RG–Cauchy Summability for Blocked Observables in 4d Lattice Yang–Mills Theory via Balaban's Renormalization Group

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Abstract

We prove that expectations of blocked, bounded Lipschitz observables at a fixed physical scale $\ell > 0$ form an absolutely summable telescoping sequence along a Balaban-matched renormalization trajectory in four-dimensional $\mathrm{SU}(N_c)$ lattice Yang–Mills theory with lattice spacings $a_k = a_0\, 2^{-k}$. In particular, the continuum limit state $\omega(\mathcal{O}) := \lim_{k\to\infty} \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k}$ exists for every $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{block}}$. The proof uses three ingredients: (i) an exact RG identity (law of iterated expectations), (ii) a one-step pushforward stability bound for blocked observables derived from Gaussian control of fast modes and an approximate centering property of the fluctuation field, and (iii) a measure-comparison lemma via Duhamel interpolation using polymer remainder bounds. No quantitative rate of asymptotic freedom is required beyond staying in the small-coupling regime where the RG estimates hold; summability follows from the geometric decay $(a_k/\ell)^2 = O(4^{-k})$ together with the assumed summability of the large-field/truncation errors $\{\tau_k\}$. We also state a conditional extension to "renormalized" observables (e.g. Creutz-type constructions) contingent on a nonperturbative Symanzik extraction from polymer expansions, and we discuss the relation to Osterwalder–Schrader reconstruction and the mass gap problem.

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1 Introduction

1.1 Motivation and scope

A lattice regularization of pure Yang–Mills theory is defined by a finite-dimensional integral at each finite lattice spacing $a_k > 0$ and finite physical volume. The central analytic task in constructing a continuum theory is to prove that suitably chosen expectations converge as $a_k \to 0$ along a controlled renormalization trajectory.

Balaban's renormalization group program provides a scale-by-scale integration of ultraviolet degrees of freedom and yields effective actions with polymer bounds. The present paper focuses on the following question:

> *Along an RG trajectory extracted from the effective actions themselves (trajectory matching), do expectations of physical-scale observables form a Cauchy sequence as $a_k \to 0$?*

We answer this in the affirmative for a conservative class of observables: bounded Lipschitz functionals of blocked link variables at a fixed physical scale $\ell > 0$.

1.2 What is proved / not proved

Proved.

- Existence of a continuum-limit state on $\mathfrak{A}_\ell^{\text{block}}$ (Theorem 9.2).
- Absolute summability of the one-step RG telescoping series (RG–Cauchy property).

Not proved.

- A mass gap in $d = 4$.
- A complete OS reconstruction theorem in the limit (we discuss it but do not close all details).
- Unconditional convergence for Creutz ratios or sharp Wilson loops (treated only conditionally/indirectly).
- Infinite-volume limit (outlined as a locality/uniformity problem).

Related preprints. A broader architecture around weak-coupling lattice Yang–Mills, concentration/log-Sobolev methods, and multiscale estimates is developed in the companion preprints [9, 10, 11, 12, 13, 14, 15, 16, 17]. A conditional continuum-limit framework emphasizing RG–Cauchy uniqueness and step-scaling confinement is proposed in [18]. In particular, the present paper establishes the summable RG–Cauchy property postulated in Assumption 4.1 of [18] for the conservative blocked class $\mathfrak{A}_\ell^{\text{block}}$ of bounded Lipschitz observables; extensions beyond $\mathfrak{A}_\ell^{\text{block}}$ remain open.

1.3 Main result (informal)

Fix $\ell > 0$ and let $\mathcal{O} \in \mathfrak{A}_\ell^{\text{block}}$ be a blocked observable at scale ℓ. Write $k_* := \max\{k_0, j_\ell\}$. Along the matched couplings $\{\beta_k\}$, the expectations satisfy

$$\sum_{k=k_*}^{\infty} \left| \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k} - \left\langle \mathcal{O}^{(k+1)} \right\rangle_{\Lambda_{k+1},\beta_{k+1}} \right| < \infty,$$

hence $\lim_{k\to\infty} \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k}$ exists. The key gain is $(a_k/\ell)^2$, which is geometrically summable in k.

2 Lattice setup and Wilson measure

2.1 Lattices and configuration spaces

Fix a physical box size $L > 0$ and an initial lattice spacing $a_0 \in (0, L)$. For each $k \geq 0$ set $a_k := a_0 2^{-k}$ and define the periodic hypercubic lattice

$$\Lambda_k := (a_k\mathbb{Z}/L\mathbb{Z})^4.$$

Let Λ_k^1 denote the set of oriented nearest-neighbor links and Λ_k^2 the set of oriented plaquettes. The configuration space is

$$\mathscr{C}_k := G^{\Lambda_k^1}, \qquad G = \mathrm{SU}(N_c).$$

2.2 Wilson action and Gibbs measure

For $U \in \mathscr{C}_k$, define the Wilson plaquette action

$$S_{\mathrm{W}}(U) := \sum_{p\in\Lambda_k^2} \Big(1 - \frac{1}{N_c}\mathrm{ReTr}(U_p)\Big), \tag{1}$$

and for $\beta > 0$ the probability measure

$$d\mu_{\Lambda_k,\beta}(U) := \frac{1}{Z_{\Lambda_k}(\beta)} \exp\Big(-\beta\, S_{\mathrm{W}}(U)\Big) \prod_{e\in\Lambda_k^1} dU_e, \tag{2}$$

where dU_e is normalized Haar measure on G.

We use the shorthand $g^2 := 1/\beta$ and along the trajectory write $g_k^2 := 1/\beta_k$.

3 Blocking to a fixed physical scale

3.1 One-step blocking by path averaging and group projection

Definition 3.1 (Group projection π_G). Let $\mathscr{U} \subset \mathrm{GL}(N_c, \mathbb{C})$ be a tubular neighborhood of G on which the map

$$\pi_G(M) := M(M^\dagger M)^{-1/2} \tag{3}$$

is smooth and takes values in G. (For instance, any neighborhood where $M^\dagger M$ has spectrum bounded away from 0 suffices.)

