The Bałaban–Dimock Structural Package

Derivation of Polymer Representation, Oscillation Bounds, and Large-Field Suppression for Lattice Yang–Mills Theory from Primary Sources

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February 2026

Abstract

We provide a self-contained derivation of the three structural hypotheses — polymer representation **(A1)**, per-link oscillation bounds with geometric decay factor **(A2)**, and large-field suppression **(B5)** — that were assumed in *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization* and in the RG–Cauchy Master Framework. All results are traced to precise equations in the primary sources: the series of papers by T. Bałaban (Commun. Math. Phys., 1984–1989) and the expository trilogy by J. Dimock (2011–2014). The translation from Bałaban's analytic norms on gauge-covariant function spaces to the per-link oscillation language used in the probabilistic framework is made explicit. Together with *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization*, this completes the unconditional discharge of the UV structural inputs for the renormalization group approach to the Yang–Mills mass gap problem at finite volume.

Contents

1 Introduction

1.1 Context and purpose

The program initiated by T. Bałaban in a series of papers [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12] constitutes the most complete rigorous construction of renormalized lattice gauge theories in four space-time dimensions. The companion expository works by J. Dimock [13, 14, 15] provide a modern translation of this program for the scalar ϕ_3^4 model, serving as a structural template.

In this paper, we extract from these primary sources the precise technical results needed to discharge the three structural hypotheses assumed in *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization* and the RG–Cauchy Master:

(A1)

Polymer representation. The irrelevant part of the effective action at scale k admits a decomposition $V_k^{\mathrm{irr}}(U) = \sum_{X\in\mathcal{P}_k} K_k(X; U|_X)$ into polymer activities that are local, analytic, and exponentially decaying in polymer size.

(A2)

Oscillation bound with geometric decay. For each lattice bond e at scale k,

$$\operatorname{osc}_e\big(K_k(X;\cdot)\big) \;\le\; C_{\mathrm{osc}}\,(L^k\eta)^{4+\alpha}\,|X|^p\,e^{-\kappa\,d_k(X)}, \tag{1}$$

where $(L^k\eta)^{4+\alpha}$ is the irrelevance factor ($\alpha > 0$), and $d_k(X)$ is the multi-scale tree distance. In the dyadic case $L = 2$, $\eta = L^{-N}$, this yields the factor 2^{-2k} for the net scaling per step (after accounting for volume).

(B5)

Large-field suppression. The **R**-operation of Bałaban replaces large-field contributions by small-field substitutes while preserving the partition function, producing residual polymer activities $\mathbf{R}^{(k)}(X)$ satisfying

$$|\mathbf{R}^{(k)}(X)| \;\le\; e^{-p_0(g_k)}\,e^{-\kappa\,d_k(X)}. \tag{2}$$

1.2 Logical architecture

The paper is organized as follows:

- §2: Lattice setup, notation, and the renormalization group transformation.
- §3: Abelian propagators, random walk expansions, and exponential decay (from [1, 2]).
- §4: Non-abelian gauge fixing, spaces of regular configurations, and the Landau gauge chart (from [4]).
- §5: The variational problem, background fields, and their analyticity (from [6, 5]).
- §6: Small-field effective actions, Ward–Takahashi identities, and the β-function (from [8]). **Discharge of (A2).**
- §7: Cluster expansions and polymer exponentiation (from [9]). **Discharge of (A1).**

- §8: Large-field renormalization and the **R**-operation (from [11, 12]). **Discharge of (B5).**

- §9: Convergence of the full expansion and UV stability (from [10]).

- §10: Translation dictionary: from Bałaban's analytic norms to the per-link oscillation $\mathrm{osc}_e(\cdot)$.

- §11: The Dimock template: structural parallels with ϕ_3^4 (from [13, 14, 15]).

- §12: Summary of derived axioms and interface with *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization* and the RG–Cauchy Master.

1.3 Notational conventions

Throughout, G denotes a compact simple Lie group (typically $G = SU(N_c)$) with Lie algebra $\mathfrak{g}$. We work on a d-dimensional periodic lattice $T_\eta = (\eta\mathbb{Z}/L^{N_{\mathrm{IR}}}\mathbb{Z})^d$ with lattice spacing $\eta = L^{-N}$, where $L \geq 2$ is the blocking factor, N is the number of UV scales, and N_{IR} the number of IR scales. The continuum limit corresponds to $N \to \infty$ (equivalently $\eta \to 0$) at fixed physical volume $(L^{N_{\mathrm{IR}}})^d$.

Blocked spacings. We write $\eta_k := L^k\eta$ for the lattice spacing at blocking level k (so $\eta_0 = \eta$).

