RG–Cauchy Summability for Blocked Observables in 4d Lattice Yang–Mills Theory via Balaban's Renormalization Group — a Conditional Summability Theorem*

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Abstract

We prove, conditionally on an explicit hypothesis ledger, that expectations of blocked, bounded Lipschitz observables at a fixed physical scale $\ell > 0$ form an absolutely summable telescoping sequence along a Balaban-matched renormalization trajectory in 4d $SU(N_c)$ lattice Yang–Mills theory with $a_k = a_0 2^{-k}$; in particular the continuum-limit state $\omega(O) = \lim_k \langle O^{(k)} \rangle$ exists on $\mathcal{A}_\ell^{\text{block}}$. The architecture is unchanged from v1 and is, we believe, correct: (i) an exact RG identity (law of iterated expectations — which, notably, resolves at the structural level the "on/off-vs-$k \to k+1$" gap flagged in the sibling audits: the one-step comparison here genuinely is a scale comparison); (ii) pushforward stability for blocked observables from approximate centering and Gaussian control of fast modes; (iii) measure comparison by Duhamel interpolation with influence control. Version 2 repairs the single broken brick: v1's Lemma 7.1 asserted the covariance bound for the Efron–Stein seminorm for arbitrary measures while proving the Doob martingale identity; per the sibling audits (2602.0070/0072 v2, same two-spin counterexample) the ES bound is false for non-product ν and the two seminorms are incomparable, so converting the ES-form input (B6) into the Doob-form covariance control requires the decoupling hypothesis (H-DEC/AT) (Dobrushin-type, verified on exact Gibbs chains at weak coupling). The main theorem is restated with the full ledger: Assumption 3.6 (blocking contraction, (H-LIP)), Assumption 5.1 with (B6) as the *conditional* Efron–Stein closure of 2602.0072 v2 and with $\sum_k \sqrt{\tau_k} < \infty$ in (B3) tied to the profile condition (H-θ) of 2602.0069 v2, plus (H-DEC/AT). Under this ledger, the one-step error is $O(4^{-k}) + O(\sqrt{\tau_k})$, absolutely summable, and Assumption 3.5 of 2602.0063 v3 — the RG–Cauchy hypothesis (H-CAUCHY), whose naive bridge is not summable (F-SUM) —

*Version 2. This paper is the "RG–Cauchy Master" of the UV sub-track, and its v1 sat at the exact junction of the two errors found in the sibling audits: its Lemma 7.1 stated the covariance bound for the *Efron–Stein* seminorm "for any ν" while proving it with *martingale (Doob) differences.* As adjudicated in 2602.0070 v2 and 2602.0072 v2, the two seminorms are incomparable for non-product measures and the ES covariance bound is false in general (two-spin counterexample: $\sigma_{\text{ES}}(X_2) = 0 < 1 = \text{Var}(X_2)$), while the Doob identity is exact but is not fed by the (B6) input, which is an ES bound. v2 repairs the junction: Lemma 7.1 is restated as the exact Doob identity plus the decoupling hypothesis (H-DEC/AT) needed to convert the ES input (B6) into Doob control; Theorem 9.2 carries the full explicit ledger. The remaining v1 inputs were already explicit (Assumptions 3.6 and 5.1) and are now cross-referenced to their audited suppliers: (B6) is the *conditional* closure of 2602.0072 v2; $\sum \sqrt{\tau_k} < \infty$ in (B3) is the profile condition (H-θ) of 2602.0069 v2; the companion references are updated from their withdrawn "unconditional" titles to the audited versions, and the cross-reference "Assumption 4.1 of [18]" is corrected to Assumption 3.5 of 2602.0063 v3. See the Note added in v2 (Section 13).

holds for the blocked class $\mathcal{A}_\ell^{\rm block}$. This is the cleanest conditional delivery of (H-CAUCHY) in the series: one decoupling hypothesis, one contraction input, one structural package, one profile condition. All mechanical steps (exact RG identity, Lipschitz iteration, pushforward stability in a Gaussian toy, telescoping arithmetic, and both counterexample adjudications at the Lemma 7.1 junction) are machine-verified in a companion suite.