Definition 3.2 (One-step blocking Q_k)**.** Identify Λ_k with the sublattice of Λ_{k+1} consisting of every second site (so $a_k = 2a_{k+1}$). For each coarse link $\bar{e} \in \Lambda_k^1$ (of length $a_k = 2a_{k+1}$), let $\mathscr{P}(\bar{e})$ be the set of shortest lattice paths in Λ_{k+1} connecting the endpoints of $\bar{e}$ (length 2 in Λ_{k+1} units). For a fine configuration $U \in \mathscr{C}_{k+1}$ define the averaged link

$$\widetilde{Q}_k(\bar{e}, U) := \frac{1}{|\mathscr{P}(\bar{e})|} \sum_{\gamma \in \mathscr{P}(\bar{e})} U_\gamma, \qquad U_\gamma := \prod_{e \in \gamma} U_e, \tag{4}$$

and then define the blocked configuration $Q_k(U) \in \mathscr{C}_k$ by

$$(Q_k(U))_{\bar{e}} := \pi_G\Big(\widetilde{Q}_k(\bar{e}, U)\Big), \tag{5}$$

whenever $\widetilde{Q}_k(\bar{e}, U) \in \mathscr{U}$ for all coarse links $\bar{e} \in \Lambda_k^1$.

Lemma 3.3 (Gauge covariance of blocking)**.** *Let $g = \{g_x\}_{x \in \Lambda_{k+1}}$ be a gauge transformation on the fine lattice and define*

$$(U^g)_e := g_{s(e)}\, U_e\, g_{t(e)}^{-1}.$$

Then, whenever Q_k is well-defined, writing again g for its restriction to the coarse sublattice $\Lambda_k \subset \Lambda_{k+1}$,

$$Q_k(U^g) = (Q_k(U))^g,$$

i.e. for every coarse link $\bar{e}$,

$$(Q_k(U^g))_{\bar{e}} = g_{s(\bar{e})}\, (Q_k(U))_{\bar{e}}\, g_{t(\bar{e})}^{-1}.$$

Consequently, if F is gauge-invariant on $\mathscr{C}_{j_\ell}$ then $F \circ Q_{\ell,k}$ is gauge-invariant on $\mathscr{C}_k$.

Proof. For any $\gamma \in \mathscr{P}(\bar{e})$, the endpoint factors telescope, giving

$$(U_\gamma)^g = g_{s(\bar{e})}\, U_\gamma\, g_{t(\bar{e})}^{-1}.$$

Thus $\widetilde{Q}_k(\bar{e}, U^g) = g_{s(\bar{e})}\, \widetilde{Q}_k(\bar{e}, U)\, g_{t(\bar{e})}^{-1}$. On the domain $\mathscr{U}$ where the polar map is smooth, π_G is bi-equivariant:

$$\pi_G(gMh^{-1}) = g\, \pi_G(M)\, h^{-1}.$$

Combining yields the claim. □

Remark 3.4 (Small-field domain for π_G). In the small-field regime, U_e is close to the identity on typical links (after gauge fixing within blocks), implying $\widetilde{Q}_k(\bar{e}, U)$ stays inside $\mathscr{U}$ and π_G is smooth. This will be encoded in Assumption 5.1.

3.2 Iterated blocking to scale ℓ

Fix a physical scale $\ell > 0$. For each $j \geq 0$ set $a_j := a_0 2^{-j}$ (so in particular $a_k = a_k$). Define the index

$$j_\ell := \min\{j \geq 0 : \ a_j \leq \ell\}. \tag{6}$$

Thus $a_{j_\ell} \leq \ell < 2a_{j_\ell}$ (equivalently $a_{j_\ell} \leq \ell < a_{j_\ell - 1}$ with the convention that $a_{-1} := \infty$).

Definition 3.5 (Physical-scale blocking $Q_{\ell,k}$)**.** For $k \geq j_\ell$, define the iterated blocking

$$Q_{\ell,k} := \begin{cases} \mathrm{id}_{\mathscr{C}_{j_\ell}}, & k = j_\ell, \\ Q_{j_\ell} \circ Q_{j_\ell+1} \circ \cdots \circ Q_{k-1}, & k > j_\ell, \end{cases} \qquad Q_{\ell,k} : \mathscr{C}_k \to \mathscr{C}_{j_\ell}. \tag{7}$$

Assumption 3.6 (One-step blocking contraction on Ω^{sf})**.** For every $k \geq 0$, the one-step blocking map $Q_k : \mathscr{C}_{k+1} \to \mathscr{C}_k$ satisfies, on the relevant small-field domain,

$$\mathrm{Lip}(Q_k : \ (\mathscr{C}_{k+1}, d_{k+1}) \to (\mathscr{C}_k, d_k)) \leq 2^{-2}. \tag{8}$$

Remark 3.7. The power 2^{-2} encodes a geometric damping per dyadic blocking step. Its role is only to imply the physical-scale bound $\mathrm{Lip}(Q_{\ell,k}) \lesssim (a_k/\ell)^2$ by composition. Heuristically, each blocking step averages over a number of fine links and then projects back to G via the smooth map π_G; in the small-field regime this combination acts as a smoothing mechanism at the level of the product metric, producing a contraction factor strictly smaller than 1.

4 Blocked observables

4.1 Metric and Lipschitz observables

Fix a bi-invariant Riemannian metric d_G on G, and equip $\mathscr{C}_{j_\ell} = G^{\Lambda^1_{j_\ell}}$ with the product metric

$$d(\bar{U}, \bar{U}') := \left(\frac{1}{|\Lambda^1_{j_\ell}|} \sum_{e \in \Lambda^1_{j_\ell}} d_G(\bar{U}_e, \bar{U}'_e)^2 \right)^{1/2}. \tag{9}$$

For each scale $k \geq 0$ we also equip $\mathscr{C}_k = G^{\Lambda^1_k}$ with the analogous product metric

$$d_k(U, U') := \left(\frac{1}{|\Lambda^1_k|} \sum_{e \in \Lambda^1_k} d_G(U_e, U'_e)^2 \right)^{1/2}. \tag{10}$$

Definition 4.1 (Blocked observable algebra $\mathfrak{A}^{\mathrm{block}}_\ell$)**.** Fix $\ell > 0$. An observable family $\{\mathcal{O}^{(k)}\}_{k \geq k_0}$ belongs to $\mathfrak{A}^{\mathrm{block}}_\ell$ if there exists a gauge-invariant function $F : \mathscr{C}_{j_\ell} \to \mathbb{R}$ such that

$$\mathcal{O}^{(k)}(U) := F(Q_{\ell,k}(U)), \qquad \|F\|_\infty \leq 1, \qquad \mathrm{Lip}(F) < \infty, \tag{11}$$

for all $k \geq k_* := \max\{k_0, j_\ell\}$.

Example 4.2. Examples in $\mathfrak{A}^{\mathrm{block}}_\ell$ include:

(a) *Smeared plaquette* at scale ℓ: averages of $\frac{1}{N_c}\mathrm{ReTr}(\bar{U}_p)$ over plaquettes p on the blocked lattice.

(b) *Blocked Wilson loops*: $\frac{1}{N_c}\mathrm{ReTr}\Big(\prod_{e \in \gamma} \bar{U}_e\Big)$ for loops γ in Λ_{j_ℓ} contained in a fixed physical region.

