Canonical Nonlinear Partial Differential Equations

Luisiana X. Cundin

February 1, 2026

Abstract

A formal, systematic approach for generating nonlinear partial differential equations is outlined, which provides a more robust, reliable method. Additionally, formal methods provide a means to test the validity and/or the veracity of proposed nonlinear partial differential equations, thereby, potentially saving researchers precious time and effort.

1 Propædeutics

Linear partial differential equations can be formally classified as either hyperbolic, parabolic or elliptic equations after substituting the declared variables of a system with more general functions, where each function substituted is dependent upon all the dependent variables, thus, the chain rule is employed, leading to a more comprehensive equation. The purpose of a canonical transform includes removing terms containing mixed partial differentials; to change the coordinate system; forming generating functions under Hamilton-Jacobi equations/systems; finding characteristics; also, point transformations, often done in mechanics; and for other reasons still; nevertheless, the present reason for summoning a canonical title is to invoke a more formal, systematic approach, better still, a more logical approach to analyzing and investigating nonlinear partial differential equations.

The typical method for deriving nonlinear equations stems from an informal, ad-hoc method, where a modeler adds desired terms to an existing linear partial differential equation, such that, added nonlinear terms are thought to add to the existing linear model additional desired nonlinear effects; unfortunately, such sloppy, unrestrained methodology leaves much to be desired. Frankly, cherry-picking, divining or adding biased, unvetted terms are wholly subject to unintended erroneous representations. Virilly, such casual approaches to any scientific endeavour should altogether be viewed as invalid, if not dangerous. A methodical approach to generating nonlinear equations promises a more formal, systematic approach, thereby, diminishing error, highlighting fundamentals, and imbuing a more comprehensive understanding, thus, raising the confidence of researchers across the scientific community.

The present thesis focuses more upon formally applying a set of linear operators to a function that can be considered linear or otherwise. The means and methods presented, within the confines of this pithy thesis, for generating a particular class of nonlinear equations, are not, by any stretch of the imagination, the only means afforded an imaginative, creative researcher for generating nonlinear equations; albeit, the class of equations discussed do have wide application in nonlinear modeling, such as, fluid dynamics, population dynamics, optics, quantum mechanics, &c. And, even though a profound interest and delight can be found with any one particular application, for example, nonlinear optics, no one individual possesses sufficient amount of time, wherewithal and skill to thoroughly contemplate and apply the means and methods, to be outlined shortly enough in this thesis, to a sundry of physical phenomena. In the pursuit of scientific knowledge, it is not so much the individual that matters, rather, what matters most is the collective endeavour of all humanity in its pursuit of greater knowledge, skills and abilities. Consequently, I invite any young researcher to consider further investigations in, what is ostensibly, a brand new world, a world barely touched, moreover, as best as I am able to tell, a very fertile world with great promise for meaningful discoveries.

2 Means and Method

As mentioned, the present thesis will be very focused, thus, only parabolic equations will be considered. Now, it is not for any particular prejudice other classes of equations are avoided, nay, for the method outlined does promise mixing together different classes of equations, whether hyperbolic, etc, both linear and nonlinear. Parabolic functions are chosen due to their importance in modeling physical phenomena, especially, the concept of energy transference; secondly, by happenstance, this formal, canonical method reveals yet another proof as to why a certain class of nonlinear partial differential equations, namely, the Newell-Whitehead-Segel equation, yields a null solution, universally.

Consider a linear parabolic operator applied to a nonlinear function raised to an arbitrary power (n), *viz.*:

$$[\mathcal{L}]u^n \Rightarrow \frac{\partial}{\partial t}u^n - \nu\frac{\partial^2}{\partial x^2}u^n + \alpha u^n = 0 \tag{1}$$

Since Green's function is a known solution, the general solution for the nonlinear case is swiftly identified as the $n-$th root of the standard linear Green's solution, *viz.*:

$$u(x,t) = (Ge^{-\alpha t})^{1/n} \tag{2}$$

By inspection, in the limit of time to the origin, the solution will reduce to the $n-$th root of a Dirac Delta distribution, which is unity at the spatial origin and zero elsewhere, therefore, the root of the solution at the origin is irrelevant. For time greater than zero (t^+), Green's function will take arbitrary values ranging from zero to unity, furthermore, in the limit of the exponent n to infinity, all values would be raised to unity in the limit of infinite responsiveness. If n goes to zero, the limit bifurcates: 1) at the time origin, the spatial distribution accumulates, thus, the n-th root at the spatial origin, unity, is determinable; otherwise, 2) Green's function ranges from zero to some number less than unity for general time, hence, the infinite root becomes indeterminate, *viz.*:

$$\delta(x) \leftarrow \ldots \lim_{\substack{n\to 0\\ t=0}} \left(Ge^{-\alpha t}\right)^{1/n} \leftarrow \underset{n=1}{Ge^{-\alpha t}} \rightarrow \lim_{\substack{n\to\infty\\ \{t|t\in(0,\infty)\}}} \left(Ge^{-\alpha t}\right)^{1/n} \ldots \rightarrow 1 \tag{3}$$

