Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization

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February 2026

Abstract

We prove a uniform Doob martingale influence bound for the irrelevant polymer remainder arising in multiscale renormalization group analyses of four-dimensional $\mathrm{SU}(N_c)$ lattice Yang–Mills theory at fixed physical volume.

Our main tool is the Doob influence seminorm $\sigma_\nu(f)^2 = \sum_i \mathbb{E}_\nu[(\Delta_i f)^2]$, which yields an exact covariance identity for *arbitrary probability measures.* Assuming a deterministic per-link oscillation estimate for polymer activities with a scale factor 2^{-2k} (imported from the Balaban renormalization group programme) and using a standard lattice-animal counting lemma (proved here), we obtain a bound $\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \leq C$ independent of the RG scale k. We then explain how this bound feeds into a Duhamel interpolation step used in RG–Cauchy convergence arguments.

Keywords: lattice gauge theory, polymer expansion, Doob martingale, influence bounds, Duhamel interpolation.

Contents

1 Introduction

1.1 The problem and the programme

The rigorous construction of four-dimensional quantum Yang–Mills theory with gauge group $\mathcal{G} = \mathrm{SU}(N_c)$ and a proof of its mass gap is a Clay Millennium Problem. The lattice provides a non-perturbative regularization, but removing the cutoff ($a \to 0$) requires controlling infinitely many scales.

Bałaban's programme (1984–1989) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] achieves this control through a multiscale renormalization group that integrates fluctuations scale by scale, maintaining uniform bounds on effective actions. Dimock's exposition (2013–2014) [12, 13, 14] reformulates the scalar prototype (ϕ^4_3) in a modern functional-analytic framework.

This paper isolates the probabilistic core needed in RG–Cauchy convergence arguments: a uniform influence bound for an irrelevant polymer remainder using Doob martingales. The multiscale RG construction providing the structural inputs (polymer locality and per-link oscillation decay) is taken from the Balaban–Dimock framework and is cited as such.

1.2 Scope and logical skeleton

The logic is:

$$\boxed{\text{Doob covariance identity}} + \boxed{\text{deterministic oscillation bounds (A1)–(A2)}} + \boxed{\text{lattice-animal counting (Lemma 1.1)}}$$
$$\Longrightarrow \boxed{\sup_{t\in[0,1]} \sigma_{\nu_k,t}(V_k^{\mathrm{irr}}) \le C} \Longrightarrow \boxed{\text{Duhamel step for RG–Cauchy estimates.}}$$

1.3 Main results

The structural inputs extracted from Bałaban's work are:

Assumption A1 (Polymer representation — A1)**.** On $\Omega_k^{\mathrm{sf}} \subset \mathcal{G}^{\Lambda_k^1}$ (the small-field domain in the sense of the Balaban RG construction, cf. [9]), the irrelevant remainder admits $V_k^{\mathrm{irr}}(U) = \sum_{X\in\mathcal{P}_k} K_k(X; U|_X)$, where each activity depends only on links inside X. Here $\mathcal{P}_k$ denotes the family of connected k-scale polymers (connected unions of d-dimensional blocks of side a_k in Λ_k), $|X|_k \in \mathbb{N}$ is the number of such blocks in X, and "$e \in X$" means that the link e belongs to $\bigcup_{B\in X} \overline{B}$.

Assumption A2 (Per-link oscillation bound — A2)**.** For every $X \in \mathcal{P}_k$ and link $e \in X$:

$$\mathrm{osc}_e(K_k(X)) \le C_{\mathrm{osc}}\, 2^{-2k}\, |X|_k^p\, e^{-\kappa|X|_k}. \tag{1}$$

The factor 2^{-2k} reflects the gap $d - d_{\mathrm{op}} = 4 - 6 = -2$ between spacetime dimension and the leading irrelevant operators (heuristically), after extraction of relevant/marginal terms in $d = 4$. We treat (1) as an imported deterministic input; see e.g. [9, 11].

Lemma 1.1 (Lattice-animal counting)**.** *There exists $C_{\mathrm{anim}} = C_{\mathrm{anim}}(d) < \infty$ such that for any fixed link e and any $n \ge 1$,*

$$|\{X \in \mathcal{P}_k :\ e \in X,\ |X|_k = n\}| \le C_{\mathrm{anim}}^n. \tag{2}$$

Proof. We give a crude exponential bound (constants are not optimized; see [19, 17, 18] for classical lattice-animal frameworks). Any connected polymer X of size $|X|_k = n$ containing a link e includes at least one block that contains e; the remaining $n-1$ blocks form a connected set in the block-adjacency graph (coordination number $2d$). A spanning tree on n blocks is explored by a walk of at most $2n$ adjacency steps, and the number of such walks is bounded by $(2d)^{2n}$. Hence (2) holds with $C_{\mathrm{anim}} = (2d)^3$. □

The main theorem is:

Theorem 1.2 (Uniform Doob influence bound)**.** *Under Assumptions A1 and A2 and Theorem 1.1, if $\kappa > \log C_{\mathrm{anim}}$, then for every RG scale $k \geq 0$ and every interpolation parameter $t \in [0,1]$:*

$$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \leq C, \tag{3}$$

where $C = C(N_c, \kappa, C_{\mathrm{osc}}, C_{\mathrm{anim}}, p, \mathcal{L}/a_0)$ is independent of k and t.