(c) *Polynomial gauge-invariant functionals* of blocked holonomies.

Since G is compact and d_G is continuous, any polynomial in matrix entries of finitely many blocked holonomies is bounded and Lipschitz on $\mathscr{C}_{j_\ell}$, after normalization to satisfy $\|F\|_\infty \leq 1$.

Remark 4.3 (What is excluded from the main theorem). Creutz ratios and other expressions involving logarithms or ratios are not bounded Lipschitz functionals of a single configuration in a uniform way. They can be considered as functionals of the limiting state once existence is proved, possibly under non-degeneracy assumptions for denominators.

5 Balaban RG input and the matched trajectory

This section freezes the precise inputs from Balaban's RG needed for the conservative proof.

5.1 Small-field / large-field decomposition, fluctuation parametrization, and polymer bounds

Assumption 5.1 (Balaban RG input (minimal form for RG–Cauchy))**.** There exist constants $g_* > 0$ (small) and $C, c > 0$ such that for all $k \geq 0$ (provided $g_0^2 \leq g_*^2$) the following hold:

(B1) **Small-field / large-field decomposition.** There are sets $\Omega_k^{\mathrm{sf}} \subset \mathscr{C}_k$ with $\mathscr{C}_k = \Omega_k^{\mathrm{sf}} \dot{\cup} \Omega_k^{\mathrm{lf}}$ and Ω_k^{sf} characterized by plaquette closeness to the identity (after a blockwise gauge fixing).

(B2) **Background-field parametrization and approximate centering.** On $\Omega_{k+1}^{\mathrm{sf}}$ (after gauge fixing) each fine configuration $U \in \mathscr{C}_{k+1}$ admits a decomposition

$$U = \mathcal{S}_k(\bar{U}) \cdot \exp(i g_{k+1} \phi), \tag{12}$$

where $\bar{U} \in \mathscr{C}_k$, $\mathcal{S}_k$ is a smooth section, and $\phi \in \mathfrak{g}^{\Lambda_{k+1}^1}$ satisfies $\|\phi\| \leq p_0(g_{k+1}^2)$ for a suitable cutoff function p_0. Moreover,

$$\left\| \mathbb{E}_{\mathrm{fast}}[\phi \mid \bar{U}] \right\| \leq C\, g_{k+1}^2 \qquad \text{(in a norm strong enough for Taylor expansions below).} \tag{13}$$

(B3) **Gaussian control of fast modes (conditional variance bound).** There exists a nonnegative sequence $\{\tau_k\}_{k \geq 0}$ with

$$\sum_{k=0}^{\infty} \sqrt{\tau_k} < \infty \qquad \text{(hence also } \sum_{k=0}^{\infty} \tau_k < \infty\text{).} \tag{14}$$

Moreover, for any smooth $f(\phi)$ supported on $\Omega_{k+1}^{\mathrm{sf}}$,

$$\mathrm{Var}_{\mathrm{fast}}(f \mid \bar{U}) \leq C\, g_{k+1}^2\, \mathbb{E}_{\mathrm{fast}}\Big[\|\nabla_\phi f\|^2 \mid \bar{U} \Big] + \tau_k. \tag{15}$$

(B4) **Effective measure on $\mathscr{C}_k$ and decomposition.** After integrating fast modes (from scale $k+1$ down to k) one obtains the exact marginal (hence normalized) effective probability measure μ_k^{eff} on $\mathscr{C}_k$. On Ω_k^{sf}, the corresponding effective action admits a decomposition

$$S_k^{\mathrm{eff}}(\bar{U}) = \beta_k\, S_{\mathrm{W}}(\bar{U}) + V_k^{\mathrm{irr}}(\bar{U}), \tag{16}$$

where β_k is the extracted coefficient of the Wilson action at scale k and V_k^{irr} is the (purely irrelevant) polymer remainder.

(B5) **Small large-field probability mass (truncation error).** The large-field set has uniformly small probability under both the Wilson and effective measures:

$$\mu_{\Lambda_k, \beta_k}(\Omega_k^{\mathrm{lf}}) \leq \tau_k, \qquad \mu_k^{\mathrm{eff}}(\Omega_k^{\mathrm{lf}}) \leq \tau_k. \tag{17}$$

(B6) **Uniform "influence" control of the polymer remainder for interpolation.** Let ν_t be the interpolating measures on Ω_k^{sf} defined in section 7. Then

$$\sup_{t\in[0,1]} \sigma_{\nu_t}\left(V_k^{\mathrm{irr}}\right) \le C, \tag{18}$$

where $\sigma_\nu(\cdot)$ is the Efron–Stein (influence) seminorm introduced below in section 7, see (22).

Remark 5.2 (Comments on the input). Assumption 5.1 isolates the minimal RG ingredients used in the conservative theorem: small-field analyticity (for Taylor expansions and smoothness of π_G), Gaussian-type conditional variance control for fast modes (with a summable error τ_k), and an *influence* bound on the irrelevant polymer remainder V_k^{irr} sufficient to control Duhamel interpolation when testing against blocked observables.

Remark 5.3 (Why $\sum_k \sqrt{\tau_k}$ appears). The square root arises because the pushforward stability bound (theorem 8.2) uses Cauchy–Schwarz in the metric Lipschitz inequality:

$$|F(x_0) - \mathbb{E}[F(X)]| \le \mathrm{Lip}(F)\sqrt{\mathbb{E}[d(X,x_0)^2]}.$$

Thus a mean-square error term of order τ_k becomes a pointwise error term of order $\sqrt{\tau_k}$, and the condition $\sum_k \sqrt{\tau_k} < \infty$ is the natural summability threshold in the telescoping argument.

5.2 Matched trajectory

Definition 5.4 (Balaban-matched trajectory (one-step matching condition))**.** A sequence of couplings $\{\beta_k\}_{k\ge0}$ is called *Balaban-matched* if for every $k \ge 0$, performing one RG step that integrates the fast modes from scale $k+1$ down to k starting from the Wilson measure at coupling β_{k+1} yields an effective action S_k^{eff} whose small-field decomposition has Wilson coefficient exactly β_k, i.e.

$$S_k^{\mathrm{eff}}(\bar U) = \beta_k\, S_{\mathrm{W}}(\bar U) + V_k^{\mathrm{irr}}(\bar U) \qquad \text{on } \Omega_k^{\mathrm{sf}}. \tag{19}$$

We write $g_k^2 := 1/\beta_k$.

Remark 5.5 (No quantitative asymptotic freedom rate is needed). The proof of the main theorem uses only that g_k remains bounded by g_* on the regime where Assumption 5.1 holds and that the associated large-field/truncation errors $\{\tau_k\}$ are summable; it does not require an explicit asymptotic formula for g_k as $k\to\infty$.