Contents

1 Introduction

1.1 Motivation and scope

Along an RG trajectory extracted from the effective actions themselves (trajectory matching), do expectations of physical-scale observables form a Cauchy sequence as $a_k \to 0$? We answer affirmatively — *conditionally on the ledger of §1.3* — for bounded Lipschitz functionals of blocked link variables at fixed physical scale $\ell > 0$.

1.2 What is proved / not proved (v2 update)

Proved (conditional on the ledger): existence of the continuum-limit state on $\mathcal{A}_\ell^{\mathrm{block}}$ (Theorem 9.2); absolute summability of the telescoping series; delivery of (H-CAUCHY) (Assumption 3.5 of 2602.0063 v3, whose cross-reference v1 misprinted as "Assumption 4.1") for the blocked class.

Not proved: the ledger itself — (H-DEC/AT) (decoupling; the single genuinely probabilistic gap), (H-LIP) (= Assumption 3.6), the Balaban package (B1)–(B6) beyond its conditional/traceable status (2602.0069/0072 v2), and (H-θ) for $\{\tau_k\}$. Also not proved, as in v1: mass gap, complete OS reconstruction, unconditional Creutz ratios, infinite-volume limit.

1.3 The hypothesis ledger (new in v2)

Hypothesis (Ledger for Theorem 9.2)**.** *(i)* (H-DEC/AT)*: Dobrushin-type decoupling of the interpolating measures ν_t — either the conditional-oscillation leakage bound (H-DEC) of 2602.0070 v2 or the approximate tensorisation (H-AT) of 2602.0072 v2 (*$\mathrm{Var}_\nu \le C_{\mathrm{AT}} \sum_e E[\mathrm{Var}_{\nu_e}]$*), uniformly in k and t. Verified on exact Gibbs chains at weak coupling (*$C_{\mathrm{AT}} \approx 1.35$ *at* $J = 0.15$*); unproven for ν_t. (ii) Assumption 3.6 ((H-LIP)). (iii) Assumption 5.1 ((B1)–(B6)), with (B6) the conditional ES closure of 2602.0072 v2 and (B3)'s $\sum \sqrt{\tau_k} < \infty$ subject to (H-θ): profile exponent $\theta_0 > 1$ (2602.0069 v2, Remark 12.1 — with $\sqrt{\tau_k} = e^{-p_0/2}$ the trichotomy applies with $A_0/2$, so $\theta_0 > 1$ still suffices and $\theta_0 \le 1$ still fails). (iv) Trajectory matching (Definition 5.4).*

2 Lattice setup and Wilson measure

Standard: $\Lambda_k = (a_k\mathbb{Z}/L\mathbb{Z})^4$, Wilson measure μ_{β_k} at matched couplings; as in v1.

3 Blocking to a fixed physical scale

Definition 3.1 (Group projection π_G)**.** *Projection from a tubular neighborhood of G in* $\mathrm{GL}(N_c, \mathbb{C})$ *onto G; smooth on the small-field domain.*

Definition 3.2 (One-step blocking Q_k)**.** *Path-averaging of fine links along the doubled lattice followed by π_G; as in v1.*

Lemma 3.3 (Gauge covariance of blocking)**.** *Q_k intertwines gauge transformations: $Q_k(U^g) = Q_k(U)^{\bar{g}}$ with $\bar{g}$ the restricted transformation. (As in v1; mechanical.)*

Remark 3.4 (Small-field domain for π_G)**.** *On Ω^{sf} the averaged matrix lies in the tubular neighborhood, so π_G is smooth; encoded in Assumption 5.1(B1).*

Definition 3.5 (Physical-scale blocking $Q_{\ell,k}$)**.** *For $k \ge j_\ell$, $Q_{\ell,k} := Q_{j_\ell} \circ \cdots \circ Q_{k-1}$ (iterated blocking down to the scale-ℓ lattice).*

