the circulation, let $C \subset K$ have parametrization $\gamma : [0,1] \to K$. By the Sobolev embedding $H^1([0,1]) \hookrightarrow L^\infty([0,1])$ applied along the curve:

$$\begin{aligned} |\Gamma_\varepsilon - \Gamma| &= \left| \oint_C (\nabla S_\varepsilon - \nabla S) \cdot dl \right| \\ &\leq L(C) \|\nabla S_\varepsilon - \nabla S\|_{L^\infty(C)} \to 0 \end{aligned} \tag{76}$$

Since each $\Gamma_\varepsilon \in 2\pi\hbar\mathbb{Z}$ (by Theorem 11 applied to $\rho_\varepsilon > 0$) and $\mathbb{Z}$ is discrete, Γ_ε is eventually constant. Therefore $\Gamma = \lim \Gamma_\varepsilon \in 2\pi\hbar\mathbb{Z}$. ☐

5.5 Corollary: Completeness of the Derivation

Corollary 17 (Schrödinger Equation from Onsager–Machlup)**.** *Under the calibration $D = \hbar/(2m)$ and the regularity assumptions stated in Theorems 11 and 15, the Onsager-Machlup variational principle uniquely determines quantum mechanical evolution via the Schrödinger equation.*

Proof. Combining Theorem 4 (hydrodynamic equations), Theorem 5 (Madelung transformation), Theorem 11 (quantization via flat connections), and Theorem 15 (extension to nodal configurations), we have:

$$\begin{aligned} \text{Onsager-Machlup} &\Rightarrow (\rho, S) \text{ with } \Gamma \in 2\pi\hbar\mathbb{Z} \\ &\Rightarrow \exists!\, \psi : i\hbar\partial_t \psi = \hat{H}\psi \end{aligned} \tag{77}$$

The wave function ψ exists, is unique (up to global phase), and satisfies the Schrödinger equation. The quantization emerges from geometric constraints (for $\rho > 0$) and the combined Hamilton-Jacobi plus current regularity requirements (at nodal zeros) rather than being postulated. ☐

5.6 Relation to L^2 Cohomology

The geometric reformulation reveals a subtle mathematical structure that clarifies the role of regularity assumptions.

Remark 9 (Continuous vs. Discrete Holonomy). On the punctured domain $\Omega^* = \Omega \setminus Z$, the space of flat U(1) connections modulo gauge transformations is classified by the first cohomology group $H^1(\Omega^*; \mathbb{R})$. For Ω simply-connected with k isolated punctures,

$$H^1(\Omega^*; \mathbb{R}) \cong \mathbb{R}^k \tag{78}$$

Each puncture contributes one real parameter: the holonomy $\alpha_i \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1)$ around that puncture.

Remark 10 (Finite Action and L^2 Conditions). The finiteness condition $S_{OM}[\rho, S] < \infty$ (assumption (i) in Theorem 15) implies that $\nabla S \in L^2_{\text{loc}}(\Omega^*)$ (Lemma 8). This constrains the connection $\omega = \nabla S/\hbar$ to lie in the L^2 de Rham cohomology class:

$$[\omega] \in H^1_{L^2}(\Omega^*) \cong \mathbb{R}^k \tag{79}$$

Importantly, L^2 cohomology classes are *continuous*: the holonomy parameters $\{\alpha_i\}$ can take any real values, not necessarily quantized.

Proposition 18 (Quantization from Current Regularity)**.** *The discretization $\alpha_i \in 2\pi\hbar\mathbb{Z}$ arises when we impose the requirement that the probability current $\mathbf{j} = \rho\nabla S/m$ extends as a C^∞ vector field over the entire domain, including nodal zeros.*

Proof. Without the current regularity requirement, the hydrodynamic equations (20)–(21) admit solutions with arbitrary holonomy $\alpha_i \in \mathbb{R}/2\pi\mathbb{Z}$ around each node. These are precisely the non-quantized solutions constructed by Reddiger and Poirier [12].

However, such solutions have singular currents at nodal zeros. By Lemma 12, the Hamilton-Jacobi equation forces $|\alpha| = \beta$ where $\rho \sim r^{2\beta}$. By Proposition 14, smooth extension of the current requires $|\alpha| \in \mathbb{N}$. The combination of these two independent conditions forces $\alpha \in \mathbb{Z}$, which by Lemma 10 implies quantized holonomy. ☐

Remark 11 (Physical Justification of Current Regularity). The requirement that $\mathbf{j} \in C^\infty(\Omega)$ is not arbitrary. It follows from the physical role of the current:

- $\mathbf{j}$ determines all measurable flow properties
- $\mathbf{j}$ appears directly in the continuity equation and its integral over surfaces gives detection rates
- The Onsager-Machlup variational principle requires well-defined first and second variations, demanding smoothness of all fields entering the action

This is a requirement on the classical observable $\mathbf{j}$, not on the wave function ψ. The logical structure is: dynamics (HJ equation) + physics of observables (current regularity) ⇒ quantization. Neither ingredient alone is equivalent to postulating single-valuedness of ψ.

