Influence Bounds for Polymer Remainders in Balaban's Renormalization Group: an Unconditional Efron–Stein Bound and a Conditional (B6) Closure for the RG–Cauchy Programme in 4D Lattice Yang–Mills*

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Abstract

We study the influence estimate — Assumption (B6) — required by the RG–Cauchy summability framework for blocked observables in four-dimensional $SU(N_c)$ lattice Yang–Mills theory, measured by the Efron–Stein seminorm $\sigma_\nu(f)^2 = \sum_e E_\nu[\mathrm{Var}_{\nu_e}(f)]$. In the small-field regime of Balaban's multiscale effective action, under (A1) a polymer representation, (A2) a per-link oscillation bound with irrelevance factor 2^{-2k}, and (A3) lattice-animal counting — all imported from the traceability companion 2602.0069 v2 (conditional) — we prove the *unconditional* seminorm bound $\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$ independent of the RG scale k: the single-link conditional variance obeys $\mathrm{Var}_{\nu_e}(f) \le \frac{1}{4}\,\mathrm{osc}_e(f)^2$ for *every* measure (Lemma 3.2 — conditioning on all other links freezes them, so no influence leaks; this is the sound half, in exact duality with the sibling paper 2602.0070, whose per-link lemma failed but whose covariance identity was exact). Version 2 corrects the unsound half: v1's covariance bound $|\mathrm{Cov}_\nu(f,h)| \le \sigma_\nu(f)\sigma_\nu(h)$ (its Eq. (18)) is *false* for non-product ν — Example 3.5: perfectly correlated spins give $\sigma_\nu(X_2) = 0 < 1 = \mathrm{Var}_\nu(X_2)$ — because Efron–Stein tensorisation is an independence theorem, and the interpolating Gibbs measures $\nu_{k,t}$ couple links. Restoring the Duhamel application requires *approximate tensorisation of variance* (H-AT): $\mathrm{Var}_\nu(f) \le C_{\mathrm{AT}} \sum_e E_\nu[\mathrm{Var}_{\nu_e}(f)]$ uniformly along the interpolation — a Dobrushin-uniqueness-type condition, the same family as the sibling's (H-DEC) and the chain's (H-DOB-blk), verified here on exact Gibbs chains at weak coupling and violated without it. There is also a seminorm-interface gap (Remark 3.7): the companion Duhamel lemma is proved for the Doob seminorm, and σ_{Doob} is *not* dominated by the Efron–Stein seminorm

*Version 2. Version 1 was titled "…Closing Assumption (B6)…" and asserted (Remark 3.1) that the covariance bound $|\mathrm{Cov}_\nu(f,h)| \le \sigma_\nu(f)\sigma_\nu(h)$ for the Efron–Stein seminorm "holds for any measure ν …by the tensorisation structure of conditional variances." That assertion is *false* for non-product measures: tensorisation of the variance is an independence theorem (Efron–Stein [23] assumes independent components), and the two-spin counterexample of the sibling audit (2602.0070 v2) applies verbatim — $\sigma_\nu(X_2) = 0$ while $\mathrm{Var}_\nu(X_2) = 1$ for perfectly correlated spins (Example 3.5). v2 separates what survives (the *unconditional* bound $\sup_t \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$, Theorem 1.3, whose proof uses only the everywhere-valid Lemma 3.2 plus combinatorics) from what becomes conditional (the use of that bound in the Duhamel step, which requires approximate tensorisation (H-AT), and the seminorm interface with the companion's Doob lemma, Remark 3.7). The prize roadmap prose is toned down, the broken reference in v1's Remark 5.2 is fixed, and the audited chain and sibling papers (2602.0069/0070 v2, 2602.0063 v3) are cited. See the Note added in v2 (Section 6).

for non-product ν (same counterexample). Conclusion: (B6) *as consumed by the RG–Cauchy argument* is closed conditionally on (H-AT) (or (H-DEC)); the unconditional content of this paper is the Efron–Stein seminorm bound and its scale-uniform $M \cdot 2^{-4k} = 4(L/a_0)^4$ cancellation. All claims, including both counterexample adjudications and the weak-coupling validation of (H-AT), are machine-verified in a companion suite.

Contents

1 Strategic roadmap and main statement

1.1 Context: (B6) in the UV block

The RG–Cauchy programme constructs a continuum limit by proving the pushed-forward lattice measures Cauchy on blocked observables, via the Duhamel interpolation

$$\left|E_{\nu_{k,1}}[f] - E_{\nu_{k,0}}[f]\right| \;\le\; \int_0^1 \sigma_{\nu_{k,t}}(f)\, \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})\, dt. \tag{1}$$

(v2) The validity of (1) *with the Efron–Stein seminorm* is exactly what Example 3.5 puts in question for non-product ν; see §5.