Assumption 3.6 (One-step blocking contraction on Ω^{sf} — (H-LIP))**.** *For every k, the one-step blocking is Lipschitz with constant 2^{-2} on the small-field domain: $d(Q_k(U), Q_k(U')) \le \frac{1}{4}\, d(U, U')$ in the blocked metric.* (v2) *Explicit input, as in v1; named (H-LIP) per the sibling audits. Suite check 3 verifies the iteration arithmetic* $\mathrm{Lip}(Q_{\ell,k}) \le (1/4)^{k-j_\ell}$ *it induces.*

Remark 3.7. *The power 2^{-2} encodes geometric damping per dyadic step; it is the engine of the $O(4^{-k})$ one-step rate.*

4 Blocked observables

Definition 4.1 (Blocked observable algebra $\mathcal{A}_\ell^{\text{block}}$)**.** *$O^{(k)}(U) := F(Q_{\ell,k}(U))$ with gauge-invariant $F : \mathcal{C}_{j_\ell} \to \mathbb{R}$, $\|F\|_\infty \le 1$, $\mathrm{Lip}(F) < \infty$, for $k \ge k_* := \max\{k_0, j_\ell\}$.*

Example 4.2. *Smeared blocked plaquettes; blocked Wilson loops in a fixed physical region.*

Remark 4.3 (What is excluded)**.** *Creutz ratios and logarithms/ratios of expectations: treated only conditionally in §11.*

5 Balaban RG input and the matched trajectory

Assumption 5.1 (Balaban RG input (minimal form); v2: statuses attached)**.** *There exist $g_* > 0$, $C, c > 0$ such that for all k (with $g_0^2 \le g_*^2$): **(B1)** small/large-field decomposition [status: traceable, 2602.0069 v2]; **(B2)** background-field parametrization with approximate centering $\|E_{\text{fast}}[\phi \,|\, \bar{U}]\| \le C g_{k+1}^2$ [input; Gaussian-toy verified mechanics, suite check 4]; **(B3)** Gaussian control of fast modes with truncation errors $\{\tau_k\}$, $\sum_k \sqrt{\tau_k} < \infty$ [*conditional on (H-θ): $\theta_0 > 1$, *2602.0069 v2]; **(B4)** effective measure with $S_k^{\text{eff}} = \beta_k S_W + V_k^{\text{irr}}$ [traceable]; **(B5)** small large-field mass $\le \tau_k$ [same status as (B3)]; **(B6)** uniform influence control $\sup_t \sigma_{\nu_t}(V_k^{\text{irr}}) \le C$ in the* Efron–Stein *seminorm [*unconditional as an ES seminorm bound, but its covariance use is conditional*: 2602.0072 v2].*

Remark 5.2 (Comments on the input)**.** *Assumption 5.1 isolates the minimal RG ingredients; v2 adds their audited statuses. None is proved here.*

Remark 5.3 (Why $\sqrt{\tau_k}$ appears — and the F-SQRT sharpening)**.** *The pushforward comparison produces τ_k under a square root, so (H-θ) applies to $e^{-p_0/2}$: the effective amplitude is $A_0/2$. The trichotomy of 2602.0069 v2 is formally unchanged ($\theta_0 > 1$ suffices, $\theta_0 < 1$ fails),* but the halved amplitude has a quantitative sting*: at the chain's representative polylog floor ($\theta_0 = 1.1$, $A_0 = 1$) the convergence crossover of $\sum \sqrt{\tau_k}$ occurs only beyond $j \gtrsim e^{(2(1+\epsilon)/A_0)^{1/(\theta_0-1)}} \sim e^{2^{10}}$ — formally convergent, practically divergent (suite check 5 shows the partial sums still growing at $N = 10^5$, while $A_0 = 6$ converges visibly). Realistically, (B3) therefore requires a power-law-strength penalty profile — i.e. (H-P0) rejoins the ledger through the back door unless A_0 is large. This sharpening (F-SQRT) is new in v2.*

Definition 5.4 (Balaban-matched trajectory)**.** *$\{\beta_k\}$ such that the extracted Wilson coefficient at each step matches the input coupling of the next (one-step matching condition).*

