Influence Bounds for Polymer Remainders in Balaban's Renormalization Group:

Closing Assumption (B6) for the RG–Cauchy Programme in 4D Lattice Yang–Mills

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Abstract

We close the missing influence estimate—Assumption (B6)—required by the RG–Cauchy summability framework for blocked observables in four-dimensional $\mathrm{SU}(N_c)$ lattice Yang–Mills theory. The influence is measured by the Efron–Stein seminorm

$$\sigma_\nu(f)^2 \;=\; \sum_{e\in\Lambda_k^1} \mathbb{E}_\nu\big[\mathrm{Var}_e^\nu(f)\big]$$

that appears in the Duhamel interpolation lemma of the companion paper [1]. We work in the small-field regime of Balaban's multiscale effective action and assume: **(A1)** a standard polymer representation for the irrelevant remainder $V_k^{\mathrm{irr}} = \sum_{X\in\mathcal{P}_k} K_k(X)$; **(A2)** an explicit per-link oscillation bound for polymer activities carrying the correct irrelevance factor 2^{-2k}; **(A3)** a lattice-animal counting estimate. Under these three verifiable hypotheses—to be discharged from Balaban's historical work in a companion compendium paper—we prove

$$\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \;\le\; C, \qquad C = C(N_c, \beta_0, \kappa, C_{\mathrm{osc}}, C_{\mathrm{anim}}, p, L/a_0) \text{ independent of the RG scale } k.$$

The proof uses only oscillation bounds and combinatorics: no log-Sobolev inequality, no mixing hypothesis, and no measure-dependent technology beyond the definition of conditional variance. This removes the only genuinely novel probabilistic input remaining in the UV block of the programme towards the Yang–Mills Millennium Prize.

Keywords: Yang–Mills mass gap, renormalization group, polymer expansion, Efron–Stein inequality, influence bounds, lattice gauge theory, Balaban programme, Clay Millennium Problem.

MSC 2020: 81T13 (primary); 81T17, 82B28, 60E15 (secondary).

Contents

1 Strategic roadmap and main statement

1.1 Context: (B6) as the UV bottleneck

The RG–Cauchy programme [1, 11] constructs a continuum limit for four-dimensional SU(N_c) Yang–Mills theory by proving that the sequence of pushed-forward lattice measures $\{\mu_k\}_{k\geq 0}$ is Cauchy with respect to a Wasserstein-type distance on blocked observables. The one-step comparison uses a Duhamel interpolation:

$$\left|\mathbb{E}_{\nu_{k,1}}[f] - \mathbb{E}_{\nu_{k,0}}[f]\right| \leq \int_0^1 \sigma_{\nu_{k,t}}(f)\, \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})\, dt, \tag{1}$$

where σ_ν is the Efron–Stein influence seminorm (see Theorem 1.2 below), f is a blocked observable whose influence is damped by the Lipschitz contraction of the blocking map Q_k, and V_k^{irr} is the irrelevant polymer remainder of the effective action at scale k.

The term $\sigma_{\nu_{k,t}}(f)$ is controlled in [1] by the geometric contraction $\mathrm{Lip}(Q_k) < 1$. The present paper supplies the *missing factor*: a uniform bound $\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \leq C$ independent of k.

Remark 1.1 (What this paper does *not* do).