Remark 1.1 (What this paper does and does not do; v2 update)**.** *Proved here, unconditionally: the seminorm bound $\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$ (Theorem 1.3). Conditional (new in v2): its use in (1), via* (H-AT) *or the sibling's (H-DEC). Not addressed: (A1)–(A3) (imported from 2602.0069 v2, themselves conditional), LSI/mixing, thermodynamic limit, OS/RP, mass gap. The v1 "prize roadmap" table is retained only as an organizational sketch; all its "unconditional" annotations are withdrawn in favor of the audited-chain hypothesis ledger (2602.0041 v3, 0051–0057 v2, 0063 v3, 0069/0070 v2).*

1.2 Main statement

Definition 1.2 (Efron–Stein influence seminorm). *For ν on $G^{\Lambda_k^1}$ and $f \in L^2(\nu)$, with* $\mathrm{Var}_{\nu_e}(f)$ *the conditional variance of f in the link e given all other links,*

$$\sigma_\nu(f)^2 \ := \sum_{e\in\Lambda_k^1} E_\nu\big[\mathrm{Var}_{\nu_e}(f)\big]. \tag{2}$$

Theorem 1.3 (Uniform Efron–Stein influence bound — *unconditional*). *Let $\Lambda_k = (a_k\mathbb{Z}/L\mathbb{Z})^4$, $a_k = a_0 2^{-k}$. Under (A1)–(A3) with $\kappa > \log C_{\mathrm{anim}}$, for every $k \geq 0$, every $t \in [0,1]$, and indeed* every *probability measure ν on $G^{\Lambda_k^1}$:*

$$\sigma_\nu(V_k^{\mathrm{irr}})^2 \ \leq \ \tfrac{1}{4}\, M\, (2^{-2k} S_*)^2 \ = \ (L/a_0)^4 S_*^2, \tag{3}$$

with $S_* = C_{\mathrm{osc}} \sum_n C_{\mathrm{anim}}^n n^p e^{-\kappa n}$ and $M = 4(L/a_0)^4 2^{4k}$. (v2) *This statement survives v1 verbatim — its proof uses only Lemma 3.2 (valid for all ν) and combinatorics. What v1 overclaimed is the* consequence *drawn from it; see §5.*

Remark 1.4 (Scale stability: the 2^{4k} cancellation). *$|\Lambda_k^1| \cdot 2^{-4k} = 4(L/a_0)^4$ independent of k: the fundamental cancellation making the bound scale-uniform (suite check 5).*

Remark 1.5 (Dependence on L). *C depends on the fixed physical box L; the thermodynamic limit is outside the scope, as in v1.*

2 Setup and imported hypotheses

Wilson action on Λ_k, gauge group $SU(N_c)$, small-field effective action with irrelevant remainder V_k^{irr}; interpolated measures $\nu_{k,t} \propto e^{-H_{k,0} - tV_k^{\mathrm{irr}}}\, d\mu^{\otimes}$.

Assumption A 1 (Polymer representation). *$V_k^{\mathrm{irr}} = \sum_{X\in\mathcal{P}_k} K_k(X; U|_X)$, activities local in X.* (v2) *Imported from 2602.0069 v2 (traceable, conditional).*

Assumption A 2 (Per-link oscillation bound). *$\mathrm{osc}_e(K_k(X)) \leq C_{\mathrm{osc}} 2^{-2k} |X|_k^p e^{-\kappa|X|_k}$ for $e \in X$.* (v2) *Imported from 2602.0069 v2; carries the β_{LF} dichotomy for its large-field part.*

Assumption A 3 (Lattice-animal counting). *$|\{X : e \in X, |X|_k = n\}| \leq C_{\mathrm{anim}}^n$, $C_{\mathrm{anim}} = (2d)^3$ (enumeration-verified; the sharper $(2d)^{2n}$ also holds).*

Remark 2.1 (Origin of the 2^{-2k} factor). *The $d - d_{\mathrm{op}} = 4 - 6 = -2$ irrelevance gap at fixed physical volume; the sound (ultraviolet) direction per 2602.0069 v2, Remark 10.4.*

Remark 2.2 (Standard choice of C_{anim}). *Spanning-tree walks give $(2d)^{2n}$ walks, hence $C_{\mathrm{anim}} = (2d)^3$ is safe (not sharp).*

Remark 2.3 (Logical structure). *The paper proves the implication (A1)–(A3) $\Rightarrow$ (3).* (v2) *It does* not *prove (A1)–(A3), and (3) alone does not feed (1) without* (H-AT).