Remark 12 (Comparison with Standard Quantum Mechanics). In standard textbook quantum mechanics, the quantization condition $\oint \nabla S \cdot dl = 2\pi n\hbar$ is typically *postulated* as a consequence of single-valuedness of $\psi = e^{iS/\hbar}$. Our derivation inverts this logic: quantization emerges from the combination of dynamical constraints (Hamilton-Jacobi equation) and regularity of observables (current smoothness), and the existence of ψ follows as a corollary (via the Madelung transformation). This resolves the Wallstrom objection by grounding quantization in two independent physical requirements, neither of which alone implies quantization.

6 Comparison with Previous Approaches

6.1 Nelson's Stochastic Mechanics

Nelson [1, 2] postulated forward and backward stochastic processes:

$$D_+x = b(x,t) = v + u \tag{80}$$
$$D_-x = b_*(x,t) = v - u \tag{81}$$

where $v = \nabla S/m$ (current velocity) and $u = (D/m)\nabla \log \rho$ (osmotic velocity). He then postulated a modified Newton's law:

$$m\frac{D(v+u)}{Dt} = -\nabla U \tag{82}$$

from which the Schrödinger equation follows.

Relation to Our Work: In our framework, the osmotic velocity emerges automatically from the quantum potential:

$$u = -\frac{1}{m}\nabla Q(\rho) = \frac{D}{m}\nabla \log \rho = \frac{\hbar}{2m^2}\nabla \log \rho \tag{83}$$

Thus Nelson's decomposition $b = v + u$ is a consequence of the Onsager-Machlup variational principle, not a postulate. Moreover, Nelson did not address the Wallstrom objection; he simply assumed single-valuedness of ψ.

6.2 Bohm's Quantum Potential

Bohm [5, 6] applied the Madelung transformation to the Schrödinger equation to obtain the quantum potential

$$Q_{Bohm} = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \tag{84}$$

where $R = \sqrt{\rho}$. He interpreted this as an additional force acting on particles with definite trajectories $x(t)$ satisfying $dx/dt = \nabla S/m$.

Relation to Our Work: We derive the identical quantum potential from the stochastic action functional. However, our ontology differs fundamentally:

- *Bohm:* Particles have definite positions; ψ is a "pilot wave."
- *Our framework:* The probability fluid ρ is fundamental; "particles" are informational structures within this fluid.

Critically, Bohm's approach is circular: he starts from the Schrödinger equation to derive Q. We derive both Q and the Schrödinger equation from a more fundamental principle.

6.3 Resolution of Wallstrom

Both Nelson and Bohm fail to address Wallstrom's objection rigorously. Our contribution is twofold: (i) Theorem 11 proves that topological stability enforces Eq. (2) for configurations with $\rho > 0$, and (ii) Theorem 15 extends this to nodal configurations by showing that the combination of the Hamilton-Jacobi constraint and current regularity independently forces integer winding numbers. This addresses the explicit non-quantized solutions constructed by Reddiger and Poirier [12], which satisfy the Hamilton-Jacobi equation but have singular currents at nodal zeros. Together, these results close the gap in stochastic derivations of quantum mechanics without additional axioms.

7 Experimental Predictions

7.1 Mesoscopic Transition and Environmental Noise

The calibration $D = \hbar/(2m)$ establishes quantum mechanics as a genuine stochastic process with physical diffusivity. This interpretation leads to testable predictions when environmental noise is present.

In a realistic experiment, the system couples to its environment (residual gas, thermal photons, electromagnetic noise). This coupling manifests as an additional diffusion term. The total effective diffusivity governing the probability fluid becomes

$$D_{eff} = D_q + D_{env} = \frac{\hbar}{2m} + D_{env} \tag{85}$$

where D_{env} characterizes the environmental noise.

Proposition 19 (Wave Packet Spreading)**.** *Environmental diffusivity contributes additively to the wave packet variance. For a Gaussian wave packet initially localized at x_0 with variance σ_0^2, the time evolution under combined quantum and environmental diffusion is*

$$\sigma^2(t) = \sigma_q^2(t) + 2D_{env}t \tag{86}$$

where $\sigma_q^2(t)$ is the standard quantum mechanical spreading.

Proof. The total stochastic process governing the system is:

$$dX_t = v(X_t,t)dt + \sqrt{2D_q}\, dW_t^{(q)} + \sqrt{2D_{env}}\, dW_t^{(env)} \tag{87}$$

where $W_t^{(q)}$ represents intrinsic quantum fluctuations and $W_t^{(env)}$ represents environmental noise.

Key Assumption: We assume $W_t^{(q)}$ and $W_t^{(env)}$ are independent Brownian motions. This is physically reasonable because quantum fluctuations arise from the vacuum substrate (calibrated by $\hbar$) while environmental noise arises from external degrees of freedom (gas collisions, thermal photons). These are distinct physical mechanisms with independent stochastic sources.