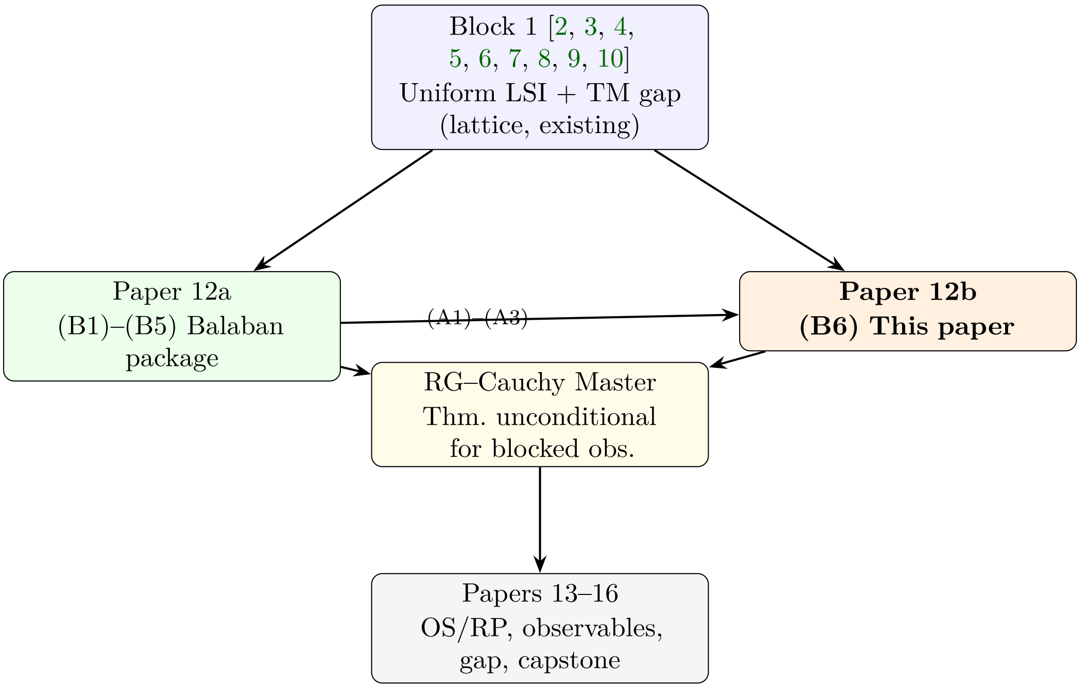
Proved here	**Not addressed here**
(B6): $\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \leq C$, all k, all t	(B1)–(B5): deferred to Paper 12a
Uses only oscillation + combinatorics	LSI, mixing, complete analyticity
Fixed physical box $L < \infty$	Thermodynamic limit $L \to \infty$
SU(N_c), Wilson action, $d = 4$	Other groups, actions, or dimensions
UV block only	OS/RP, mass gap, capstone

1.2 Prize roadmap: the 6-paper plan

The Clay Millennium Prize for the Yang–Mills existence and mass gap problem is \$1,000,000 USD. The present programme organises the proof into three blocks of two papers each:

Block	**Paper**	**Content**
UV	12b (this paper)	Close (B6): influence bound for V_k^{irr}
	12a	Close (B1)–(B5): Balaban package with uniform constants
Axiomatic	13	OS/RP compatibility and Hilbert-space reconstruction
	15	Renormalized local observables (Symanzik fields)
IR + Synthesis	14	Thermodynamic limit and continuum mass gap
	16	Capstone: OS0–OS4 plus mass gap assembled

The logical dependencies are:



1.3 Main theorem

Definition 1.2 (Efron–Stein influence seminorm)**.** Let ν be a probability measure on $G^{\Lambda_k^1}$ and let $f \in L^2(\nu)$. The *conditional variance of f in the link e* is the random variable

$$\mathrm{Var}_e^{\nu}(f) \;:=\; \mathbb{E}_\nu\Big[\big(f - \mathbb{E}_\nu[f \mid U_{e'},\, e' \neq e]\big)^2 \;\Big|\; U_{e'},\, e' \neq e\Big]. \tag{2}$$

The *influence seminorm* (cf. [23, 24]) is

$$\sigma_\nu(f)^2 \;:=\; \sum_{e \in \Lambda_k^1} \mathbb{E}_\nu\big[\mathrm{Var}_e^{\nu}(f)\big]. \tag{3}$$

Theorem 1.3 (Uniform influence bound — Assumption (B6) closed)**.** *Let $\Lambda = (\mathbb{Z}/L'\mathbb{Z})^4$ with $L' = L/a_0 \in 2\mathbb{N}$ and dyadic lattice spacings $a_k = a_0\, 2^{-k}$, so that $\Lambda_k = (a_k\mathbb{Z}/L\mathbb{Z})^4$. Let $G = \mathrm{SU}(N_c)$ and $\beta \geq \beta_0(N_c)$. Let $\nu_{k,t}$ $(t \in [0,1])$ be the interpolated small-field measures on Ω_k^{sf} defined in Section 2.3. Assume* **??** *A1–A3.*

Then there exists a finite constant

$$C \;=\; C(N_c, \beta_0, \kappa, C_{\mathrm{osc}}, C_{\mathrm{anim}}, p, L/a_0)$$

*such that for **every** RG scale $k \geq 0$ and **every** $t \in [0,1]$,*

$$\boxed{\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \;\leq\; C.} \tag{4}$$

Explicitly, one may take

$$C \;=\; 2\left(\frac{L}{a_0}\right)^2 \sqrt{C_1}, \qquad C_1 \;=\; \frac{1}{4}\left(C_{\mathrm{osc}} \sum_{n=1}^{\infty} C_{\mathrm{anim}}^n\, n^p\, e^{-\kappa n}\right)^2. \tag{5}$$

The series converges whenever $\kappa > \log C_{\mathrm{anim}}$.