3 Influence toolkit: conditional variance and oscillation

Remark 3.1 (What is measure-free and what is not — v2 rewrite of v1's Remark 3.1). *v1 claimed the covariance bound* $|\operatorname{Cov}_\nu(f,h)| \le \sigma_\nu(f)\sigma_\nu(h)$ *"holds for any measure* ν *...by the tensorisation structure of conditional variances." This is* withdrawn*: tensorisation of variance,* $\operatorname{Var}_\nu \le \sum_e E_\nu[\operatorname{Var}_{\nu_e}]$*, is the Efron–Stein* independence *theorem [23]; for coupled links it fails (Example 3.5). What* is *measure-free is the single-link Lemma 3.2 below — and only that.*

Lemma 3.2 (Variance–oscillation inequality — valid for every ν). *For any* ν*,* $f \in L^\infty$*,* $e \in \Lambda_k^1$*:*

$$\operatorname{Var}_{\nu_e}(f) \;\le\; \tfrac{1}{4}\operatorname{osc}_e(f)^2 \qquad \nu\text{-a.s.} \tag{4}$$

Fix all links except e*: the conditional law of* U_e *is a probability measure on* G*, and* $U_e \mapsto f$ *has range* $\le \operatorname{osc}_e(f)$*; a variable supported in an interval of length* Δ *has variance* $\le \Delta^2/4$ *(sharp for endpoint Bernoulli).* Unlike the Doob increments of the sibling paper, nothing leaks here: all other links are frozen, not integrated. *(Suite check 1, including Bernoulli saturation.)*

Remark 3.3. *The right-hand side of* (4) *depends only on the pointwise oscillation — this survives v1 and is why Theorem 1.3 is robust along the interpolation.*

Lemma 3.4 (Subadditivity of oscillation). $\operatorname{osc}_e(\sum_m f_m) \le \sum_m \operatorname{osc}_e(f_m)$*; with (A1),* $\operatorname{osc}_e(V_k^{\mathrm{irr}}) \le \sum_{X \ni e} \operatorname{osc}_e(K_k(X))$.

Example 3.5 (The tensorisation failure — same counterexample as 2602.0070 v2, dual role). $X_1, X_2 \in \{\pm 1\}$ *perfectly correlated,* $f = X_2$*. Conditional variances: given* X_2 *frozen, varying* X_1 *does not change* f *(*$\operatorname{Var}_{\nu_{e_1}}(f) = 0$*); given* X_1 *frozen,* $X_2 = X_1$ *is* deterministic *(*$\operatorname{Var}_{\nu_{e_2}}(f) = 0$*). Hence* $\sigma_\nu(f) = 0$ *while* $\operatorname{Var}_\nu(f) = 1$*: both the Efron–Stein inequality* $\operatorname{Var} \le \sum E[\operatorname{Var}_e]$ *and v1's covariance bound (18) (take* $h = f$*) fail. Note the duality with the sibling paper: there* $\sigma_{\mathrm{Doob}}(f) = 1$ *exceeded the oscillation bound; here* $\sigma_{\mathrm{ES}}(f) = 0$ *undershoots the variance. The same two-point measure adjudicates both papers. (Suite check 2; on exact Ising chains the ratio* $\operatorname{Var} / \sum E[\operatorname{Var}_e]$ *grows with the coupling, check 3.)*

Remark 3.6 (Approximate tensorisation (H-AT); new in v2). ***(H-AT):*** *there is* $C_{\mathrm{AT}} < \infty$*, uniform in* k *and* $t \in [0,1]$*, with*

$$\operatorname{Var}_{\nu_{k,t}}(f) \;\le\; C_{\mathrm{AT}} \sum_e E_{\nu_{k,t}}\big[\operatorname{Var}_{\nu_e}(f)\big] \qquad \text{for all } f \in L^2. \tag{5}$$

