Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization — a Corrected and Conditional Influence Bound*

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February 2026 (v2: July 2026)

Abstract

We study a uniform Doob martingale influence bound for the irrelevant polymer remainder arising in multiscale renormalization group analyses of four-dimensional $SU(N_c)$ lattice Yang–Mills theory at fixed physical volume, via the Doob influence seminorm $\sigma_\nu(f)^2 = \sum_i E_\nu[(\Delta_i f)^2]$ and its exact covariance identity. Version 2 corrects a genuine error of v1: the increment–oscillation inequality $E[(\Delta_i f)^2 \,|\, \mathcal{F}_{i-1}] \le \frac{1}{4}\,\mathrm{osc}_{e_i}(f)^2$ was asserted for *arbitrary* probability measures; it is false in general (Example 3.4: two perfectly correlated spins, $f = X_2$, give $E[(\Delta_1 f)^2] = 1$ while $\mathrm{osc}_{e_1}(f) = 0$), because the Doob increment collects influence transmitted through correlations. The correct, measure-independent statement uses the *conditional* oscillation $\widetilde{\mathrm{osc}}_i(f)$ (Lemma 3.5); passing from $\widetilde{\mathrm{osc}}_i$ back to osc_{e_i} requires a decoupling hypothesis (H-DEC) bounding the influence-leakage matrix, of Dobrushin type — plausible for the interpolating Gibbs measures $\nu_{k,t}$ in the small-field weak-coupling regime, but unproven, and structurally akin to the (H-DOB-blk) family of the audited chain. Under (H-DEC), the imported oscillation input (A2) (now cited from 2602.0069 v2: traceable, conditional — β_{LF} dichotomy and all), and the lattice-animal lemma (proved here, verified exactly), the main theorem holds: $\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$ uniformly in the RG scale k, by the exact scale cancellation $M \cdot 2^{-4k} = 4(L/a_0)^4$. The Duhamel interface then delivers a one-step rate $\delta_k = O(4^{-k})$ — precisely the geometrically summable rate that Assumption 3.5 of 2602.0063 v3 requires (its Remark 3.7 with $\eta = 2$) — *conditionally* on (H-DEC) + (A2) + the assumed blocking contraction (H-LIP). v1's closing claim "this establishes the RG–Cauchy property" is softened accordingly: this paper supplies the leading candidate for closing (H-CAUCHY), not its proof. All quantitative claims, including the counterexample and the Dobrushin-corrected bound on exact Gibbs chains, are adjudicated in a companion suite.

*Version 2. Version 1's Lemma 3.4 and Proposition 3.5 asserted the increment–oscillation inequality *for arbitrary probability measures*, and Remark 3.3 presented this measure-independence as the key advantage of the method. That assertion is *false*: a two-point counterexample (Example 3.4, machine-verified) shows the Doob increment picks up influence through correlations, so the plain oscillation bound fails for non-product measures — including the interpolating Gibbs measures $\nu_{k,t}$ used in this paper. v2 prints the counterexample, replaces the lemma by its correct conditional-oscillation form (Lemma 3.5), and restates Theorem 1.2 conditionally on a decoupling hypothesis (H-DEC) of Dobrushin type (Remark 3.6) — the same family of conditions flagged across the audited chain. The structural inputs (A1)–(A2) are now cited from their traceability companion 2602.0069 v2 (conditional), and the references to the audited chain are updated. See the Note added in v2 (Section 6).

Contents

1 Introduction

1.1 The problem and the programme

The rigorous construction of 4d quantum Yang–Mills theory and its mass gap is a Clay Millennium Problem. Bałaban's programme [1–11] controls the continuum limit scale by scale; Dimock [12–14] gives the modern φ^4_3 template. This note isolates the probabilistic core needed in RG–Cauchy convergence arguments: a uniform influence bound for the irrelevant polymer remainder via Doob martingales.

1.2 Scope and logical skeleton

(v2) The corrected logic is:

$$\text{Doob covariance identity} + \underbrace{\text{conditional-oscillation bound}}_{\text{exact}} + \underbrace{\text{(H-DEC)}}_{\text{decoupling}} + \underbrace{\text{(A1)–(A2)}}_{\text{2602.0069 v2}} + \text{animal counting}$$

$$\Longrightarrow \sup_t \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C \Longrightarrow \text{Duhamel step: } \delta_k = O(4^{-k}) \text{ (candidate for (H-CAUCHY) of 2602.0063 v3).}$$

1.3 Main results

Assumption 1 (Polymer representation — A1)**.** *On the small-field domain $\Omega_k^{\mathrm{sf}} \subset G^{\Lambda_k}$, $V_k^{\mathrm{irr}}(U) = \sum_{X\in\mathcal{P}_k} K_k(X; U|_X)$, each activity depending only on links inside X.* (v2) *Imported from 2602.0069 v2 (traceable; conditional on the transcription of the primary sources).*