Remark 1.4 (Scale stability: the 2^{4k} cancellation). The number of links satisfies $|\Lambda_k^1| = 4(L/a_k)^4 = 4(L/a_0)^4\, 2^{4k}$, which grows with k. The per-link influence, however, decays as 2^{-4k} thanks to the irrelevance factor 2^{-2k} in Assumption A2. The product $|\Lambda_k^1| \times 2^{-4k} = 4(L/a_0)^4$ is independent of k. This is the fundamental cancellation making (B6) scale-uniform.

Remark 1.5 (Dependence on L and the thermodynamic limit). The constant C depends on the physical box size L, as is standard for finite-volume results. Uniformity as $L \to \infty$ is a separate problem belonging to Paper 14 (thermodynamic limit). For the UV telescoping at fixed L, Theorem 1.3 is all that is needed.

2 Setup and imported hypotheses

2.1 Lattice Yang–Mills theory

We fix the gauge group $G = \mathrm{SU}(N_c)$ and the spacetime dimension $d = 4$. At RG scale $k \geq 0$ the lattice spacing is $a_k = a_0\, 2^{-k}$ and the lattice is the periodic box

$$\Lambda_k \;=\; \big(a_k\mathbb{Z} \,/\, L\mathbb{Z}\big)^4, \tag{6}$$

where $L > 0$ is the fixed physical side length. We write Λ_k^1 for the set of positively oriented links and Λ_k^2 for the set of positively oriented plaquettes. A *gauge configuration* at scale k is a map $U\colon \Lambda_k^1 \to G$, $e \mapsto U_e$. For each $e \in \Lambda_k^1$ we write $\bar{e}$ for the same geometric edge with the opposite orientation, and we set $U_{\bar{e}} := U_e^{-1}$.

For each plaquette $p = (e_1, e_2, e_3, e_4) \in \Lambda_k^2$ the *holonomy* is

$$U_p \;=\; U_{e_1}\, U_{e_2}\, U_{e_3}^{-1}\, U_{e_4}^{-1} \;\in\; G. \tag{7}$$

The *Wilson action* at inverse coupling $\beta > 0$ is

$$S_{\mathrm{W}}(U) \;=\; \sum_{p\in\Lambda_k^2} \Big(1 - \tfrac{1}{N_c}\,\mathrm{Re}\,\mathrm{Tr}(U_p)\Big), \tag{8}$$

and the lattice Yang–Mills measure is

$$d\mu_{\Lambda_k,\beta}(U) \;=\; \frac{1}{Z_{\Lambda_k,\beta}}\; e^{-\beta\, S_{\mathrm{W}}(U)} \prod_{e\in\Lambda_k^1} dU_e, \tag{9}$$

where dU_e denotes the normalised Haar measure on G.

2.2 Small-field regime and effective action

Balaban's RG programme decomposes configurations according to plaquette fluctuations. At each scale k there is a *small-field domain*

$$\Omega_k^{\mathrm{sf}} \;=\; \{U \in G^{\Lambda_k^1} : \|U_p - \mathbb{1}\|_{\mathrm{HS}} \leq p_k \text{ for all scale-}k\text{ plaquettes } p\}, \tag{10}$$

where the threshold $p_k = c_0\, g_k^{1-\delta}$ $(0 < \delta < \frac{1}{4},\ c_0 = c_0(N_c))$ satisfies $p_k \to 0$ as $k \to \infty$ by *asymptotic freedom*:

$$g_k^2 \;=\; \frac{g_0^2}{1 + b_0\, g_0^2\, k} \xrightarrow{k\to\infty} 0, \qquad g_0^2 = \frac{2N_c}{\beta}, \quad b_0 = b_0(N_c) > 0. \tag{11}$$

On Ω_k^{sf} (after axial gauge fixing within blocks) the effective action at scale k has the form

$$S_k^{\mathrm{eff}}(U) \;=\; \beta_k\, S_{\mathrm{W}}(U) + V_k^{\mathrm{irr}}(U), \tag{12}$$

where β_k is the running coupling and V_k^{irr} is the *irrelevant remainder*: it contains all non-Gaussian contributions of engineering dimension ≥ 6 in $d = 4$.