Remark 5.5 (No quantitative AF rate needed)**.** *Only $g_k \le g_*$ on the regime where Assumption 5.1 holds; consistent with the window bookkeeping of 2602.0063 v3 (Lemma 1.4 there: the dyadic trajectory fits the audited window).*

6 Lemma A1: exact RG identity

Definition 6.1 (RG operator)**.** *$(\mathcal{R}_k f)(\bar{U}) := E[f \,|\, Q_k(\cdot) = \bar{U}]$ under the fine measure.*

Lemma 6.2 (Exact RG identity)**.** *For bounded measurable f on the fine lattice: $\langle f\rangle_{k+1} = \langle \mathcal{R}_k f\rangle_k^{\mathrm{eff}}$ — the law of iterated expectations.* (v2) *Exact; suite check 1. Structurally, this is what makes the present one-step comparison a genuine* scale *comparison, resolving the "on/off-vs-k→k+1" identification gap flagged in 2602.0070/0072 v2: the Duhamel step below compares the exact effective measure with its Wilson truncation* at the same scale*, and Lemma 6.2 transports the result across scales.*

7 Lemma A3: measure comparison by interpolation

Lemma 7.1 (Covariance bound by influence — v2 CORRECTED)**.** *(a)* Exact (Doob form): *for any ν and $f, g \in L^2(\nu)$, $\mathrm{Cov}_\nu(f,g) = \sum_i E[\Delta_i f\, \Delta_i g]$, whence $|\mathrm{Cov}_\nu(f,g)| \le \sigma_\nu^{\mathrm{Doob}}(f)\, \sigma_\nu^{\mathrm{Doob}}(g)$. (b)* Conditional (Efron–Stein form): *under* (H-AT)*, $|\mathrm{Cov}_\nu(f,g)| \le C_{\mathrm{AT}}\, \sigma_\nu^{\mathrm{ES}}(f)\, \sigma_\nu^{\mathrm{ES}}(g)$.* (v1 stated (b) without C_{AT} "for any ν" and proved (a): the martingale-difference proof sketch delivers the Doob identity, not the ES bound. The two seminorms are incomparable for non-product ν — the two-spin counterexample of the sibling audits gives $\sigma_{\mathrm{ES}}(X_2) = 0 < 1 = \mathrm{Var}(X_2) = \sigma^{\mathrm{Doob}}(X_2)^2$ — so (b) genuinely needs (H-AT); this was the single broken brick of v1. Suite check 2.)

Lemma 7.2 (Duhamel interpolation bound (influence form); conditional)**.** *Assume Assumption 5.1(B6)* and (H-AT)*. With $\nu_t \propto e^{-\beta_k S_W - t V_k^{\mathrm{irr}}}$ on Ω_k^{sf}:*

$$\big|\langle f\rangle_{\nu_1} - \langle f\rangle_{\nu_0}\big| \;\le\; C_{\mathrm{AT}} \left(\sup_t \sigma_{\nu_t}^{\mathrm{ES}}(f)\right)\left(\sup_t \sigma_{\nu_t}^{\mathrm{ES}}(V_k^{\mathrm{irr}})\right). \tag{1}$$

Lemma 7.3 (Large-field truncation error)**.** $|\langle f\rangle_\mu - \langle f\rangle_{\mu(\cdot|\Omega^{\mathrm{sf}})}| \le 2\|f\|_\infty \mu(\Omega^{\mathrm{lf}}) \le 2\|f\|_\infty \tau_k$. *(As in v1; exact.)*

Lemma 7.4 (Physical-scale Lipschitz bound for iterated blocking)**.** *Under Assumption 3.6:* $\mathrm{Lip}(Q_{\ell,k}) \le (1/4)^{k-j_\ell} = (a_k/a_{j_\ell})^2$*; hence per-link oscillations of $f = F \circ Q_{\ell,k}$ are damped by* $\mathrm{Lip}(F)(a_k/\ell)^2$*-type factors. (Suite check 3.)*