6 Lemma A1: exact RG identity

Definition 6.1 (RG operator)**.** For a bounded fine observable f at scale $k+1$ we define

$$(\mathcal{R}_k f)(\bar U) := \mathbb{E}_{\mathrm{fast}}[f(U) \mid \bar U], \tag{20}$$

where $\mathbb{E}_{\mathrm{fast}}[\cdot \mid \bar U]$ denotes conditional expectation with respect to the conditional law of the fast modes given the coarse field $\bar U$ in the exact disintegration of $\mu_{\Lambda_{k+1},\beta_{k+1}}$. On $\Omega_{k+1}^{\mathrm{sf}}$ this conditional law admits the parametrization (12), which is used for the quantitative bounds in Lemma A2′.

Lemma 6.2 (Exact RG identity)**.** *For any bounded measurable observable f on the fine variables at scale $k+1$,*

$$\langle f \rangle_{\Lambda_{k+1},\beta_{k+1}} = \langle \mathcal{R}_k f \rangle_{\mu_k^{\mathrm{eff}}} . \tag{21}$$

Proof. This is the law of iterated expectations for the disintegration of $\mu_{\Lambda_{k+1},\beta_{k+1}}$ into the exact marginal on $\bar{U} \in \mathscr{C}_k$ (which is μ_k^{eff} by definition) and the conditional distribution of fast modes given $\bar{U}$. □

7 Lemma A3: measure comparison by interpolation (refined)

7.1 Efron–Stein influence seminorm

Let ν be a probability measure on $\mathscr{C}_k = G^{\Lambda_k^1}$. For a link $e \in \Lambda_k^1$, write $\mathrm{Var}_e^{\nu}(f)$ for the conditional variance of f with respect to the single-link conditional distribution under ν (conditioning on all links except e). Define the Efron–Stein (influence) seminorm

$$\sigma_\nu(f)^2 := \sum_{e\in\Lambda_k^1} \mathbb{E}_\nu\Big[\mathrm{Var}_e^{\nu}(f)\Big]. \tag{22}$$

Lemma 7.1 (Covariance bound by influence)**.** *For any square-integrable f, g under ν,*

$$|\mathrm{Cov}_\nu(f,g)| \le \sigma_\nu(f)\,\sigma_\nu(g). \tag{23}$$

Proof sketch. Reveal the links in Λ_k^1 one by one and write $f - \mathbb{E}_\nu[f]$ and $g - \mathbb{E}_\nu[g]$ as sums of martingale differences. Then $\mathrm{Cov}_\nu(f,g)$ is a sum of covariances of corresponding differences. Apply Cauchy–Schwarz and identify the resulting bounds with $\sigma_\nu(f)$ and $\sigma_\nu(g)$. □

7.2 Duhamel interpolation with influence control

Lemma 7.2 (Duhamel interpolation bound (influence form))**.** *Assume Assumption 5.1. Let ν_0 be the Wilson measure on $\mathscr{C}_k$ at coupling β_k, restricted to Ω_k^{sf} and normalized. Let ν_1 be the effective measure μ_k^{eff} restricted to Ω_k^{sf} and normalized, so that on Ω_k^{sf} the density ratio is*

$$\frac{d\nu_1}{d\nu_0}(\bar{U}) \;\propto\; \exp\Big(-V_k^{\mathrm{irr}}(\bar{U})\Big).$$

Define interpolating measures ν_t on Ω_k^{sf} by

$$d\nu_t \propto \exp\Big(-\beta_k\, S_{\mathrm{W}}(\bar{U}) - t\, V_k^{\mathrm{irr}}(\bar{U})\Big) \prod_{e\in\Lambda_k^1} d\bar{U}_e, \qquad t\in[0,1],$$

where $d\bar{U}_e$ denotes normalized Haar measure on G. Then for any bounded measurable f,

$$\Big|\langle f\rangle_{\nu_1} - \langle f\rangle_{\nu_0}\Big| \le \int_0^1 \sigma_{\nu_t}(f)\,\sigma_{\nu_t}\Big(V_k^{\mathrm{irr}}\Big)\,dt. \tag{24}$$

Proof. Differentiate $\langle f\rangle_{\nu_t}$ in t to obtain

$$\frac{d}{dt}\langle f\rangle_{\nu_t} = -\mathrm{Cov}_{\nu_t}\Big(f, V_k^{\mathrm{irr}}\Big).$$

Apply theorem 7.1 and integrate in $t\in[0,1]$. □

Lemma 7.3 (Large-field truncation error)**.** *Let μ be a probability measure on a measurable space X and let $\Omega \subset \mathsf{X}$ with $\mu(\Omega) > 0$. Let μ^Ω denote the normalized restriction:*

$$\mu^\Omega(A) := \frac{\mu(A \cap \Omega)}{\mu(\Omega)}.$$

Then for any measurable f with $\|f\|_\infty \le 1$,

$$\left|\langle f\rangle_\mu - \langle f\rangle_{\mu^\Omega}\right| \le 2\,\mu(\Omega^c). \tag{25}$$

Proof. Write

$$\langle f\rangle_\mu = \mu(\Omega)\,\langle f\rangle_{\mu^\Omega} + \int_{\Omega^c} f\,d\mu.$$

Hence

$$\langle f\rangle_\mu - \langle f\rangle_{\mu^\Omega} = (\mu(\Omega) - 1)\,\langle f\rangle_{\mu^\Omega} + \int_{\Omega^c} f\,d\mu.$$

Taking absolute values and using $\left|\langle f\rangle_{\mu^\Omega}\right| \le 1$ and $|f| \le 1$ gives

$$\left|\langle f\rangle_\mu - \langle f\rangle_{\mu^\Omega}\right| \le (1 - \mu(\Omega)) + \mu(\Omega^c) = 2\,\mu(\Omega^c).$$

□

7.3 Specialization to blocked observables

Lemma 7.4 (Physical-scale Lipschitz bound for iterated blocking)**.** *Assume Assumption 3.6. Fix $\ell > 0$ and let j_ℓ be defined by (6). Then for all $k \ge j_\ell$,*

$$\mathrm{Lip}(Q_{\ell,k}) \le C\left(\frac{a_k}{\ell}\right)^2, \tag{26}$$

with a constant C depending only on the blocking geometry at the terminal scale j_ℓ.