2.3 Interpolated measures for the Duhamel argument

The RG–Cauchy comparison [1] interpolates between a "pure Wilson" measure and the full effective measure by defining, for $t \in [0,1]$,

$$d\nu_{k,t}(U) \;\propto\; \exp\bigl(-\beta_k\, S_{\mathrm{W}}(U) - t\, V_k^{\mathrm{irr}}(U)\bigr)\, \mathbf{1}_{\Omega_k^{\mathrm{sf}}}(U) \prod_{e\in\Lambda_k^1} dU_e. \tag{13}$$

At $t = 0$ this is the Wilson measure restricted to Ω_k^{sf}; at $t = 1$ it is the full small-field effective measure. The Duhamel identity gives

$$\mathbb{E}_{\nu_{k,1}}[f] - \mathbb{E}_{\nu_{k,0}}[f] \;=\; -\int_0^1 \mathrm{Cov}_{\nu_{k,t}}(f,\, V_k^{\mathrm{irr}})\, dt. \tag{14}$$

2.4 Imported hypotheses

The present paper derives Theorem 1.3 from three hypotheses. Each is a standard output of Balaban's RG once the irrelevant remainder is expressed in properly normalised dimensionless variables. A companion paper (Paper 12a) will verify them from Balaban's original work [12, 13, 14, 15, 16, 17, 18, 19] and Dimock's reformulations [20, 21, 22].

Assumption A1 (Polymer representation and locality)**.** On Ω_k^{sf}, the irrelevant remainder admits a polymer expansion

$$V_k^{\mathrm{irr}}(U) \;=\; \sum_{X\in\mathcal{P}_k} K_k(X;\, U|_X), \tag{15}$$

where $\mathcal{P}_k$ is a countable collection of *polymers* (connected subsets of the block lattice at scale k), each activity $K_k(X;\cdot)$ depends only on link variables $\{U_e : e \in X\}$.

Assumption A2 (Per-link oscillation bound with irrelevance scaling)**.** There exist constants $\kappa > 0$, $C_{\mathrm{osc}} > 0$, and an integer $p \ge 0$, all independent of k and L, such that for every $X \in \mathcal{P}_k$ and every link $e \in X$,

$$\mathrm{osc}_e\bigl(K_k(X)\bigr) \;:=\; \sup_{\substack{U,U' \\ U\equiv U' \text{ off } e}} \bigl|K_k(X;U|_X) - K_k(X;U'|_X)\bigr| \;\le\; C_{\mathrm{osc}}\, 2^{-2k}\, |X|_k^p\, e^{-\kappa\,|X|_k}, \tag{16}$$

where $|X|_k$ denotes the number of scale-k blocks in X.

Remark 2.1 (Origin of the 2^{-2k} factor). In $d = 4$, irrelevant terms in the effective action have engineering dimension ≥ 6. When expressed in *dimensionless* variables (field fluctuations measured in units of a_k), each such term carries a prefactor $a_k^{6-4} = a_k^2$ relative to the marginal (dimension-4) terms. Since $a_k = a_0\, 2^{-k}$, this gives $a_k^2 = a_0^2\, 2^{-2k}$. The constant a_0^2 is absorbed into C_{osc}.

This is the single most important structural fact in the proof: the 2^{-2k} in the oscillation bound yields a factor 2^{-4k} after squaring in the variance–oscillation inequality, which exactly compensates the 2^{4k} growth of $|\Lambda_k^1|$.

Assumption A3 (Lattice-animal counting)**.** There exists $C_{\mathrm{anim}} < \infty$, independent of k and L, such that for every link $e \in \Lambda_k^1$ and every $n \ge 1$,

$$\bigl|\{X \in \mathcal{P}_k : e \in X,\ |X|_k = n\}\bigr| \;\le\; C_{\mathrm{anim}}^n. \tag{17}$$

No specific value of C_{anim} is needed in this paper beyond finiteness.

Remark 2.2 (A standard explicit choice of C_{anim} for bounded-degree lattices). If polymers are defined as connected vertex-sets in a graph of maximum degree Δ, then the number of connected vertex-sets of size n containing a fixed vertex is at most $(e(\Delta-1))^{n-1}$ (hence also at most $(e\Delta)^{n-1}$); see, e.g., [27, pp. 129–130]. Consequently one may take, for instance, $C_{\text{anim}} := e\Delta$ in Assumption A3. For the nearest-neighbour block lattice in $d=4$ one has $\Delta = 2d = 8$, hence one may take $C_{\text{anim}} \le 8e$.

Remark 2.3 (Logical structure of the paper). The logical content of this paper is the implication

$$\textbf{(A1)} + \textbf{(A2)} + \textbf{(A3)} \implies \textbf{(B6)}.$$

Once Paper 12a establishes **??** A1–A3 from Balaban's work, Assumption (B6) becomes unconditional.