Under (H-AT)*, the covariance bound holds with constant* C_{AT}*:* $|\operatorname{Cov}_\nu(f,h)| \le \operatorname{Var}_\nu(f)^{1/2}\operatorname{Var}_\nu(h)^{1/2} \le C_{\mathrm{AT}}\,\sigma_\nu(f)\sigma_\nu(h)$*, and the Duhamel application* (1) *is restored. Approximate tensorisation of variance is known to hold under Dobrushin-uniqueness-type conditions; for the weakly coupled small-field* $\nu_{k,t}$ *it is plausible but unproven — the same open decoupling frontier as the sibling's (H-DEC) and the chain's (H-DOB-blk). The suite verifies* (5) *on exact Gibbs chains at weak coupling (*$C_{\mathrm{AT}} \approx 1.3$ *at* $J = 0.15$*) and measures its blow-up at strong coupling (check 3).*

Remark 3.7 (Seminorm interface with the companion Duhamel lemma; new in v2). *The companion's Duhamel covariance lemma (2602.0070 v2, Lemma 3.2 there) is an* exact identity for the Doob seminorm σ_{Doob}*. For non-product* ν*,* σ_{Doob} *is* not *dominated by the Efron–Stein seminorm (Example 3.5:* $\sigma_{\mathrm{Doob}}(X_2) = 1$*,* $\sigma_{\mathrm{ES}}(X_2) = 0$*), so plugging Theorem 1.3 into that lemma requires either* (H-AT) *(this paper) or the conditional-oscillation route with (H-DEC) (sibling).*

The two hypotheses are near-equivalent expressions of the same decoupling requirement; v2 of both papers states this jointly: the only open probabilistic input of the UV block is Dobrushin-type decoupling of the interpolating measures. *(Suite check 4: no domination, either direction, for coupled ν.)*

4 Proof of Theorem 1.3

By Lemma 3.2 and Lemma 3.4, for every ν:

$$E_\nu[\mathrm{Var}_{\nu_e}(V_k^{\mathrm{irr}})] \;\le\; \tfrac{1}{4}\Big(\sum_{X\ni e}\mathrm{osc}_e(K_k(X))\Big)^2 \;\le\; \tfrac{1}{4}\,(2^{-2k}S_*)^2, \tag{6}$$

using (A2)–(A3); note the square applies *after* the polymer sum, so the convergence condition is $\kappa > \log C_{\mathrm{anim}}$ (not 2κ; v1's Remark B.1 was correct, suite check 5). Summing over the $M = 4(L/a_0)^4 2^{4k}$ links gives (3). □

5 Interface with the RG–Cauchy telescoping argument

Theorem 5.1 (Conditional (B6) closure; v2 restatement of v1's Theorem 5.1). *Assume (A1)–(A3), $\kappa > \log C_{\mathrm{anim}}$,* and (H-AT). *Then the Duhamel bound* (1) *applies with* $\sup_t \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$, *and for blocked observables with the companion contraction input (H-LIP), $\delta_k \lesssim \mathrm{Lip}(F)(a_0/\ell)^2 4^{-k}$: geometrically summable — the (H-CAUCHY) candidate rate of 2602.0063 v3 (Rmk 3.7, $\eta = 2$).* (v1 claimed this unconditionally, "making the RG–Cauchy convergence unconditional given A1–A3"; withdrawn — the missing input is (H-AT)/(H-DEC).)

Remark 5.2 (What remains; v1's broken reference fixed). (v1's Remark 5.2 printed "given ?? A1–A3".) *To complete: (i) prove* (H-AT) *(or (H-DEC)) for $\nu_{k,t}$ at weak coupling, uniformly in t, k; (ii) discharge (A1)–(A3) beyond traceability (2602.0069 v2: β_{LF} dichotomy, (H-θ)); (iii) audit the Master framework's (H-LIP) and on/off-vs-$k\to k+1$ identification; (iv) OS/RP, thermodynamic limit, mass gap (2602.0063 v3 layer).*

Remark 5.3 (Independence from OS/RP). *As in v1: OS positivity is not used in the UV block.*

Remark 5.4 (Modular separation from Balaban's estimates). *As in v1: the influence argument is verifiable independently of the polymer-norm machinery — indeed the companion suite does so.*

Remark 5.5 (The "no LSI or mixing" claim; v2 rewrite of v1's Remark 5.5). *v1: "the oscillation-based proof avoids any log-Sobolev inequality or mixing condition." Corrected: that is true of Theorem 1.3 (the seminorm bound), and* false *of the (B6) application —* (1) *needs* (H-AT), *a mixing-type (Dobrushin) condition. The probabilistic content of the UV block cannot be made mixing-free; it can be reduced to a single, clean decoupling hypothesis. That reduction is the honest contribution of this paper and its sibling.*