Under independence, the quadratic variation of the total noise term is:

$$\langle (dX_t - v\,dt)^2\rangle = 2D_q\,dt + 2D_{env}\,dt = 2(D_q + D_{env})\,dt \tag{88}$$

The Fokker-Planck equation for the probability density $\rho(x,t)$ is therefore:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(v\rho) + (D_q + D_{env})\nabla^2\rho \tag{89}$$

For a Gaussian wave packet, the variance evolves according to:

$$\frac{d\sigma^2}{dt} = 2D_{eff} + \frac{\langle(\nabla S)^2\rangle}{m^2} \tag{90}$$

where the first term is diffusive spreading and the second is quantum mechanical spreading from the phase gradient. Identifying $D_{eff} = D_q + D_{env}$ and noting that quantum spreading contributes $\sigma_q^2(t) = \sigma_0^2 + 2D_q t + \int_0^t \langle(\nabla S)^2\rangle/(m^2)\,dt'$, we obtain the stated result by separation. □

7.2 Interference Visibility

In a matter-wave interferometer, interference visibility is determined by the overlap between wave packets traveling along different paths. Environmental diffusion broadens each wave packet beyond the quantum minimum, reducing this overlap.

Physical Criterion: Interference visibility is significantly suppressed when the environmental diffusion length $L_{diff} = \sqrt{2D_{env}\tau}$ becomes comparable to the coherence length ℓ_c of the interferometer, where τ is the characteristic time for path separation.

For typical setups, ℓ_c is set by the spatial separation between paths. When $L_{diff} \sim \ell_c$, the wave packets spread sufficiently that their tails no longer overlap coherently, destroying the interference pattern.

Remark 13. This prediction is qualitatively distinct from standard decoherence theory. While standard quantum mechanics models decoherence as phase randomization, the stochastic framework predicts an additional spatial spreading of the probability distribution itself. Both effects coexist, but the diffusive contribution provides a characteristic signature: suppression scales with $\sqrt{D_{env}\tau}$ rather than decoherence time scales.

7.3 Candidate Systems

Promising experimental systems for testing Eq. (85) include:

1. Fullerenes in controlled backgrounds: Matter-wave interferometry with C_{60} or C_{70} molecules where background gas pressure can be varied systematically. The environmental diffusivity D_{env} scales with collision rate, allowing direct tests of the additive diffusion hypothesis.

2. Levitated nanoparticles: Optically trapped silica spheres (10–100 nm diameter) in ultra-high vacuum. Environmental contributions come from residual gas collisions and photon recoil. Both can be independently controlled and measured.

3. Superconducting circuits: SQUIDs with tunable coupling to engineered thermal baths. The effective mass can be varied electronically, testing the m-dependence in Eq. (25).

Remark 14 (Falsifiability). The key testable prediction is: increasing environmental isolation (reducing D_{env}) should proportionally sharpen interference patterns for a given mass. Deviations from this scaling would falsify the stochastic substrate hypothesis.

8 Conclusion

We have shown that the hydrodynamic formulation derived from the Onsager-Machlup functional admits only integer phase circulation around isolated nodes through a combination of geometric and regularity requirements.

The key result is that quantization does not require:

- postulating single-valuedness of a wave function,
- imposing ad hoc smoothness conditions,
- or introducing additional dynamical constraints.

Instead, quantization follows from two independent ingredients whose combination forces integer winding:

1. **Dynamics:** The Hamilton-Jacobi equation binds the vanishing order β to the winding number: $|\alpha| = \beta$.
2. **Observable regularity:** Smoothness of the probability current requires $|\alpha| \in \mathbb{N}$.

Together, these imply $\alpha \in \mathbb{Z}$, yielding quantization.

Neither condition alone implies quantization—the Reddiger-Poirier non-quantized solutions satisfy (1) but violate (2)—making the derivation genuinely non-circular.

Thus, the quantization condition

$$\oint \nabla S \cdot dl = 2\pi n\hbar \tag{91}$$

emerges from the combination of dynamical constraints and physical requirements on observables, rather than being postulated.

This resolves the Wallstrom objection within the Onsager-Machlup framework without postulating wave function single-valuedness. The logical structure is: dynamics + physics of observables ⇒ quantization, where neither ingredient alone is equivalent to the Wallstrom condition.

We emphasize that the result does not rely on additional topological input or on the existence of a global complex structure. It follows from combining two independent physical requirements, neither of which presupposes quantization.

Future work should extend this to:

- Relativistic field theory (covariant Onsager-Machlup functional)
- Many-body systems (configuration space topology and entanglement)
- Quantum gravity (stochastic geometry)

The physical interpretation—that reality is a probability fluid optimizing coherence—offers a resolution of foundational puzzles while making testable predictions for mesoscopic systems.

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