3 Influence toolkit: conditional variance and oscillation

3.1 Single-link conditional variance

We recall the definitions from Theorem 1.2. For a probability measure ν on $G^{\Lambda_k^1}$, a link $e \in \Lambda_k^1$, and $f \in L^2(\nu)$, the conditional variance $\mathrm{Var}_e^\nu(f)$ is the random variable (2) obtained by integrating f over the marginal of U_e with all other links frozen.

The seminorm $\sigma_\nu(f)^2 = \sum_e \mathbb{E}_\nu[\mathrm{Var}_e^\nu(f)]$ is exactly the quantity entering the Duhamel covariance bound in [1]:

$$|\mathrm{Cov}_\nu(f,h)| \;\le\; \sigma_\nu(f)\,\sigma_\nu(h). \tag{18}$$

Remark 3.1 (No mixing or LSI needed). The bound (18) holds for *any* measure ν on a product of compact groups, by the tensorisation structure of conditional variances. No log-Sobolev inequality, no mixing condition, and no spectral gap enter the proof of (B6). These will be needed later (for the mass gap and the thermodynamic limit), but not here.

3.2 Conditional variance bounded by oscillation

Lemma 3.2 (Variance–oscillation inequality)**.** *For any probability measure ν on $G^{\Lambda_k^1}$, any $f \in L^\infty(\nu)$, and any link $e \in \Lambda_k^1$,*

$$\mathrm{Var}_e^\nu(f) \;\le\; \frac{1}{4}\,\mathrm{osc}_e(f)^2 \qquad \nu\text{-a.s.}, \tag{19}$$

where $\mathrm{osc}_e(f)$ denotes the deterministic single-link oscillation

$$\mathrm{osc}_e(f) \;:=\; \sup_{\substack{U,U' \\ U\equiv U' \text{ off } e}} \left|f(U) - f(U')\right|,$$

cf. (16)*.*

Proof. Fix an arbitrary configuration of all links except e. The conditional law of U_e under ν is a probability measure on G. The function $U_e \mapsto f(U_e, U_{\neq e})$ takes values in an interval of length at most $\mathrm{osc}_e(f)$. For any random variable Y supported in $[a, a+\Delta]$ one has

$$\mathrm{Var}(Y) \;\le\; \frac{\Delta^2}{4}$$

(sharp for the Bernoulli on endpoints $\{a, a+\Delta\}$). Applying this with $\Delta = \mathrm{osc}_e(f)$ gives (19). ☐

Remark 3.3. The inequality (19) is *measure-free*: the right-hand side depends only on the pointwise oscillation of f, not on ν. This is why our proof of (B6) is robust under the interpolation $t \in [0,1]$.

3.3 Subadditivity and polymer localisation of oscillation

Lemma 3.4 (Subadditivity of oscillation)**.** *For any functions $f_1, \ldots, f_m$ on $G^{\Lambda_k^1}$,*

$$\operatorname{osc}_e\Big(\sum_{i=1}^m f_i\Big) \;\le\; \sum_{i=1}^m \operatorname{osc}_e(f_i). \tag{20}$$

If f_i does not depend on U_e, then $\operatorname{osc}_e(f_i) = 0$.

Proof. For the first claim: by the triangle inequality,

$$\Big|\sum_i f_i(U) - \sum_i f_i(U')\Big| \;\le\; \sum_i |f_i(U) - f_i(U')|\,.$$

Taking the supremum over (U, U') agreeing off e on both sides gives (20). The second claim is immediate. □

Corollary 3.5 (Oscillation of V_k^{irr} localises to polymers through e)**.** *Under Assumption A1,*

$$\operatorname{osc}_e(V_k^{\mathrm{irr}}) \;\le\; \sum_{\substack{X\in\mathcal{P}_k\\ e\in X}} \operatorname{osc}_e(K_k(X)). \tag{21}$$

Proof. Apply Theorem 3.4 to $V_k^{\mathrm{irr}} = \sum_{X\in\mathcal{P}_k} K_k(X)$. The terms with $e \notin X$ vanish by locality (Assumption A1). □

4 Proof of Theorem 1.3: closing (B6)

4.1 Per-link conditional variance bound

Lemma 4.1 (Uniform per-link bound)**.** *Under* **??** *A1–A3, for every link $e \in \Lambda_k^1$, every scale $k \ge 0$, and every $t \in [0,1]$,*

$$\mathbb{E}_{\nu_{k,t}}[\operatorname{Var}_e^{\nu_{k,t}}(V_k^{\mathrm{irr}})] \;\le\; C_1\, 2^{-4k}, \tag{22}$$

where

$$C_1 \;:=\; \frac{1}{4}\, S_*^2, \qquad S_* \;:=\; C_{\mathrm{osc}} \sum_{n=1}^{\infty} C_{\mathrm{anim}}^n\, n^p\, e^{-\kappa\, n}. \tag{23}$$

The series S_* converges whenever $\kappa > \log C_{\mathrm{anim}}$.