Assumption 2 (Per-link oscillation bound — A2)**.** *For every $X \in \mathcal{P}_k$ and link $e \in X$:*

$$\operatorname{osc}_e(K_k(X)) \;\le\; C_{\rm osc}\, 2^{-2k}\, |X|_k^p\, e^{-\kappa |X|_k}. \tag{1}$$

(v2) *Imported from 2602.0069 v2: traceable to [7,9,11], conditional per that paper's Remarks 8.7 and 10.4 (the $\beta_{\rm LF}$ dichotomy affects the large-field part; the 2^{-2k} factor is the fine-lattice irrelevance factor at fixed physical volume, which is the sound direction).*

Lemma 1.1 (Lattice-animal counting)**.** *There exists $C_{\rm anim} = C_{\rm anim}(d) < \infty$ such that for any fixed link e and $n \ge 1$,*

$$|\{X \in \mathcal{P}_k : e \in X,\ |X|_k = n\}| \;\le\; C_{\rm anim}^n. \tag{2}$$

A spanning-tree walk argument gives $C_{\rm anim} = (2d)^3$ (machine-verified against exact enumeration in the companion suites of 2602.0069 v2 and this paper: the true counts satisfy the sharper $(2d)^{2n}$).

Theorem 1.2 (Uniform Doob influence bound; v2: conditional)**.** *Assume A1, A2, Lemma 1.1,* and the decoupling hypothesis (H-DEC) of Remark 3.6. *If $\kappa > \log C_{\rm anim}$, then for every RG scale $k \ge 0$ and every $t \in [0,1]$:*

$$\sigma_{\nu_{k,t}}(V_k^{\rm irr}) \;\le\; C, \tag{3}$$

with $C = C(N_c, \kappa, C_{\rm osc}, C_{\rm anim}, p, L/a_0, b_*)$ independent of k and t. (v1 asserted this without (H-DEC), relying on the false measure-independent Lemma 3.4; see Example 3.4.)

2 Set-up and imported structural inputs

$\Lambda_k = (a_k\mathbb{Z}/L\mathbb{Z})^4$, $a_k = a_0 2^{-k}$, links $U_e \in G = SU(N_c)$, $M = |\Lambda_k^1| = 4(L/a_0)^4 2^{4k}$. The probabilistic arguments use only A1, A2, Lemma 1.1, and — new in v2 — (H-DEC); all other Balaban details are cited via 2602.0069 v2.

3 Doob influence seminorm and uniform bound

3.1 The Doob martingale influence seminorm

Definition 3.1 (Doob influence seminorm)**.** *Fix an ordering $e_1, \dots, e_M$ of the links, filtration $\mathcal{F}_i := \sigma(U_{e_1}, \dots, U_{e_i})$, increments*

$$\Delta_i f := E_\nu[f \,|\, \mathcal{F}_i] - E_\nu[f \,|\, \mathcal{F}_{i-1}], \qquad \sigma_\nu(f)^2 := \sum_{i=1}^M E_\nu[(\Delta_i f)^2]. \tag{4}$$

3.2 Exact covariance identity

Lemma 3.2 (Covariance via Doob increments)**.** *For any probability measure ν and $f, h \in L^2(\nu)$:*

$$\operatorname{Cov}_\nu(f,h) = \sum_{i=1}^M E_\nu[\Delta_i f\, \Delta_i h], \qquad |\operatorname{Cov}_\nu(f,h)| \le \sigma_\nu(f)\,\sigma_\nu(h), \tag{5}$$

by orthogonality of martingale differences and Cauchy–Schwarz. This identity is genuinely measure-independent and survives v1 unchanged *(suite check 1).*

Remark 3.3 (What is and is not measure-independent — v2 rewrite of v1's Remark 3.3)**.** *v1 claimed the entire route needs "no independence, mixing, or log-Sobolev assumptions." Only the covariance identity* (5) *has that property. The passage from increments to per-link oscillations does* not*: it is exactly where dependence re-enters (Example 3.4). The corrected route quantifies that re-entry through* (H-DEC).

3.3 Increment–oscillation inequality: the v1 error and its correction

v1's Lemma 3.4 asserted, *for any* ν: $E_\nu[(\Delta_i f)^2|\mathcal{F}_{i-1}] \le \frac{1}{4}\operatorname{osc}_{e_i}(f)^2$.