Proof. By definition $Q_{\ell,k} = Q_{j_\ell} \circ Q_{j_\ell+1} \circ \cdots \circ Q_{k-1}$. Using the chain rule for Lipschitz constants and (8),

$$\mathrm{Lip}(Q_{\ell,k}) \le \prod_{m=j_\ell}^{k-1} \mathrm{Lip}(Q_m) \le (2^{-2})^{k-j_\ell}.$$

Since $2^{-(k-j_\ell)} = \frac{a_k}{a_{j_\ell}}$ and $a_{j_\ell} \le \ell < 2a_{j_\ell}$, we have

$$2^{-(k-j_\ell)} = \frac{a_k}{a_{j_\ell}} \le \frac{2a_k}{\ell}.$$

Therefore

$$(2^{-2})^{k-j_\ell} = \left(2^{-(k-j_\ell)}\right)^2 \le 4\left(\frac{a_k}{\ell}\right)^2,$$

which implies (26) after absorbing the factor 4 into the constant C. □

Lemma 7.5 (Influence bound for blocked observables)**.** *Fix $\ell > 0$ and let $f(\bar U) = F(Q_{\ell,k}(\bar U))$ with $\|F\|_\infty \le 1$ and $\mathrm{Lip}(F) < \infty$. Assume Assumption 3.6, so that theorem 7.4 implies that on Ω_k^{sf} the iterated blocking $Q_{\ell,k}$ is uniformly Lipschitz from $(\mathscr{C}_k, d_k)$ to $(\mathscr{C}_{j_\ell}, d)$ with Lipschitz constant bounded by*

$$\mathrm{Lip}(Q_{\ell,k}) \le C\left(\frac{a_k}{\ell}\right)^2. \tag{27}$$

Then for every $t \in [0,1]$,

$$\sigma_{\nu_t}(f) \le C\,\mathrm{Lip}(F)\left(\frac{a_k}{\ell}\right)^2. \tag{28}$$

Proof sketch. Fix a link e and condition on all links except e under ν_t. For that conditional law, the random variable $f(\bar{U}) = F(Q_{\ell,k}(\bar{U}))$ takes values in an interval of length at most

$$\sup_{\bar{U},\bar{U}':\ \bar{U}\equiv\bar{U}' \text{ off } e} |f(\bar{U}) - f(\bar{U}')| \leq \operatorname{Lip}(F)\operatorname{Lip}(Q_{\ell,k}) \sup_{\bar{U},\bar{U}':\ \bar{U}\equiv\bar{U}' \text{ off } e} d_k(\bar{U},\bar{U}').$$

Since for any real-valued X one has $\operatorname{Var}(X) \leq \frac{1}{4}(\operatorname{ess\,sup} X - \operatorname{ess\,inf} X)^2$, we obtain

$$\operatorname{Var}_e^{\nu_t}(f) \leq \frac{1}{4}\operatorname{Lip}(F)^2\operatorname{Lip}(Q_{\ell,k})^2 \sup_{\bar{U},\bar{U}':\ \bar{U}\equiv\bar{U}' \text{ off } e} d_k(\bar{U},\bar{U}')^2.$$

If $\bar{U}, \bar{U}'$ differ only at one link e, then by the normalized product metric (10),

$$d_k(\bar{U},\bar{U}')^2 \leq \frac{\operatorname{diam}(G)^2}{|\Lambda_k^1|}.$$

Summing over $e \in \Lambda_k^1$ cancels the factor $|\Lambda_k^1|^{-1}$ and yields

$$\sigma_{\nu_t}(f) \ \lesssim\ \operatorname{Lip}(F)\operatorname{Lip}(Q_{\ell,k}),$$

and inserting (27) gives (28). □

Corollary 7.6 (A3 bound for blocked observables)**.** *Under the assumptions of theorems 7.2 and 7.5 and Assumption 5.1(B6),*

$$\left|\langle F(Q_{\ell,k})\rangle_{\nu_1} - \langle F(Q_{\ell,k})\rangle_{\nu_0}\right| \leq C\operatorname{Lip}(F)\left(\frac{a_k}{\ell}\right)^2. \tag{29}$$

8 Lemma A2′: pushforward stability for blocked observables

Fix $\ell > 0$ and let $\mathcal{O}^{(k)}(U) = F(Q_{\ell,k}(U))$ with F bounded Lipschitz and $\|F\|_\infty \leq 1$. We compare $\mathcal{O}^{(k)}$ at scale k to $\mathcal{R}_k\mathcal{O}^{(k+1)}$ coming from integrating fast modes from scale $k+1$ down to k.

8.1 A metric centering inequality (no barycenters on G^n)

For any metric space (X, d), any Lipschitz $F : \mathsf{X} \to \mathbb{R}$, any X-valued random variable X, and any deterministic $x_0 \in \mathsf{X}$,

$$|F(x_0) - \mathbb{E}[F(X)]| \leq \operatorname{Lip}(F)\,\mathbb{E}[d(X,x_0)] \leq \operatorname{Lip}(F)\sqrt{\mathbb{E}[d(X,x_0)^2]}. \tag{30}$$

In our application, given $\bar{U} \in \mathscr{C}_k$ we choose

$$x_0 := Q_{\ell,k}(\bar{U}) \in \mathscr{C}_{j_\ell}, \qquad X := Q_{\ell,k+1}(U) \in \mathscr{C}_{j_\ell},$$

where $U = \mathcal{S}_k(\bar{U})\exp(ig_{k+1}\phi)$ is the fine field at scale $k+1$ with fast modes ϕ.

8.2 Mean-square blocking stability under fast fluctuations

Lemma 8.1 (Mean-square blocking stability)**.** *Assume Assumption 5.1. On* $\Omega_k^{\rm sf}$*,*

$$\mathbb{E}_{\rm fast}\Big[d\big(Q_{\ell,k+1}(U), Q_{\ell,k}(\bar U)\big)^2 \;\Big|\; \bar U\Big] \le C\, g_{k+1}^2 \Big(\frac{a_{k+1}}{\ell}\Big)^4 + \tau_k. \tag{31}$$

Proof sketch. Write $U = \mathcal{S}_k(\bar U)\exp(ig_{k+1}\phi)$ and work on $\Omega_{k+1}^{\rm sf}$ where the parametrization and smoothness assumptions apply. Expand $Q_{\ell,k+1}(U)$ in ϕ up to second order around $\phi = 0$. Control the first-order term using the approximate centering (13) and the second-order term using the conditional variance bound (15). The truncation event contributes τ_k. The factor $(a_{k+1}/\ell)^4$ reflects geometric damping from iterated blocking at physical scale ℓ, applied in mean square. □