Proof. We proceed in three steps.

Step 1: Variance to oscillation. By Theorem 3.2,

$$\mathbb{E}_{\nu_{k,t}}[\operatorname{Var}_e^{\nu_{k,t}}(V_k^{\mathrm{irr}})] \;\le\; \frac{1}{4} \operatorname{osc}_e(V_k^{\mathrm{irr}})^2. \tag{24}$$

Step 2: Polymer localisation. By Theorem 3.5,

$$\operatorname{osc}_e(V_k^{\mathrm{irr}}) \;\le\; \sum_{\substack{X\in\mathcal{P}_k\\ e\in X}} \operatorname{osc}_e(K_k(X)). \tag{25}$$

Step 3: Oscillation bound + counting. Group the sum by polymer size $n = |X|_k$:

$$\begin{aligned}
\sum_{\substack{X\in\mathcal{P}_k\\ e\in X}} \operatorname{osc}_e(K_k(X)) \;&=\; \sum_{n=1}^{\infty} \sum_{\substack{X\in\mathcal{P}_k\\ e\in X,\ |X|_k=n}} \operatorname{osc}_e(K_k(X)) \\
&\le\; \sum_{n=1}^{\infty} C_{\mathrm{anim}}^n \cdot C_{\mathrm{osc}}\, 2^{-2k}\, n^p\, e^{-\kappa\, n} \qquad (26) \\
&=\; 2^{-2k}\, S_*, \qquad (27)
\end{aligned}$$

where in (26) we used Assumption A3 (at most $C_{\rm anim}^n$ polymers of size n through e) and Assumption A2 (each has oscillation at most $C_{\rm osc}\,2^{-2k}\,n^p\,e^{-\kappa n}$).

Combining (24), (25), and (27):

$$\mathbb{E}_{\nu_{k,t}}[\mathrm{Var}_e^{\nu_{k,t}}(V_k^{\rm irr})] \;\le\; \frac{1}{4}\,(2^{-2k}\,S_*)^2 \;=\; C_1\,2^{-4k}. \tag{28}$$

Convergence of S_*. The ratio test gives $C_{\rm anim}^n\,n^p\,e^{-\kappa n} = n^p\,e^{-(\kappa-\log C_{\rm anim})\,n}$, which is summable if and only if $\kappa > \log C_{\rm anim}$. In Balaban's estimates, $\kappa \sim c\,\beta/N_c$ for large β, so this holds for $\beta \ge \beta_0(N_c)$ with an explicit finite β_0. □

4.2 Summation over links and scale cancellation

Proof of Theorem 1.3. By definition of the seminorm (3),

$$\sigma_{\nu_{k,t}}(V_k^{\rm irr})^2 \;=\; \sum_{e\in\Lambda_k^1} \mathbb{E}_{\nu_{k,t}}[\mathrm{Var}_e^{\nu_{k,t}}(V_k^{\rm irr})]. \tag{29}$$

Applying Theorem 4.1 to every link:

$$\sigma_{\nu_{k,t}}(V_k^{\rm irr})^2 \;\le\; |\Lambda_k^1|\,C_1\,2^{-4k}. \tag{30}$$

The key cancellation. The number of positively oriented links in Λ_k is

$$|\Lambda_k^1| \;=\; d\left(\frac{L}{a_k}\right)^d \;=\; d\left(\frac{L}{a_0}\right)^d 2^{dk} \;=\; 4\left(\frac{L}{a_0}\right)^4 2^{4k} \qquad (d=4). \tag{31}$$

Therefore

$$|\Lambda_k^1|\;\cdot\;2^{-4k} \;=\; 4\left(\frac{L}{a_0}\right)^4, \tag{32}$$

which is **independent of** k.

Substituting (32) into (30):

$$\sigma_{\nu_{k,t}}(V_k^{\rm irr})^2 \;\le\; 4\left(\frac{L}{a_0}\right)^4 C_1 \;=:\; C^2, \tag{33}$$

so that

$$\sigma_{\nu_{k,t}}(V_k^{\rm irr}) \;\le\; C \;=\; 2\left(\frac{L}{a_0}\right)^2 \sqrt{C_1},$$

which is independent of k and of $t\in[0,1]$.