Example 3.4 (Two-point counterexample — the v1 lemma is false)**.** *Let* $X_1, X_2 \in \{\pm 1\}$ *be* perfectly correlated *(*$X_2 = X_1$ *a.s., each symmetric), and* $f = X_2$*. Then* f *does not depend on the first coordinate:* $\operatorname{osc}_{e_1}(f) = 0$*. But* $\Delta_1 f = E[X_2|X_1] - E[X_2] = X_1$*, so* $E[(\Delta_1 f)^2] = 1 > \frac{1}{4}\operatorname{osc}_{e_1}(f)^2 = 0$*. The Doob increment collects influence transmitted through the correlation. The same mechanism operates, quantitatively attenuated, under any non-product* ν *— including the interpolating Gibbs measures* $\nu_{k,t}$ *of Section 4, whose Hamiltonian couples links. (Machine-verified, together with the fact that for* product *measures the v1 bound does hold: suite check 2.)*

Lemma 3.5 (Corrected increment–oscillation inequality)**.** *For any* ν*,* $f \in L^\infty$*, and* i*, define the* conditional oscillation

$$\widetilde{\operatorname{osc}}_i(f) \;:=\; \operatorname*{ess\,sup}_{\mathcal{F}_{i-1}} \operatorname{osc}\big(u \mapsto E_\nu[f \,|\, \mathcal{F}_{i-1}, U_{e_i} = u]\big). \tag{6}$$

Then

$$E_\nu[(\Delta_i f)^2 \,|\, \mathcal{F}_{i-1}] \;\le\; \tfrac{1}{4}\,\widetilde{\operatorname{osc}}_i(f)^2 \qquad \nu\text{-a.s.}, \tag{7}$$

and hence $\sigma_\nu(f)^2 \le \frac{1}{4}\sum_i \widetilde{\operatorname{osc}}_i(f)^2$*. This is exact and measure-independent (a variable supported in an interval of length* Δ *has variance* $\le \Delta^2/4$*); the difference from v1 is that* $\widetilde{\operatorname{osc}}_i$*, not* osc_{e_i}*, is the correct modulus. For product* ν*,* $\widetilde{\operatorname{osc}}_i(f) \le \operatorname{osc}_{e_i}(f)$ *by conditional Jensen, recovering v1's Proposition 3.5 in the only regime where it was true.*

Remark 3.6 (Decoupling hypothesis (H-DEC); new in v2)**.** *For non-product* ν*, conditioning on* $U_{e_i} = u$ *tilts the law of the future links; a change of* u *propagates with influence coefficients* b_{ij} *(the oscillation of the conditional law of* U_{e_j}*,* $j > i$*, under a change of* U_{e_i} *— a one-sided Dobrushin matrix). Standard telescoping gives*

$$\widetilde{\operatorname{osc}}_i(f) \;\le\; \operatorname{osc}_{e_i}(f) + \sum_{j>i} b_{ij}\, \operatorname{osc}_{e_j}(f). \tag{8}$$

(H-DEC): uniformly in k, t and the ordering, $\|b^{(k,t)}\|_{\ell^2\to\ell^2} \le b_* < \infty$*. Then*

$$\sigma_\nu(f)^2 \;\le\; \tfrac{1}{4}(1+b_*)^2 \sum_i \operatorname{osc}_{e_i}(f)^2. \tag{9}$$

For the Gibbs interpolation $\nu_{k,t} \propto e^{-H_{k,0} - tV_k^{\mathrm{irr}}} d\mu^{\otimes}$ *at weak coupling in the small-field regime,* b_{ij} *is expected to decay exponentially in the link distance with small prefactor, making* $b_* = O(1)$ *plausible — but this is* unproven*, and it is structurally the same kind of condition as the (H-DOB-blk) family flagged throughout the audited chain (2602.0053/0054 v2). The companion suite (check 3) verifies* (7)–(9) *on exact Gibbs chains and exhibits the violation of the v1 bound at strong coupling.*

3.4 Polymer localization of oscillation

By subadditivity and A1:

$$\mathrm{osc}_{e_i}(V_k^{\mathrm{irr}}) \;\le\; \sum_{X \ni e_i} \mathrm{osc}_{e_i}(K_k(X)). \tag{10}$$

3.5 Proof of Theorem 1.2

Proof. Step 1 (Per-link bound). By A2, (10) and Lemma 1.1: $\mathrm{osc}_{e_i}(V_k^{\mathrm{irr}}) \le 2^{-2k} S_*$ with $S_* = C_{\mathrm{osc}} \sum_{n\ge1} C_{\mathrm{anim}}^n n^p e^{-\kappa n} < \infty$ for $\kappa > \log C_{\mathrm{anim}}$.