Lemma 7.5 (Influence bound for blocked observables)**.** *With Lemma 7.4 and the link-count bookkeeping: $\sup_t \sigma_{\nu_t}^{\mathrm{ES}}(f) \le C\,\mathrm{Lip}(F)\,(a_k/\ell)^2 \cdot (\text{volume factor})$. Combined with (B6) and (1):*

$$\big|\langle F(Q_{\ell,k})\rangle_{\nu_1} - \langle F(Q_{\ell,k})\rangle_{\nu_0}\big| \;\le\; C\, C_{\mathrm{AT}}\,\mathrm{Lip}(F)\left(\frac{a_k}{\ell}\right)^2. \tag{2}$$

8 Lemma A2′: pushforward stability

Lemma 8.1 (Mean-square blocking stability)**.** *Under (B2)–(B3), on Ω_k^{sf} the blocked variable moves by $O(g_{k+1}^2) + O(g_{k+1}\sqrt{\cdot})$ in mean square under the fast integration. (Mechanics verified in a Gaussian toy: suite check 4.)*

Proposition 8.2 (Lemma A2′: pushforward stability)**.** *Under Assumption 5.1, for $f = F \circ Q_{\ell,k}$: $|\langle \mathcal{R}_k f\rangle_k^{\mathrm{eff}} - \langle f'\rangle_k^{\mathrm{eff}}| \le C\,\mathrm{Lip}(F)\,(a_k/\ell)^2 (g_k^2 + \sqrt{\tau_k})$, where f' is the scale-k representative. (As in v1, with the same proof; the $\sqrt{\tau_k}$ is where Remark 5.3 enters.)*

9 Main theorem

Remark 9.1 (Why blocking removes the logarithmic obstruction)**.** *For general observables the one-step error is $O(g_k^2) \sim c/k$ — not summable (this is exactly F-SUM of 2602.0063 v3). For blocked observables the contraction (H-LIP) supplies the extra $(a_k/\ell)^2$: geometric. As in v1, now with the audited cross-reference.*

Theorem 9.2 (RG–Cauchy summability for blocked observables; v2: conditional on the ledger)**.** *Assume the ledger of §1.3:* (H-DEC/AT)*, Assumption 3.6, Assumption 5.1 (with its v2 statuses), and trajectory matching. Then for every $O = \{F \circ Q_{\ell,k}\} \in \mathcal{A}_\ell^{\text{block}}$:*

$$\sum_{k \geq k_*} \left| \langle O^{(k+1)} \rangle_{k+1} - \langle O^{(k)} \rangle_k \right| \;\leq\; C \,\mathrm{Lip}(F) \sum_{k \geq k_*} \left[\left(\frac{a_k}{\ell} \right)^2 + \sqrt{\tau_k} \right] \;<\; \infty, \tag{3}$$

so $\omega(O) := \lim_k \langle O^{(k)} \rangle_k$ exists. Consequently, Assumption 3.5 of 2602.0063 v3 ((H-CAUCHY)) holds for the blocked class $\mathcal{A}_\ell^{\text{block}}$ under this ledger *— the cleanest conditional delivery of (H-CAUCHY) in the series.* (v1 asserted this with Lemma 7.1 unconditioned and cited it as "Assumption 4.1 of [18]"; both corrected.)

10 Uniformity in volume (outline)

Assumption 10.1 (Uniformity in volume)**.** *As in v1: locality/uniformity input for the $L \to \infty$ limit.*

Proposition 10.2 (Infinite-volume limit (conditional))**.** *Under Theorem 9.2 and Assumption 10.1. (As in v1.)*

11 Conditional extension to renormalized observables

Definition 11.1 (Renormalized class (schematic))**.** *As in v1.*

Remark 11.2. *Intentionally schematic; Creutz-type expressions require the Symanzik input below. Cf. the Layer-2 treatment of 2602.0063 v3.*

Assumption 11.3 (Polymer-to-local operator map (Symanzik input))**.** *As in v1; unproven input.*

Theorem 11.4 (Conditional convergence for the renormalized class)**.** *Under the ledger plus Assumption 11.3. (As in v1, with the ledger substituted for the bare Assumption 5.1.)*

12 Discussion: OS reconstruction and the mass gap

As in v1: OS positivity of the lattice theory is standard; the blocked limit state's RP and reconstruction are treated in 2602.0063 v3 (conditional); the mass gap does not follow here (it requires the transfer-gap input, 2602.0054 v2, conditional on (H-DOB-blk)+(H-P0)). Ratios and logarithms of expectations require strictly positive denominators in the limit — conditional.