So, what does it mean: the function (u^n) is meant to represent the response for a nonlinear system, thus, 1) in the limit of infinite sensitivity, $n \to \infty$, the medium or space instantaneously responds to any impulse given in the space, spreading and replicating that impulse throughout the entire space; 2) in the limit of zero sensitivity, $n = 0$, the space is completely unresponsive, save at the point of an impulse, caveat, only true at the time origin, otherwise, the response is indeterminate. Another perspective is to think of nonlinearity as memory, thus: 1) a highly sensitive medium would be instantly susceptible to any impulse, moreover, that impression would be *known* or *felt* throughout the space, immediately replicated or transferred throughout the entire domain, eternally remembered; 2) as for the case of dull memory or responsivity[1], the impulse is only known at the point of impact, nowhere else, thus, the space has absolutely no memory whatsoever; the indeterminacy for general time can be perceived as the model cannot predict if any energy is transferred from the point of impact, and, if so, how it is transferred. To be clear: think of the linear heat equation, whose solution is Green's solution, the diffusion constant, equally, the diffusivity of the medium is a measure of how susceptible, responsive and to what degree it remembers a past influence, thus, one may model a medium's receptiveness by either varying the diffusivity and/or medium response coefficient (α), or, by considering a nonlinear function (u^n). In the case of Schrödinger's equation: the introduction of an imaginary number forces the contemplation of a complex number in either limit, which is no simple task; but, the absolute modulus can be implemented to provide insight.

3 Method of validating nonlinear equations

A formal approach to generating nonlinear partial differential equations aids in producing far more meaningful, descriptive equations with non-zero solutions; additionally, when relating canonical nonlinear partial differential equations to more commonly encountered nonlinear equation, for example, the Newell-Whitehead-Segel equation, opaque topological pathologies become painfully obvious.

Consider applying a linear operator to a sum of functions, one a linear function, the latter possesses nonlinear responsiveness, modeled by raising to an arbitrary power (n). Since linear operators distribute over a series, a series of linear and nonlinear functions will form a set of linear and nonlinear partial differential equations, *viz.*:

$$\begin{aligned} &[\mathcal{L}](u + w^n) = \\ &\left(c_1 \frac{\partial}{\partial t} u - c_2 \frac{\partial^2}{\partial x^2} u + c_3 u\right) + \left(c_4 \frac{\partial}{\partial t} w^n - c_5 \frac{\partial^2}{\partial x^2} w^n + c_6 w^n\right) = 0 \end{aligned} \tag{4}$$

Since integrals distribute, each bracketed set of partial differential equation need be solved independently to satisfy the overall equation, where a homogeneous equation is presently considered. By inspection, the bracketed equations

[1]Responsivity measures the input–output gain of a detector system. For a system that responds linearly to its input, there is a unique responsivity. For nonlinear systems, the responsivity is the local slopes.

yield the solutions readily, *viz.*:

$$u = \frac{e^{-x^2/(\pi(c_2/c_1)t)}}{\sqrt{4\pi(c_2/c_1)t}} e^{-(c_3/c_1)t} \tag{5}$$

$$w = \left(\frac{e^{-x^2/(\pi(c_5/c_4)t)}}{\sqrt{4\pi(c_5/c_4)t}} e^{-(c_6/c_4)t} \right)^{1/n} , \tag{6}$$

both functions (u, w) are purposely labeled differently, otherwise, it can cause confusion, where the reader could misread one singular function and overlook a bracketed partial differential equation set, although, all labels may be changed at any point in the derivation, *inept*, function (w) may be relabeled function (u) at will.

Now, inspecting equation (4), if both differentials involving the nonlinear function (w) are zeroed out... the equation reduces to the well-known Newell-Whitehead-Segel equation. To achieve this reduction, the leading coefficients must be zeroed. Do keep in mind, the function (w) can always be relabeled function (u) after all manipulations are complete.

Consider the spatial gradient coefficient (c_5) some positive, real number, in the limit of the temporal coefficient (c_4) to zero, the w solution approaches zero; conversely, if the temporal coefficient remains positive and real and the diffusivity goes to zero, the w solution contracts to a delta-distribution in the limit of zero curvature $(c_5 \to 0)$. For the former case, the tangent with respect to time is forced to be everywhere zero, in other words, a flat plane is created in space [1]. In the latter case, the space is contracted to a singleton.

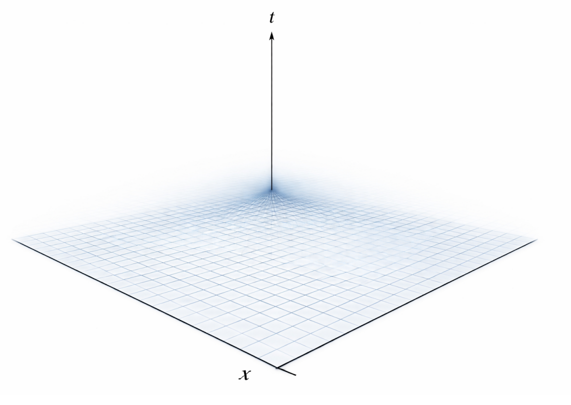

Figure 1: AI-generated schematic illustration of the Riemann surface associated with $u(x, t) = a\partial_t - v\partial_x^2$ in the limit $a, v \to 0$.