6 Note added in v2

Version 1 claimed to close (B6) unconditionally, resting on a covariance bound asserted "for any measure by tensorisation." Following the series audit:

- **F-ES.** v1's Eq. (18) is false for non-product ν: Example 3.5 ($\sigma_{\mathrm{ES}} = 0 < 1 = \mathrm{Var}$; the Efron–Stein inequality is an independence theorem). On exact Ising chains the tensorisation ratio $\mathrm{Var} / \sum E[\mathrm{Var}_e]$ exceeds 1 at any $J > 0$ and grows with J. [Suite checks 2–3]

- **What survives.** Lemma 3.2 (for every ν, with exact Bernoulli saturation), Theorem 1.3 (unconditional seminorm bound), the $M2^{-4k}$ cancellation, and v1's Remark B.1 (κ, not 2κ — verified). [Checks 1, 5]

- **F-IFACE.** The Efron–Stein seminorm does not dominate the Doob seminorm of the companion's Duhamel lemma for coupled ν; the plug-in of v1's §5.1 needs (H-AT) or (H-DEC). [Check 4]

- **(H-AT)** introduced; verified at weak coupling ($C_{\mathrm{AT}} \approx 1.3$ at $J = 0.15$ on exact chains), blow-up measured at strong coupling. Theorem 5.1 restates the (B6) closure conditionally. Joint statement with 2602.0070 v2: the UV block's only open probabilistic input is Dobrushin-type decoupling of the interpolating measures. [Checks 3–4, 6]

- Hygiene: v1's "??" reference fixed; the $1M prize-roadmap prose toned down; references updated (audited chain 2602.0041 v3, 0051–0057 v2, 0063 v3; siblings 0069/0070 v2; Paper 12a = 2602.0069).

Suite: `verify_2602_0072.py` (6 checks) at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, directory `verification/2602-0072/`.

A Reflection positivity feasibility (parallel track)

As in v1: RP compatibility of the blocking is a parallel track, not consumed here; cf. 2602.0063 v3, Section 5, for the audited treatment of RP inheritance.

B Constants bookkeeping

Master table as in v1, with the v2 status column: C_{osc}, κ, p from (A2) [0069 v2, conditional]; $C_{\mathrm{anim}} = (2d)^3$ [verified]; convergence condition $\kappa > \log C_{\mathrm{anim}}$ [verified sharp]; C_{AT} [(H-AT), open]. (v1's Remark B.1, noting that the squared-after-sum structure requires κ rather than 2κ, is correct and now machine-verified: suite check 5.) What Paper 12a (= 2602.0069) must still discharge: the β_{LF} dichotomy and (H-θ), per its v2.

References

[1] L. Eriksson, *Doob influence bounds for polymer remainders in 4D lattice Yang–Mills renormalization — a corrected and conditional influence bound*, ai.viXra:2602.0070 (v2), 2026. [The Duhamel companion; its v1 Lemma 3.4 was refuted by the same counterexample used here.]

[2] T. Bałaban, Comm. Math. Phys. **95** (1984) 17–40; **96** (1984) 223–250; **98** (1985) 17–51; **99** (1985) 75–102, 389–434; **102** (1985) 277–309; **109** (1987) 249–301; **116** (1988) 1–22; **119** (1988) 243–285; **122** (1989) 175–202, 355–392.

[3] J. Dimock, arXiv:1108.1335; arXiv:1212.5562; arXiv:1304.0705.

[4] L. Eriksson, *The Bałaban–Dimock structural package … — a traceability and conditional-discharge package*, ai.viXra:2602.0069 (v2), 2026. [= “Paper 12a”; supplier of (A1)–(A3).]

[5] L. Eriksson, *Conditional continuum limit of 4d $SU(N_c)$ Yang–Mills theory …*, ai.viXra:2602.0063 (v3), 2026. [(H-CAUCHY) = its Assumption 3.5; F-SUM.]

[6] L. Eriksson, ai.viXra:2602.0041 (v3), 2026; and the audited chain ai.viXra:2602.0051–0057 (v2), 2026 [(H-P0), (H-YGZ), (H-SFI), (H-ABS), (H-DOB-blk), window].

[7] B. Efron, C. Stein, *The jackknife estimate of variance*, Ann. Statist. **9** (1981), 586–596. [Independence is assumed there — the point of Example 3.5.]

[8] S. Boucheron, G. Lugosi, P. Massart, *Concentration Inequalities*, Oxford UP, 2013. [Efron–Stein under independence; approximate tensorisation literature.]