13 Note added in v2

The series audit reached this paper last, with two counterexamples already in hand (2602.0070/0072 v2). Findings:

- **F-JUNCTION.** v1's Lemma 7.1 stated the ES covariance bound for arbitrary ν and proved the Doob identity; the ES form is false in general and the seminorms are incomparable (the two-spin counterexample applies verbatim at both ends). Corrected: Lemma 7.1(a) exact (Doob), (b) conditional on (H-AT); Lemma 7.2 and Theorem 9.2 carry the hypothesis. [Suite check 2]

- **What the architecture gets right** (and the audit confirms): the exact RG identity makes the one-step comparison a genuine scale comparison — the "on/off-vs-$k \to k+1$" gap flagged in the siblings is resolved *structurally* here [check 1]; the (H-LIP) contraction supplies the geometric $(a_k/\ell)^2$ that defeats F-SUM for blocked observables [check 3]; the telescoping arithmetic is exact [check 5].

- **Ledger consolidation.** (B6) $\to$ conditional ES closure (2602.0072 v2: unconditional as a seminorm bound, conditional in its covariance use); (B3)/(B5) $\sum \sqrt{\tau_k} < \infty \to$ (H-θ) with $A_0/2$ (2602.0069 v2) [check 5]; blocking contraction $\to$ (H-LIP); decoupling $\to$ (H-DEC/AT) [checks 2, 6]. Under the ledger, (H-CAUCHY) of 2602.0063 v3 holds for $\mathcal{A}_\ell^{\text{block}}$ — v1's cross-reference "Assumption 4.1 of [18]" corrected to Assumption 3.5.

- **Reference hygiene.** v1's [13] ("Unconditional uniform LSI. . . "), [15] (". . . unconditional mass gap"), [17] (". . . unconditional closure. . . ") cited withdrawn v1 titles — all updated to the audited versions (2602.0041 v3, 0051–0057 v2, 0063 v3, 0069/0070/0072 v2). With this replacement, *every paper of the programme is audited*: no unaudited companion remains.

Suite: `verify_2602_0073.py` (6 checks) at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, directory `verification/2602-0073/`.

A Absence of dimension-5 operators

As in v1 (symmetry orientation; no hypothesis load).

B Dimension-6 operators (orientation)

As in v1.

C A metric Lipschitz inequality (no barycenters)

As in v1; mechanics folded into suite check 4.

References

[1] T. Bałaban, Comm. Math. Phys. **95** (1984)–**122** (1989) (the eleven-paper program; precise equation-level citations in ai.viXra:2602.0069 (v2)).

[2] J. Dimock, arXiv:1108.1335; 1212.5562; 1304.0705.

[3] L. Eriksson, *The Bałaban–Dimock structural package ... — a traceability and conditional-discharge package*, ai.viXra:2602.0069 (v2), 2026.

[4] L. Eriksson, *Doob influence bounds ... — a corrected and conditional influence bound*, ai.viXra:2602.0070 (v2), 2026.

[5] L. Eriksson, *Influence bounds ... — an unconditional Efron–Stein bound and a conditional (B6) closure*, ai.viXra:2602.0072 (v2), 2026.

[6] L. Eriksson, *Conditional continuum limit of 4d* $SU(N_c)$ *Yang–Mills theory ...*, ai.viXra:2602.0063 (v3), 2026. [(H-CAUCHY) = Assumption 3.5 there; F-SUM; window compatibility.]

[7] L. Eriksson, ai.viXra:2602.0041 (v3); 2602.0046; 2602.0051–0057 (all v2), 2026. [The audited infrared/LSI chain; open hypotheses (H-P0), (H-YGZ), (H-DOB-blk), (H-SFI), (H-ABS), window.]

[8] K. Osterwalder, E. Seiler, Ann. Physics **110** (1978), 440–471; E. Seiler, LNP 159, Springer, 1982.