The above analysis is yet another proof the only possible solution to the Newell-Whitehead-Segel equation is the null solution, regardless of nonlinear exponent value. Taking a step back, equation (4) shows the impossibility of what is being demanded by the NWS equation, for after removing all partials involving the nonlinear function, the remaining solution demands a representation of a surface that is simultaneously flat and curved! If one peruses published solutions to the NWS equation, planar solutions are commonly reported, typically, very steep planes with deformations near the origin.

4 General Sol'n to Nonlinear Functions

Consider a general nonlinear function (u^n) operated upon by some linear operator ($\mathcal{L}$):

$$[\mathcal{L}]u^n = f \Rightarrow \frac{\partial}{\partial t}u^n - \nu\frac{\partial^2}{\partial x^2}u^n + \alpha u^n = f(x,t) \tag{7}$$

The general solution is the n-th root of Green's function, *viz.*:

$$u(x,t) = \left(Ge^{-\alpha t}\right)^{1/n} \tag{8}$$

Considering initial condition, $y(x,0)$, and forcing function (f); the particular solution is the following:

$$u(x,t) = y(x,0) * (Ge^{-\alpha t})^{1/n} + (Ge^{-\alpha t})^{1/n} * \int f(x,t) * (Ge^{-\alpha t})^{-1/n} dt, \tag{9}$$

where an asterisk ($*$) signifies spatial convolution.

It is possible to consider the reciprocal case, where the power of nonlinearity is an n-th root, *viz.*:

$$[\mathcal{L}]u^{1/n} = f \Rightarrow \frac{\partial}{\partial t}u^{\frac{1}{n}} - \nu\frac{\partial^2}{\partial x^2}u^{\frac{1}{n}} + \alpha u^{\frac{1}{n}} = f(x,t) \tag{10}$$

Similarly, the general solution is the n-th power of Green's function, *viz.*:

$$u(x,t) = \left(Ge^{-\alpha t}\right)^{n} \tag{11}$$

Considering initial condition, $y(x,0)$, and forcing function (f); the particular solution is the following:

$$u(x,t) = y(x,0) * (Ge^{-\alpha t})^{n} + (Ge^{-\alpha t})^{n} * \int f(x,t) * (Ge^{-\alpha t})^{-n} dt, \tag{12}$$

where an asterisk ($*$) signifies spatial convolution.

It is possible to generate an expanded form for a given nonlinear equation by simply operating through all derivatives, for example, equation (7) yields the following expanded form, after division, *viz.*:

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} - \nu(n-1)\frac{1}{u}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\alpha}{n}u = 0 \tag{13}$$

Take particular note of the third term, which is either the square of the spatial tangent or can be considered the curvature of the function (u) after distributing the square to the derivative, then divided by the unknown function. The above is true for the homogeneous case; in other words, division through an inhomogeneous equation by the unknown function would transfer to the forcing function.

5 Closing

The method outlined for formally generating nonlinear equations can be applied to higher dimensional equations with an appropriate linear operator, although, the solutions are similar, essentially, the two or three dimensional Green's solution is raised or rooted appropriately. In addition, the linear operator considered may be transformed to any desired coordinate system, thus, the solution is again Green's function of the coordinate system considered with requisite power. It is also possible to apply this method of generating nonlinear equations to both hyperbolic and elliptic operators. For hyperbolic equations, the particular solution requires the appropriate Wronskian, which would involve powered Green's solutions. In the case of elliptic equations, given the solutions are logarithmic in nature, I imagine the nonlinear solutions to be very rich and complex after an appropriate power is added for the solution.

Considerable time and effort can be wasted pursuing solutions to nonlinear equations with null solutions; fortunately, a method has been described for checking the validity of a given proposed nonlinear equation. In addition, the same method provides to young researchers a general method for generating new, meaningful nonlinear equations for a sundry of physical phenomena.

Lastly, it is important to explicitly state what is meant by nonlinear responsivity, summarily, a number added is often represented by the number occurrences times that number ($a + a \cdots = na$), and, when that number is added together equal to the number, the exponent form emerges ($aa = a^2$); thus, for a linear medium response (au), the leading coefficient (a) indicates the number of additional responses the medium presents to an impulse, further still, if a medium response is in powered fashion, exponential forms emerge (u^n).

References

[1] Luisiana Cundin. "Newell-Whitehead-Segel Equation: An Exact, Generalized Solution". In: (2024). arXiv: 2409.01501 [math-ph]. URL: https://arxiv.org/abs/2409.01501.