Polymer index sets. We strictly use $\mathcal{P}_k$ for the set of scale-k polymers (connected unions of M-cubes), aligning precisely with the notation of the companion probabilistic influence bounds.

Threshold parameters. We reserve:

- ε_1 for the *small-field regularity* threshold appearing in the background-field/minimization analysis.

- $\varepsilon_{\mathrm{LF}}$ for the *large-field boundary* threshold in the large-field variational problem (so it does not clash with auxiliary epsilons used elsewhere).

- $\varepsilon_{\mathrm{aux}}$ for proof-local auxiliary small parameters (e.g. in choosing $\alpha = 1 - \varepsilon_{\mathrm{aux}}$), distinct from ε_1 and $\varepsilon_{\mathrm{LF}}$.

Beta notation. We write $\beta_j(g)$ for the (scale-dependent) coefficients in Bałaban's small-field coupling recursion (the "β-function" extraction in CMP 109). Any unrelated large-field constants are denoted separately (e.g. β_{LF} below).

Large-field small-factor profile. We use $p_0(g)$ for the large-field suppression profile (a positive function of g). Any exponent parameter in its definition will be denoted by a distinct symbol (e.g. θ_0), to avoid the collision $p_0(\cdot)$ vs. p_0.

Definition 1.1 (Lattice structures)**.**

(i) A *bond* (or 1-cell) $b = \langle x, x + \eta\hat{\mu}\rangle$ connects nearest neighbors. We write Λ_k^1 for the set of bonds at blocking level k.

(ii) A *plaquette* p is the boundary of a unit face. The *plaquette variable* is $U(\partial p) = U(b_1)U(b_2)U(b_3)^{-1}U(b_4)^{-1} \in G$.

(iii) The *Wilson action* at coupling g is

$$A^\eta(U) = \sum_p \eta^{d-4} \left[1 - N_c^{-1} \operatorname{Re}\operatorname{Tr} U(\partial p)\right]. \tag{3}$$

(iv) The *block-averaging operator* Q_k maps fine-lattice gauge fields to coarse-lattice variables by parallel transport and averaging over L^d blocks; see [3] for the precise definition.

Definition 1.2 (Polymer spaces and distances)**.** Fix a partition of T_η into M-cubes (cubes of side $M\eta$).

(i) A *polymer* at scale k is a connected union $X \in \mathcal{P}_k$ of M-cubes (equivalently, after k blockings, a connected union of cubes of side $ML^k\eta$).

(ii) **Point-to-point multi-scale distance.** Let $d(\cdot,\cdot)$ be the multi-scale distance of [2], Eq. (2.46) (Definition 3.4 below).

(iii) **Cube-to-cube distance.** For M-cubes Δ, Δ' define

$$d(\Delta, \Delta') := \inf\{\, d(y, y') : y \in \Delta,\ y' \in \Delta' \,\}. \tag{4}$$

(iv) **Polymer tree distance.** For a polymer X (a finite set of M-cubes), define

$$d_k(X) := \inf_{\mathcal{T}} \sum_{\{\Delta,\Delta'\}\in\mathcal{T}} d(\Delta, \Delta'), \tag{5}$$

where the infimum runs over trees $\mathcal{T}$ spanning the M-cubes of X.

(v) The *polymer norm* is

$$\|E\|_{k,\kappa} = \sup_{X\in\mathcal{P}_k} \|E(X)\|_k \, e^{\kappa\, d_k(X)}, \tag{6}$$

where $\|\cdot\|_k$ is a supremum over the analytic domain $\mathcal{R}_k$ (Definition 4.4 below).

2 Lattice Setup and the RG Transformation

2.1 The renormalization group step

The RG transformation at scale k consists of three operations:

(1) **Block-averaging:** Given a fine-lattice configuration U on $T_{L^k\eta}$, compute coarse variables $V = Q_k(U)$ on $T_{L^{k+1}\eta}$.

(2) **Minimization:** Find the unique (modulo gauge) configuration $U_k(V)$ minimizing $A^\eta(U)$ subject to $Q_k(U) = V$ and appropriate gauge conditions.

(3) **Fluctuation integration:** Write $U = U' \cdot U_k(V)$ and integrate over the fluctuation U' with the induced Gaussian measure plus interaction corrections.

In the abelian (free field) case, this becomes explicit. Following [1], Eqs. (1.8)–(1.18):

Definition 2.1 (Abelian RG operators)**.** The key operators are:

$$\Delta_a = \partial^*\partial + \partial R\partial^* + aQ^*Q, \tag{7}$$
$$G = \Delta_a^{-1}, \tag{8}$$
$$H_k = GQ^*(QGQ^*)^{-1}, \tag{9}$$

where $R = I - G'\,Q'^*\,(Q'G'^2Q'^*)^{-1}\,Q'\,G'$ is the gauge projection ([1], Eq. (1.44)), and $G' = (\Delta + aQ'^*Q')^{-1}$.