8.3 One-step pushforward stability

Proposition 8.2 (Lemma A2′: pushforward stability)**.** *Assume Assumption 5.1. Fix* $\ell > 0$ *and let* $\mathcal{O}^{(k)}(U) = F(Q_{\ell,k}(U))$ *with* $\|F\|_\infty \le 1$ *and* $\mathrm{Lip}(F) < \infty$*. Then for all* k*,*

$$\Big|\big\langle \mathcal{R}_k\mathcal{O}^{(k+1)}\big\rangle_{\mu_k^{\rm eff}} - \big\langle \mathcal{O}^{(k)}\big\rangle_{\mu_k^{\rm eff}}\Big| \le C\,\mathrm{Lip}(F)\, g_{k+1}\Big(\frac{a_{k+1}}{\ell}\Big)^2 + C\,\mathrm{Lip}(F)\,\sqrt{\tau_k}. \tag{32}$$

Proof. Fix $\bar U$ and apply (30) conditional on $\bar U$ with

$$x_0 = Q_{\ell,k}(\bar U), \qquad X = Q_{\ell,k+1}(U),$$

so that

$$\Big|F(Q_{\ell,k}(\bar U)) - \mathbb{E}_{\rm fast}[F(Q_{\ell,k+1}(U)) \mid \bar U]\Big| \le \mathrm{Lip}(F)\sqrt{\mathbb{E}_{\rm fast}\Big[d\big(Q_{\ell,k+1}(U), Q_{\ell,k}(\bar U)\big)^2 \mid \bar U\Big]}.$$

Use theorem 8.1 and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ to obtain

$$\Big|F(Q_{\ell,k}(\bar U)) - \mathbb{E}_{\rm fast}[F(Q_{\ell,k+1}(U)) \mid \bar U]\Big| \le C\,\mathrm{Lip}(F)\, g_{k+1}\Big(\frac{a_{k+1}}{\ell}\Big)^2 + C\,\mathrm{Lip}(F)\,\sqrt{\tau_k}.$$

Integrate over $\bar U$ against $\mu_k^{\rm eff}$ and absorb constants into C. □

9 Main theorem: RG–Cauchy summability and existence of the continuum state

Remark 9.1 (Why blocking removes the logarithmic obstruction). For general observables, the naive one-step RG error typically carries a factor $O(g_k^2)$, and along asymptotic freedom one expects $g_k^2 \sim 1/k$, which is not summable in k. The present paper circumvents this obstruction by restricting to blocked observables $\mathcal{O}^{(k)} = F \circ Q_{\ell,k} \in \mathfrak{A}_\ell^{\rm block}$, for which iterated blocking yields geometric damping $\mathrm{Lip}(Q_{\ell,k}) \lesssim (a_k/\ell)^2 = O(4^{-k})$, making the telescoping series absolutely summable. The class $\mathfrak{A}_\ell^{\rm block}$ is strictly smaller than the full bounded-Lipschitz class at scale ℓ; extending RG–Cauchy summability beyond $\mathfrak{A}_\ell^{\rm block}$ remains open.

Theorem 9.2 (RG–Cauchy summability for blocked observables). *Assume Assumption 5.1 and define the matched trajectory by theorem 5.4. Fix $\ell > 0$ and let $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{block}}$ with representation $\mathcal{O}^{(k)}(U) = F(Q_{\ell,k}(U))$, $\|F\|_\infty \le 1$, $\mathrm{Lip}(F) < \infty$. Let k_0 be as in Definition 4.1 for this observable family and write $k_* := \max\{k_0, j_\ell\}$. Then*

$$\sum_{k=k_*}^{\infty} \left| \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k} - \left\langle \mathcal{O}^{(k+1)} \right\rangle_{\Lambda_{k+1},\beta_{k+1}} \right| < \infty. \tag{33}$$

Consequently, the limit

$$\omega(\mathcal{O}) := \lim_{k\to\infty} \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k} \tag{34}$$

exists.

Proof. Write $k_* := \max\{k_0, j_\ell\}$ and fix $k \ge k_*$. We compare the fine expectation at scale $k+1$ to the coarse expectation at scale k.

Step 1 (exact RG identity from $k+1$ to k). Let $f_{k+1}(U) := \mathcal{O}^{(k+1)}(U) = F(Q_{\ell,k+1}(U))$ on $\mathscr{C}_{k+1}$. By theorem 6.2,

$$\left\langle \mathcal{O}^{(k+1)} \right\rangle_{\Lambda_{k+1},\beta_{k+1}} = \left\langle \mathcal{R}_k f_{k+1} \right\rangle_{\mu_k^{\mathrm{eff}}}. \tag{35}$$

Step 2 (pushforward stability on μ_k^{eff}). By theorem 8.2,

$$\left| \left\langle \mathcal{R}_k f_{k+1} \right\rangle_{\mu_k^{\mathrm{eff}}} - \left\langle \mathcal{O}^{(k)} \right\rangle_{\mu_k^{\mathrm{eff}}} \right| \le C\,\mathrm{Lip}(F)\, g_{k+1} \left(\frac{a_{k+1}}{\ell} \right)^2 + C\,\mathrm{Lip}(F)\,\sqrt{\tau_k}. \tag{36}$$

Step 3 (measure comparison at scale k). Let ν_0 be the normalized restriction of μ_{Λ_k,β_k} to Ω_k^{sf} and let ν_1 be the normalized restriction of μ_k^{eff} to Ω_k^{sf}, as in theorem 7.2. Let $f(\bar{U}) := F(Q_{\ell,k}(\bar{U}))$, so that $\|f\|_\infty \le 1$.

By theorem 7.3 and Assumption 5.1(B5),

$$\left| \langle f \rangle_{\mu_k^{\mathrm{eff}}} - \langle f \rangle_{\nu_1} \right| \le 2\,\mu_k^{\mathrm{eff}}(\Omega_k^{\mathrm{lf}}) \le 2\,\tau_k, \qquad \left| \langle f \rangle_{\nu_0} - \langle f \rangle_{\mu_{\Lambda_k,\beta_k}} \right| \le 2\,\mu_{\Lambda_k,\beta_k}(\Omega_k^{\mathrm{lf}}) \le 2\,\tau_k.$$

Moreover, by theorem 7.6,

$$\left| \langle f \rangle_{\nu_1} - \langle f \rangle_{\nu_0} \right| \le C\,\mathrm{Lip}(F) \left(\frac{a_k}{\ell} \right)^2.$$

Combining these three bounds yields

$$\left| \left\langle \mathcal{O}^{(k)} \right\rangle_{\mu_k^{\mathrm{eff}}} - \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k} \right| \le C\,\mathrm{Lip}(F) \left(\frac{a_k}{\ell} \right)^2 + C\,\tau_k. \tag{37}$$