Uniformity in t. The bound (28) uses only the deterministic oscillation $\mathrm{osc}_e(V_k^{\rm irr})$, which does not depend on the measure $\nu_{k,t}$. The conditional variance $\mathrm{Var}_e^{\nu_{k,t}}$ is bounded by $(1/4)\,\mathrm{osc}_e(V_k^{\rm irr})^2$ for *every* measure $\nu_{k,t}$ (Theorem 3.2), hence the final bound is uniform in $t\in[0,1]$. □

5 Interface with the RG–Cauchy telescoping argument

5.1 Direct plug-in into the Duhamel bound

The RG–Cauchy paper [1] uses the Duhamel identity (14) together with the influence covariance bound (18) to obtain

$$\Big|\mathbb{E}_{\nu_{k,1}}[f]-\mathbb{E}_{\nu_{k,0}}[f]\Big| \;\le\; \int_0^1 \sigma_{\nu_{k,t}}(f)\,\sigma_{\nu_{k,t}}(V_k^{\rm irr})\,dt \;\le\; \sup_t \sigma_{\nu_{k,t}}(f)\;\cdot\;C. \tag{34}$$

For blocked observables $f(\bar{U}) = F(Q_{\ell,k}(\bar{U}))$, the paper [1] proves

$$\sigma_{\nu_{k,t}}(f) \;\le\; \mathrm{Lip}(F)\,\Big(\frac{a_k}{\ell}\Big)^2 \;\cdot\; (\text{volume factor}). \tag{35}$$

Combining (34) and (35) with $a_k = a_0\,2^{-k}$, the one-step error at scale k is

$$\delta_k \;\lesssim\; \mathrm{Lip}(F)\,\Big(\frac{a_0}{\ell}\Big)^2\,2^{-2k}\;\cdot\;C. \tag{36}$$

5.2 Geometric summability

Corollary 5.1 (RG–Cauchy convergence for blocked observables)**.** *The telescoping sum converges geometrically:*

$$\sum_{k=0}^{\infty}\delta_k \;\lesssim\; \mathrm{Lip}(F)\,\Big(\frac{a_0}{\ell}\Big)^2\,C\sum_{k=0}^{\infty}2^{-2k} \;=\; \frac{4}{3}\,\mathrm{Lip}(F)\,\Big(\frac{a_0}{\ell}\Big)^2\,C \;<\; \infty. \tag{37}$$

This proves that the sequence of lattice expectations $\{\mathbb{E}_{\mu_k}[f]\}_{k\ge0}$ is Cauchy for every blocked observable of finite Lipschitz norm.

Remark 5.2 (What remains for a complete proof). Theorem 5.1 makes the RG–Cauchy convergence unconditional, *given* **??** A1–A3. To complete the programme one must:

- (i) **Paper 12a:** Verify the three assumptions from Balaban's work. This converts the conditional result into an unconditional continuum limit for blocked observables at fixed L.
- (ii) **Paper 13:** Establish Osterwalder–Schrader positivity (or a gradient-flow substitute) and reconstruct the physical Hilbert space.
- (iii) **Paper 14:** Take the thermodynamic limit $L \to \infty$ and prove that the continuum mass gap $m_{\mathrm{phys}} > 0$ persists.
- (iv) **Papers 15–16:** Extend to renormalised local observables and assemble the full OS axioms plus mass gap.

5.3 Discussion of alternative approaches

Remark 5.3 (Independence from OS/RP). Osterwalder–Schrader positivity is essential for the eventual Hilbert-space reconstruction, but it is not on the critical path for closing the UV telescoping. The Duhamel comparison and RG–Cauchy convergence are made unconditional at fixed L without solving OS/RP compatibility of the blocking map. OS/RP runs as a parallel feasibility track (cf. Section A).

Remark 5.4 (Modular separation from Balaban's estimates). The present paper intentionally isolates the probabilistic mechanism into three explicit assumptions (**??** A1–A3). This modular design allows the influence-theoretic argument to be verified independently of the polymer norm estimates, and creates a clean interface for Paper 12a.

Remark 5.5 (No need for LSI or mixing in the UV block). The oscillation-based proof avoids all measure-dependent technology: no log-Sobolev inequality, no spectral gap, and no mixing condition. This makes Theorem 1.3 simultaneously simpler, more robust, and free of any suspicion of circularity with the lattice-level estimates of Block 1 [2, 6]. Log-Sobolev and mixing techniques will reappear in the IR block (Papers 14–16), where they are genuinely needed for the thermodynamic limit and mass gap.