2 Set-up and imported structural inputs

We work on the four-torus $\Lambda_k = (a_k\mathbb{Z}/\mathcal{L}\mathbb{Z})^4$ with $a_k = a_0\,2^{-k}$ (dyadic blocking). The gauge group is $\mathcal{G} = \mathrm{SU}(N_c)$ and link variables are $U_e \in \mathcal{G}$ for $e \in \Lambda_k^1$. Here Λ_k^1 denotes the set of (oriented) nearest-neighbour links (1-cells) of the lattice Λ_k. Recall that $\mathcal{P}_k$ and $|X|_k$ were defined in Assumption A1.

The probabilistic arguments below only use:

- the locality/polymer decomposition of the remainder V_k^{irr} on the small-field domain (Assumption A1),
- a deterministic per-link oscillation bound with a geometric scale factor 2^{-2k} (Assumption A2),
- lattice-animal entropy bounds (Lemma 1.1).

All other details of the Balaban renormalization group construction (gauge fixing, background fields, random-walk expansions, **R**-operation, and the inductive polymer scheme) are orthogonal to the present note and are deferred to a separate exposition.

3 Doob influence seminorm and uniform bound

This section is self-contained and constitutes the novel probabilistic contribution.

3.1 The Doob martingale influence seminorm

Definition 3.1 (Doob influence seminorm)**.** Fix an ordering $e_1, \ldots, e_M$ of the links in Λ_k^1 and define the filtration $\mathcal{F}_i := \sigma(U_{e_1}, \ldots, U_{e_i})$ with $\mathcal{F}_0$ trivial. For $f \in L^2(\nu)$, the martingale increments are

$$\Delta_i f := \mathbb{E}_\nu[f \mid \mathcal{F}_i] - \mathbb{E}_\nu[f \mid \mathcal{F}_{i-1}]. \tag{4}$$

The Doob influence seminorm is

$$\sigma_\nu(f)^2 := \sum_{i=1}^{M} \mathbb{E}_\nu[(\Delta_i f)^2]. \tag{5}$$

3.2 Exact covariance identity

Lemma 3.2 (Covariance via Doob increments)**.** *For any probability measure ν and $f, h \in L^2(\nu)$:*

$$\mathrm{Cov}_\nu(f,h) = \sum_{i=1}^{M} \mathbb{E}_\nu[\Delta_i f\,\Delta_i h], \tag{6}$$

and hence $|\mathrm{Cov}_\nu(f,h)| \leq \sigma_\nu(f)\,\sigma_\nu(h)$.

Proof. The decomposition $f - \mathbb{E}_\nu[f] = \sum_i \Delta_i f$ is a martingale difference decomposition. By orthogonality of martingale differences, $\mathbb{E}_\nu[\Delta_i f\,\Delta_j h] = 0$ for $i \neq j$. The covariance identity follows; Cauchy–Schwarz gives the bound. ☐

Remark 3.3 (No mixing or LSI needed)*.* This identity holds for *any* measure ν, with no independence, mixing, or log-Sobolev assumptions. This is the key advantage over Efron–Stein-type inequalities that require independent resampling.

3.3 Increment–oscillation inequality

Lemma 3.4. *For any ν, any $f \in L^\infty$, and any i:*

$$\mathbb{E}_\nu[(\Delta_i f)^2 \mid \mathcal{F}_{i-1}] \le \tfrac{1}{4}\,\mathrm{osc}_{e_i}(f)^2 \quad \nu\text{-a.s.}, \tag{7}$$

where $\mathrm{osc}_{e_i}(f) := \sup_{U \equiv U' \text{ off } e_i} |f(U) - f(U')|$.

Proof. Conditioned on $\mathcal{F}_{i-1}$, the increment $\Delta_i f$ is a function of the single variable U_{e_i} taking values in an interval of length at most $\mathrm{osc}_{e_i}(f)$. For any random variable supported in $[a, a+\Delta]$: $\mathrm{Var} \le \Delta^2/4$. □

Proposition 3.5 (Measure-independent Doob bound)**.** *For any probability measure ν and any $f \in L^\infty(\nu)$,*

$$\sigma_\nu(f)^2 \le \tfrac{1}{4}\sum_{i=1}^{M} \mathrm{osc}_{e_i}(f)^2. \tag{8}$$

Proof. Take unconditional expectations in Theorem 3.4 and sum over $i = 1, \dots, M$. □

3.4 Polymer localization of oscillation

By subadditivity of oscillation and the locality of polymer activities (Assumption A1):

$$\mathrm{osc}_{e_i}(V_k^{\mathrm{irr}}) \le \sum_{X \ni e_i} \mathrm{osc}_{e_i}(K_k(X)). \tag{9}$$