Step 2 (From oscillations to increments — corrected). By Lemma 3.5 and (H-DEC) ((9)):

$$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})^2 \;\le\; \tfrac{1}{4}(1+b_*)^2\, M\, (2^{-2k} S_*)^2. \tag{11}$$

Step 3 (Scale cancellation). With $M = 4(L/a_0)^4 2^{4k}$:

$$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})^2 \;\le\; (1+b_*)^2 (L/a_0)^4 S_*^2, \tag{12}$$

independent of k and t (the t-uniformity uses only that the route is deterministic-oscillation-based — *given* (H-DEC) uniformly in t). □

4 Interface: From Influence Bound to RG–Cauchy Convergence

With $\nu_{k,t} \propto \exp(-H_{k,0} - tV_k^{\mathrm{irr}})\, d\mu$, differentiation under the integral gives the Duhamel identity

$$E_{\nu_{k,1}}[f] - E_{\nu_{k,0}}[f] \;=\; -\int_0^1 \mathrm{Cov}_{\nu_{k,t}}(f, V_k^{\mathrm{irr}})\, dt, \tag{13}$$

(exact; suite check 5) and, by Lemma 3.2 and Theorem 1.2,

$$\big|E_{\nu_{k,1}}[f] - E_{\nu_{k,0}}[f]\big| \;\le\; C \sup_t \sigma_{\nu_{k,t}}(f). \tag{14}$$

For blocked observables $f = F \circ Q_{\ell,k}$ we *assume* (**(H-LIP)**, part of the companion Master framework, unaudited) a deterministic contraction $\mathrm{Lip}(Q_{\ell,k}) \le C(a_k/\ell)^2$; then the one-step error obeys $\delta_k \lesssim (a_k/\ell)^2 = O(4^{-k})$: geometrically summable.

Remark 4.1 (What this does and does not establish; v2 rewrite)**.** *v1 concluded "this establishes the RG–Cauchy property for blocked observables." Corrected:* conditionally on (H-DEC)+(A2)+(H-LIP), *this route delivers a one-step rate $\delta_k = O(4^{-k})$ — precisely a rate of the sufficient class $\varepsilon(a) \le C(a/\ell)^\eta$ ($\eta = 2$) identified in 2602.0063 v3, Remark 3.7, for its Assumption 3.5 (the RG–Cauchy hypothesis (H-CAUCHY), whose naive $O(g_k^2)$ bridge is not summable: F-SUM). This paper is therefore the leading* candidate *for closing (H-CAUCHY), not its proof. Two further gaps remain: the Duhamel step compares the remainder switched on/off* at fixed k*; identifying that difference with the $k \to k+1$ RG–Cauchy difference is part of the (unaudited) Master framework; and the $\nu_{k,t}$-uniformity of* (H-DEC) *along the interpolation is open.*

5 Conclusion and Checklist

5.1 What is established

Item	Content	Status (v2)
Lemma 3.2	covariance identity	exact, any ν (verified)
Example 3.4	v1 Lemma 3.4 false for general ν	exact counterexample
Lemma 3.5	conditional-oscillation bound	exact, any ν (verified)
(H-DEC)	leakage control $b_* < \infty$	*hypothesis* (Dobrushin-type)
(A1)–(A2)	structural inputs	2602.0069 v2: traceable, conditional
Lemma 1.1	animal entropy	proved; enumeration-verified
Theorem 1.2	$\sup_t \sigma \le C$	conditional on (H-DEC)+(A2)
$\delta_k = O(4^{-k})$	candidate (H-CAUCHY) rate	conditional, + (H-LIP)

5.2 What remains

(1) Prove (H-DEC) for $\nu_{k,t}$ at weak coupling (Dobrushin coefficients of the interpolating Gibbs measures, uniformly in t and k). (2) Settle the β_{LF} dichotomy of 2602.0069 v2 (else (H-P0) enters through (A2)'s large-field part). (3) Audit the Master framework: (H-LIP) and the on/off-vs-$k \to k+1$ identification. (4) OS/RP reconstruction, thermodynamic limit, mass gap (the 2602.0063 v3 layer).

6 Note added in v2

Version 1 contained a genuine mathematical error, found in the series audit: Lemma 3.4/Proposition 3.5 asserted the increment–oscillation bound for arbitrary measures, and Remark 3.3 advertised this as the method's key advantage. Example 3.4 (two correlated spins, $E[(\Delta_1 f)^2] = 1$ vs. $\frac{1}{4}\,\mathrm{osc}^2 = 0$) refutes it; the suite verifies both the counterexample and the validity of the v1 bound in the product case, quantifies the violation on exact Gibbs chains at strong coupling, and validates the corrected route (Lemma 3.5 + (H-DEC)) at weak coupling. Theorem 1.2 is restated conditionally; the scale-cancellation arithmetic (12) and the covariance/Duhamel identities survive exactly. The interface claim is softened to "leading candidate for (H-CAUCHY) of 2602.0063 v3." References updated: [15] = 2602.0041 (v3); [16] cited 2602.0051 under its withdrawn pre-audit title ("...unconditional closure...") — corrected to the v2 series erratum; added: 2602.0069 (v2) as the (A1)–(A2) supplier, 2602.0063 (v3), and the audited chain block. Suite: `verify_2602_0070.py` (6 checks) at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, directory `verification/2602-0070/`.

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