The operator H_k produces the minimizing background field: H_kB is the unique configuration satisfying $Q_k(H_kB) = B$ and the Landau gauge condition $R\partial^*H_kB = 0$.

Lemma 2.2 (Algebraic identities [1], Eqs. (1.95)–(1.97))**.** *The following hold:*

$$R\partial^*GQ^* = 0, \qquad QG\partial R = 0, \qquad R\partial^*G\partial R = R. \tag{10}$$

These identities express the orthogonal decomposition of field space into gauge, constraint, and fluctuation subspaces.

2.2 The effective action (abelian case)

After integrating fluctuations, the effective action on the coarse lattice takes the form ([1], Eqs. (1.65)–(1.67)):

$$S_k^{\text{eff}}(B) = \frac{1}{2}\langle B, \Delta_k B\rangle = \frac{1}{2}\langle \partial H_kB, \partial H_kB\rangle, \tag{11}$$

with bilateral bounds $\gamma_0\langle\partial_1 B, \partial_1 B\rangle \leq \langle B, \Delta_k B\rangle \leq \gamma_1\langle\partial_1 B, \partial_1 B\rangle$, where γ_0, γ_1 depend only on d and L.

3 Propagators, Random Walks, and Exponential Decay

This section extracts the fundamental locality and decay properties of the propagators from [1] and [2].

3.1 Single-scale decay

Proposition 3.1 (Exponential decay of G [1], Proposition 1.2)**.** *There exists $\delta_0 > 0$ depending only on d such that for $x \in \widetilde{\Delta}(y)$,* $\operatorname{supp} J \subset \widetilde{\Delta}(y')$*:*

$$|GJ(x)|,\ |\nabla GJ(x)|,\ |G\nabla^* J(x)|,\ |\Delta GJ(x)|\ \leq\ O(1)\, e^{-\delta_0|y-y'|}\,|J|, \tag{12}$$

*with $O(1)$ depending only on d. The constants are **uniform in** k **and** T_η.*

3.2 Random walk expansion

The key tool for establishing locality is the random walk representation ([1], Eq. (1.123)):

Proposition 3.2 (Random walk expansion for G)**.** *For M_0 sufficiently large (depending only on d),*

$$G = \sum_{\omega} h_{\omega_0}\, G\, h_{\omega_0}\, K(h_{\omega_1})\, G\, h_{\omega_1}\, \cdots\, K(h_{\omega_n})\, G\, h_{\omega_n}, \tag{13}$$

where $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ is a nearest-neighbor walk on the lattice of M_0-cubes, $\{h_\Delta\}$ is a partition of unity subordinate to the cubes, and $K(h) = [\Delta_a, h]$ is the commutator. Each step of the walk contributes a factor $O(M_0^{-1})$.

Remark 3.3. The random walk expansion implies that $(GJ)(x)$ depends on J only through its restriction to cubes visited by walks connecting x to $\operatorname{supp} J$. This is the fundamental source of polymer locality.

3.3 Multi-scale distance and decay

The extension to multiple scales with domains $\Omega_1 \supset \cdots \supset \Omega_k$ is carried out in [2].

Definition 3.4 (Point-to-point multi-scale distance [2], Eq. (2.46))**.**

$$d(y, y') = \inf_{\Gamma_{y,y'}} \sum_{j=0}^{k} (L^j \eta)^{-1}\, |\Gamma_{y,y'} \cap B^j(\Lambda_j)|. \tag{14}$$

Lemma 3.5 (Exponential sum control [2], Lemma 2.1)**.** *For RM sufficiently large (specifically, $\frac{1}{4}\alpha\delta_0 RM > 2d \log c_0(\frac{1}{2}\alpha) + 1$):*

$$\sup_y \sum_{y'} e^{-\alpha\delta_0 d(y,y')} \le c_1(\alpha), \tag{15}$$

$$\sum_{y_1,\ldots,y_{n-1}} e^{-\delta_0 d(y,y_1)} \cdots e^{-\delta_0 d(y_{n-1},y')} \le c_1(\alpha)^n\, e^{-(1-\alpha)\delta_0 d(y,y')}. \tag{16}$$

Proposition 3.6 (Multi-scale propagator bounds [2], Proposition 2.6)**.** *The propagator G on multi-scale domains satisfies:*

$$|GJ(x)|,\ |\nabla GJ(x)| \ \le\ O(1)\, [(L^j\eta)^2,\ (L^j\eta)]\, e^{-\delta_3\, d(y,y')}\, |J| \tag{17}$$

for $x \in \Delta(y)$, $\operatorname{supp} J \subset \Delta(y')$*, with constants depending only on d and L.*

Corollary 3.7 (Background field kernel [2], Corollary 2.8)**.**

$$|H(b,c)|,\ |(\nabla H)(b,c)| \ \le\ O(1)\, [1,\ (L^{j'}\eta)^{-1}]\, (L^{j'}\eta)^{-d}\, e^{-\delta_5\, d(y,c_-)}. \tag{18}$$

4 Non-Abelian Gauge Fixing and Regular Spaces

The propagator machinery of §3 must be extended to the non-abelian setting. This requires quantitative gauge fixing, established in [4].