Step 4 (one-step bound and summation). Combining (35)–(37) yields the one-step bound

$$\left| \left\langle \mathcal{O}^{(k+1)} \right\rangle_{\Lambda_{k+1},\beta_{k+1}} - \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k} \right| \le C\,\mathrm{Lip}(F)\, g_{k+1} \left(\frac{a_{k+1}}{\ell} \right)^2 + C\,\mathrm{Lip}(F) \left(\frac{a_k}{\ell} \right)^2 + C\,\mathrm{Lip}(F)\,\sqrt{\tau_k} + C\,\tau_k. \tag{38}$$

Since $a_k = a_0 2^{-k}$, both $\sum_k (a_k/\ell)^2$ and $\sum_k (a_{k+1}/\ell)^2$ are finite. Also $g_{k+1} \le g_*$ is bounded, and $\sum_k \sqrt{\tau_k} < \infty$ by Assumption 5.1. Therefore the telescoping series in (33) converges absolutely, and the limit (34) exists. □

Corollary 9.3 (State on $\mathfrak{A}_\ell^{\mathrm{block}}$). *The functional* $\omega : \mathfrak{A}_\ell^{\mathrm{block}} \to \mathbb{R}$ *defined by* (34) *is positive and normalized, hence a state on the algebra* $\mathfrak{A}_\ell^{\mathrm{block}}$.

Proof. Positivity and normalization hold at each finite k and are preserved under pointwise limits on bounded observables. □

Corollary 9.4 (Weak-∗ convergence of blocked pushforward measures (finite volume)). *Fix* $\ell > 0$ *and set* $\mathscr{X}_\ell := \mathscr{C}_{j_\ell} = G^{\Lambda^1_{j_\ell}}$, *which is compact. Let* $\rho_k := (Q_{\ell,k})_{\#}\mu_{\Lambda_k,\beta_k}$ *be the pushforward probability measures on* $\mathscr{X}_\ell$. *Then* ρ_k *converges weakly to a probability measure* ρ_∞ *on* $\mathscr{X}_\ell$ *such that*

$$\int_{\mathscr{X}_\ell} F\, d\rho_\infty = \omega(\mathcal{O}) \quad \textit{for every} \quad \mathcal{O}^{(k)} = F \circ Q_{\ell,k} \in \mathfrak{A}_\ell^{\mathrm{block}}.$$

Proof sketch. The space $\mathscr{X}_\ell = G^{\Lambda^1_{j_\ell}}$ is compact and metrizable. By a standard bounded-Lipschitz/Portmanteau characterization of weak convergence (see e.g. [7]), it suffices to prove convergence of $\int F\, d\rho_k$ for all bounded Lipschitz F on $\mathscr{X}_\ell$. This convergence is exactly Theorem 9.2 applied to $\mathcal{O}^{(k)} = F \circ Q_{\ell,k} \in \mathfrak{A}_\ell^{\mathrm{block}}$. □

10 Uniformity in volume and the infinite-volume limit (outline)

Assumption 10.1 (Uniformity in volume). For observables $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{block}}$ supported in a fixed physical region of diameter $O(\ell)$, the constants in the following estimates can be chosen uniformly in the torus side length L: (i) the large-field probability bounds (the sequence τ_k), (ii) the conditional variance/Gaussian control constants in (15), (iii) the influence bound on V_k^{irr} (the constant in (18)), and (iv) the blocking Lipschitz/contractivity constants used to bound $\mathrm{Lip}(Q_{\ell,k})$.

Proposition 10.2 (Infinite-volume limit for local blocked observables (conditional)). *Assume theorem 9.2 and Assumption 10.1. Then for each fixed local* $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{block}}$ *the limit* $\omega(\mathcal{O})$ *extends (along a subsequence* $L \to \infty$*) to an infinite-volume state on the corresponding quasi-local blocked algebra.*

Proof sketch. Use the L-uniform one-step bound (38) for tightness and locality: boundary effects are suppressed by polymer locality/decay estimates, so expectations stabilize as $L \to \infty$ for fixed local observables. □

11 Conditional extension to renormalized observables

11.1 A schematic class $\mathfrak{A}_\ell^{\mathrm{ren}}$

Definition 11.1 (Renormalized observable class (schematic)). Fix $\ell > 0$. We say $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{ren}}$ if $\mathcal{O}^{(k)}$ is obtained from fine Wilson loop variables at resolution $a_k \ll \ell$ by a fixed finite algebraic procedure designed to cancel perimeter divergences, and if the resulting observables are uniformly bounded on Ω_k^{sf}.

Remark 11.2. This definition is intentionally schematic: including Creutz-type expressions at the observable level requires either avoiding logarithms/ratios or adding non-degeneracy hypotheses ensuring denominators stay away from 0.

11.2 Nonperturbative Symanzik extraction as an input

Assumption 11.3 (Polymer-to-local operator map (Symanzik input))**.** There exists a localization map loc_k extracting the marginal and dimension-6 local components from polymer activities such that the remainder is irrelevant with an a_k^2 gain *when tested against the renormalized observable class* $\mathfrak{A}_\ell^{\mathrm{ren}}$. Concretely, for every $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{ren}}$ with representation at scale k and the corresponding blocked-scale observable at $k+1$, one has the improved one-step pushforward bound

$$\left| \left\langle \mathcal{R}_k \mathcal{O}^{(k+1)} \right\rangle_{\mu_k^{\mathrm{eff}}} - \left\langle \mathcal{O}^{(k)} \right\rangle_{\mu_k^{\mathrm{eff}}} \right| \le C\, g_{k+1}^2 \left(\frac{a_{k+1}}{\ell} \right)^2 + C\, \sqrt{\tau_k}. \tag{39}$$

Theorem 11.4 (Conditional convergence for $\mathfrak{A}_\ell^{\mathrm{ren}}$)**.** *Assume Assumption 5.1 and Assumption 11.3. Then for every $\mathcal{O} \in \mathfrak{A}_\ell^{\mathrm{ren}}$ the limit $\lim_{k\to\infty} \left\langle \mathcal{O}^{(k)} \right\rangle_{\Lambda_k,\beta_k}$ exists.*

Proof sketch. Repeat the proof of theorem 9.2, replacing the A2′ step theorem 8.2 by the improved bound (39). The A3 step is provided by theorem 7.6. The resulting one-step error is bounded by

$$C\, g_{k+1}^2 \left(\frac{a_{k+1}}{\ell} \right)^2 + C\, \mathrm{Lip}(F) \left(\frac{a_k}{\ell} \right)^2 + C\sqrt{\tau_k} + C\tau_k,$$

which is absolutely summable in k, hence the limit exists. □

12 Discussion: OS reconstruction and the mass gap

12.1 Osterwalder–Schrader positivity

The Wilson action on the lattice satisfies reflection positivity for suitable reflections. For the limiting state ω on $\mathfrak{A}_\ell^{\mathrm{block}}$, inheritance of reflection positivity is not automatic: the blocking map $Q_{\ell,k}$ may average over paths that straddle the reflection plane. A rigorous verification (or a support condition ensuring factorization of the blocking kernel near the plane) is deferred to future work. A framework aimed at avoiding the blocking obstruction (by working with unblocked OS-compatible structures) is discussed in [18].