3.5 Proof of Theorem 1.2

Proof. **Step 1 (Per-link bound).** Combining Theorem 3.4 with Assumption A2 and Theorem 1.1:

$$\mathbb{E}_{\nu_{k,t}}[(\Delta_i V_k^{\mathrm{irr}})^2] \le \tfrac{1}{4}\Big(\sum_{X \ni e_i} \mathrm{osc}_{e_i}(K_k(X))\Big)^2 \le \tfrac{1}{4}(2^{-2k} S_*)^2 = C_1\, 2^{-4k}, \tag{10}$$

where $S_* = C_{\mathrm{osc}} \sum_{n \ge 1} C_{\mathrm{anim}}^n n^p e^{-\kappa n} < \infty$ when $\kappa > \log C_{\mathrm{anim}}$.

Step 2 (Scale cancellation). Summing over $i = 1, \dots, M$,

$$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})^2 = \sum_{i=1}^{M} \mathbb{E}_{\nu_{k,t}}[(\Delta_i V_k^{\mathrm{irr}})^2] \le M\, C_1\, 2^{-4k}. \tag{11}$$

(This is precisely Theorem 3.5 applied to $f = V_k^{\mathrm{irr}}$, combined with the bound from Step 1.) In $d = 4$ at fixed physical volume $\mathcal{L}^4$, the number of links is $M = |\Lambda_k^1| = 4(\mathcal{L}/a_0)^4\, 2^{4k}$, hence

$$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})^2 \le 4(\mathcal{L}/a_0)^4\, 2^{4k}\, C_1\, 2^{-4k} = 4(\mathcal{L}/a_0)^4\, C_1, \tag{12}$$

which is independent of k and t. □

4 Interface: From Influence Bound to RG–Cauchy Convergence

Fix $k \ge 0$. Let $\nu_{k,0}$ be the reference effective measure at scale k (with the irrelevant remainder switched off), and for $t \in [0,1]$ define the interpolating family $\nu_{k,t}$ by

$$d\nu_{k,t}(U) := \frac{1}{Z_{k,t}} \exp(-H_{k,0}(U) - t\, V_k^{\mathrm{irr}}(U))\, d\mu(U), \tag{13}$$

where $\mu := \bigotimes_{e \in \Lambda_k^1} \mu_{\mathrm{Haar}}$ is the product Haar measure on $\mathcal{G}^{\Lambda_k^1}$ and $Z_{k,t}$ normalizes to a probability measure.

For any bounded observable f, differentiation under the integral gives the Duhamel interpolation identity

$$\mathbb{E}_{\nu_{k,1}}[f] - \mathbb{E}_{\nu_{k,0}}[f] = -\int_0^1 \mathrm{Cov}_{\nu_{k,t}}(f, V_k^{\mathrm{irr}})\, dt. \tag{14}$$

Using Theorem 3.2 and Cauchy–Schwarz,

$$\left|\mathbb{E}_{\nu_{k,1}}[f] - \mathbb{E}_{\nu_{k,0}}[f]\right| \le \int_0^1 \sigma_{\nu_{k,t}}(f)\, \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})\, dt \le \Big(\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})\Big)\Big(\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(f)\Big). \tag{15}$$

By Theorem 1.2, $\sup_{t\in[0,1]} \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$. This uniformity in t is immediate because the route to Theorem 1.2 uses only deterministic oscillation bounds and Theorem 3.5, and therefore does not require any stability of functional inequalities along the interpolation.

For blocked observables $f = F \circ Q_{\ell,k}$, we assume (as part of the companion RG framework) a deterministic contraction estimate of the form $\mathrm{Lip}(Q_{\ell,k}) \le C(a_k/\ell)^2$. Under this input, the one-step error is $\delta_k \lesssim (a_k/\ell)^2 = \mathcal{O}(4^{-k})$, which is geometrically summable: $\sum_k \delta_k < \infty$.

This establishes the RG–Cauchy property for blocked observables. It is a key input toward the existence of a continuum limit in the companion framework (the remaining steps are addressed in the companion works [15, 16]).

5 Conclusion and Checklist

5.1 What is established

Item	Content	Source
(A1)	Polymer representation	[8, 9]
(A2)	Oscillation bound $\le C_{\mathrm{osc}}\, 2^{-2k}\, \lvert X\rvert_k^p e^{-\kappa\lvert X\rvert_k}$	[7, 9, 11]
Lemma 1.1	Lattice-animal entropy $\le C_{\mathrm{anim}}^n$	This paper
(Main)	$\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$	Theorem 1.2 (this paper)

5.2 What remains

1. **Paper 12a** (companion): Verify (A1)–(A2) from Bałaban's original work with explicit uniform constants.
2. **Papers 13–16**: OS/RP reconstruction, renormalized observables, thermodynamic limit, mass gap.

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