A Reflection positivity feasibility (parallel track)

This paper does not use Osterwalder–Schrader (OS) positivity. The full Clay package, however, requires an OS-compatible construction and a rigorous bridge from lattice reflection positivity to the reconstructed Hamiltonian in the continuum.

The current programme runs a separate feasibility check:

(a) Half-plane blocking. Construct a blocking map Q_k^{hp} that averages only over links in the temporal half-space $\{x \in \Lambda : x_0 \geq 0\}$. Verify simultaneously: gauge covariance, Lipschitz contraction $\mathrm{Lip}(Q_k^{\mathrm{hp}}) < 1$, and factorisation with respect to temporal reflection.

(b) Gradient-flow alternative. If the half-plane blocking fails to preserve contraction, replace the geometric blocking by Yang–Mills gradient flow (Wilson flow) [25, 26], which preserves RP automatically and defines smooth gauge-invariant observables.

(c) Scope impact. The gradient-flow pivot would require one additional paper but does not affect Paper 12b: the oscillation-based proof of (B6) is independent of the blocking architecture.

This appendix serves as a placeholder for the parallel RP feasibility document. It is not used anywhere in the proofs of this paper.

B Constants bookkeeping

B.1 Master table of constants

Symbol	Meaning	Depends on	Source
β_0	Minimum inverse coupling	N_c	Block 1 [2, 3, 4, 5, 6, 7, 8, 9, 10]
κ	Polymer decay rate	N_c, β_0	Assumption A2
C_{osc}	Oscillation prefactor	N_c, β_0	Assumption A2
p	Polynomial exponent in osc. bound	—	Assumption A2
C_{anim}	Lattice-animal constant	d	Assumption A3
S_*	Polymer oscillation series	$\kappa, C_{\mathrm{osc}}, C_{\mathrm{anim}}, p$	(23)
C_1	$S_*^2/4$	S_*	(23)
C	Main bound	$C_1, L/a_0$	(5)

B.2 The convergence condition

The only structural convergence condition is that polymer decay beats lattice-animal growth:

$$\kappa \; > \; \log C_{\mathrm{anim}}. \tag{38}$$

Once an explicit value of C_{anim} is fixed, this requires $\kappa > \log C_{\mathrm{anim}}$.

Example (nearest-neighbour block lattice). If polymers are connected subsets of a graph of maximum degree Δ, one may take $C_{\mathrm{anim}} := e\Delta$. In $d = 4$ with nearest-neighbour adjacency one has $\Delta = 8$, hence $\kappa > \log(8e)$ is sufficient.

In Balaban's estimates, $\kappa \sim c\,\beta/N_c$ for large β. Therefore (38) holds for $\beta \geq \beta_0(N_c)$ with an *explicit, finite* threshold β_0.

Remark B.1. Note the contrast with the earlier formulation that required $2\kappa > \log C_{\mathrm{anim}}$: the factor of 2 arose from squaring the oscillation *after* summing, rather than summing the squared oscillation. In the present proof (Theorem 4.1), we first bound the oscillation sum $\mathrm{osc}_e(V_k^{\mathrm{irr}}) \le 2^{-2k} S_*$ and then square. The convergence of S_* requires only $\kappa > \log C_{\mathrm{anim}}$, not $2\kappa > \log C_{\mathrm{anim}}$. This is a (mild) improvement.

B.3 What Paper 12a must discharge

To make Theorem 1.3 unconditional, the companion Paper 12a must verify:

Assumption	Content to verify
Assumption A1	Polymer representation of V_k^{irr} on Ω_k^{sf}, with activities depending only on links inside each polymer.
Assumption A2	Per-link oscillation bound with the 2^{-2k} irrelevance factor. This is the most delicate verification: it requires tracking the engineering dimension through Balaban's extraction of irrelevant terms.
Assumption A3	Lattice-animal counting in the block lattice Λ_k. This is a standard combinatorial fact.

B.4 Notation dictionary

This paper	RG–Cauchy [1]	Balaban [13]
V_k^{irr}	V_k^{irr}	R_k
$K_k(X)$	$\Phi_k(X)$	$K(X,U)$
$\mathcal{P}_k$	$\mathcal{P}_k$	polymer collection
$\lvert X\rvert_k$	$\lvert X\rvert_k$	$\lvert X\rvert$
Ω_k^{sf}	Ω_k^{sf}	small-field region
p_k	p_k	ε_k
g_k	g_k	$g(k)$
$\nu_{k,t}$	$\nu_{k,t}$	—
$\sigma_\nu(f)$	$\sigma_\nu(f)$	—
Q_k	Q_k (blocking map)	—
$\mathrm{osc}_e(f)$	(oscillation)	—

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