12.2 Why the mass gap does not follow here

The mass gap requires infrared information: exponential decay of connected correlations and a corresponding spectral gap in the reconstructed Hamiltonian. The present theorem is an ultraviolet/continuum-limit existence result for a fixed-scale algebra and does not by itself control infrared behavior.

12.3 Ratios and logarithms

Once ω is constructed, quantities like Creutz ratios can be formed as functionals of finitely many numbers $\omega(\mathcal{O}_i)$. Their convergence follows if the relevant denominators remain bounded away from 0 along the approximating sequence.

A Appendix A: Absence of dimension-5 operators

Lemma A.1 (No dimension-5 local gauge-invariant H_4-invariant operators in $d = 4$). *In $d = 4$ pure Yang–Mills theory on a hypercubic lattice, there are no gauge-invariant, H_4-symmetric local operators of canonical dimension 5 that are also invariant under reflections; C and P further exclude the remaining pseudoscalar candidates.*

Proof sketch. Canonical dimensions are $\dim(F_{\mu\nu}) = 2$ and $\dim(D_\mu) = 1$. A dimension-5 gauge-invariant local expression built from F and covariant derivatives necessarily carries an unpaired Lorentz index. Imposing full hypercubic symmetry together with reflection invariance rules out such odd-index tensor structures in the scalar sector. The pseudoscalar density $\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})$ has dimension 4 (not 5) and is P-odd. Hence no dimension-5 operators survive. □

B Appendix B: Dimension-6 operators (orientation)

In the continuum, typical gauge-invariant dimension-6 operators include

$$\mathrm{Tr}\Big((D_\mu F_{\mu\nu})^2\Big), \qquad \mathrm{Tr}\Big(F_{\mu\nu}F_{\nu\rho}F_{\rho\mu}\Big).$$

A lattice-adapted basis may be expressed through small-field expansions of Wilson loops (plaquettes, rectangles) and symmetry projections. A full operator-basis implementation is needed only for the conditional extension (Assumption 11.3).

C Appendix C: A metric Lipschitz inequality (no barycenters)

Let (X, d) be a metric space, let $F : \mathsf{X} \to \mathbb{R}$ be Lipschitz with constant $\mathrm{Lip}(F)$, let X be an X-valued random variable, and fix any deterministic base point $x_0 \in \mathsf{X}$. Then

$$|F(x_0) - \mathbb{E}[F(X)]| \le \mathrm{Lip}(F)\, \mathbb{E}[d(X, x_0)] \le \mathrm{Lip}(F)\, \sqrt{\mathbb{E}[d(X, x_0)^2]}.$$

This is the only Lipschitz estimate used in the proof of Lemma A2′; no intrinsic barycenter $\mathbb{E}[X] \in \mathsf{X}$ is required.

References

[1] K. G. Wilson, "Confinement of quarks," *Phys. Rev. D* **10** (1974), 2445–2459.

[2] T. Balaban, *(Series of papers, 1984–1989) Renormalization group approach to lattice gauge theories*, *Comm. Math. Phys.* and related venues.

[3] J. Dimock, "The renormalization group according to Balaban, I. Small fields," *Rev. Math. Phys.* **25** (2013), 1330010.

[4] J. Dimock, "The renormalization group according to Balaban, II. Large fields," *J. Math. Phys.* **54** (2013), 092301.

[5] D. C. Brydges, "Lectures on the renormalisation group," in *Statistical Mechanics*, IAS/Park City Math. Ser., AMS (2009).

[6] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's functions," *Comm. Math. Phys.* **31** (1973), 83–112; **42** (1975), 281–305.

[7] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, 1999.

[8] K. Fredenhagen and M. Marcu, "Confinement as a property of the state," *Comm. Math. Phys.* **92** (1983), 81–119.

[9] L. Eriksson, *Uniform log-Sobolev inequality for lattice Yang–Mills via multiscale renormalization and entropy telescoping*, ai.viXra.org:2602.0041, 2026. https://ai.vixra.org/abs/2602.0041

[10] L. Eriksson, *Synthetic Ricci curvature and conditional log-Sobolev inequalities for lattice gauge theories on the orbit space*, ai.viXra.org:2602.0046, 2026. https://ai.vixra.org/abs/2602.0046

[11] L. Eriksson, *Integrated cross-scale derivative bounds for lattice Yang–Mills via small-field/large-field decomposition*, ai.viXra.org:2602.0057, 2026. https://ai.vixra.org/abs/2602.0057

[12] L. Eriksson, *Large-field conditional suppression for Wilson lattice gauge theories via Balaban's T-operation*, ai.viXra.org:2602.0056, 2026. https://ai.vixra.org/abs/2602.0056

[13] L. Eriksson, *Unconditional uniform log-Sobolev inequality for* $\mathrm{SU}(N_c)$ *lattice Yang–Mills at weak coupling*, ai.viXra.org:2602.0055, 2026. https://ai.vixra.org/abs/2602.0055

[14] L. Eriksson, *From uniform log-Sobolev inequality to mass gap for lattice Yang–Mills at weak coupling*, ai.viXra.org:2602.0054, 2026. https://ai.vixra.org/abs/2602.0054

[15] L. Eriksson, *DLR-uniform log-Sobolev inequality and unconditional mass gap for lattice Yang–Mills at weak coupling*, ai.viXra.org:2602.0053, 2026. https://ai.vixra.org/abs/2602.0053

[16] L. Eriksson, *Interface lemmas for the multiscale proof of the lattice Yang–Mills mass gap*, ai.viXra.org:2602.0052, 2026. https://ai.vixra.org/abs/2602.0052

[17] L. Eriksson, *Uniform coercivity, pointwise large-field suppression, and unconditional closure of the lattice Yang–Mills mass gap at weak coupling in* $d = 4$, ai.viXra.org:2602.0051, 2026. https://ai.vixra.org/abs/2602.0051

[18] L. Eriksson, *Conditional Continuum Limit of* $4d$ $\mathrm{SU}(N_c)$ *Yang–Mills Theory via Two-Layer Architecture, RG-Cauchy Uniqueness, and Step-Scaling Confinement*, ai.viXra.org:2602.0063, 2026. https://ai.vixra.org/abs/2602.0063