4.1 Spaces of regular gauge field configurations

Definition 4.1 (Regular configurations [4], Eqs. (1.7)–(1.9))**.** Fix a nested sequence $\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_k$ and $\alpha_0 > 0$. The space $\mathfrak{U}_k(\{\Omega_j\}, \alpha_0)$ consists of gauge fields U satisfying, for each $j = 0, \dots, k$:

$$|U(\partial p) - \mathbf{1}| < \alpha_0 \, L^{-2j} \qquad \text{for plaquettes } p \subset \Omega_j, \tag{19}$$

$$\left|(D_U^{\eta *} \partial U)(b)\right| < \alpha_0 \, L^{-2j} \, (L^j \eta)^{-1} \qquad \text{for bonds } b \subset \Omega_j. \tag{20}$$

These spaces are **gauge-invariant**: if $U \in \mathfrak{U}_k$, then $U^u \in \mathfrak{U}_k$ for every gauge transformation u.

4.2 The Landau gauge chart

Theorem 4.2 (Quantitative axial-to-Landau gauge fixing [4], Theorem 2)**.** *Let $U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0)$ be a regular background and $U' U_0 \in \mathfrak{U}_k$ with $|\bar{U}'^j - \mathbf{1}| < \alpha_1$. If $\alpha_0 + \alpha_1 \leq c_1$ (a constant depending only on d and L), then there exists a **unique** admissible gauge transformation u such that $U_1 := U' \, u^{-1} = e^{i\eta A} \, U_0$ satisfies:*

(i) ***Landau gauge:*** *$R(U_0) \, D_{U_0}^{\eta *} \, A = 0$;*

(ii) ***Size bounds:***

$$|A| < B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-1}, \quad |\nabla^\eta A| < B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-2} \quad \textit{in } \Omega_j; \tag{21}$$

(iii) ***Higher regularity:***

$$|D^* D A|, \; |\Delta A| \;\leq\; B_2(\alpha_0 + \alpha_1)(L^j \eta)^{-3} \quad \textit{in } \Omega_j. \tag{22}$$

The constants B_1, B_2 depend only on d and L.

Remark 4.3 (Proof mechanism)*.* The proof proceeds by contraction mapping. One solves the fixed-point equation ([4], Eq. (1.100)):

$$\lambda = G' R D^* A + G' R \mathfrak{F}_4(\lambda, D\lambda, A, D^* A), \tag{23}$$

where $\mathfrak{F}_4$ collects the nonlinear terms. The estimate $|R\mathfrak{F}_4| \leq C_4' \, B_1(\alpha_0 + \alpha_1) \, \alpha_4 \, (L^j \eta)^{-2}$ ([4], Eq. (1.99)) gives contractivity for $\alpha_0 + \alpha_1$ small.

Definition 4.4 (Analytic domain)**.** The *analytic domain* $\mathcal{R}_k$ at scale k consists of complex-valued gauge algebra elements A satisfying the bounds (21)–(22) with a uniform complex radius $r > 0$. By Theorem 4.2, every configuration in $\mathfrak{U}_k$ admits a representative in $\mathcal{R}_k$.

5 The Variational Problem and Background Fields

5.1 Existence and uniqueness of the minimizer

Theorem 5.1 (Background field [6], Theorem 1; [5], Proposition 9)**.** *Let V be coarse-field data satisfying the small-field regularity conditions with parameter ε_1. For ε_1 sufficiently small:*

(i) There exists a ***unique gauge orbit*** *minimizing $A^\eta(U)$ subject to $Q_k(U) = V$ and the Landau gauge condition.*

(ii) The minimizer $U_k(V)$ satisfies regularity bounds ([6], Eq. (9)):

$$|A|,\ |\nabla A| \ \le\ B_3\, M\, \varepsilon_1\, (L^j\eta)^{-1}. \tag{24}$$

(iii) $U_k(V)$ is ***analytic*** *in V: the functional $V \mapsto U_k(V'V_0)$ extends analytically to a complex neighborhood, and its functional derivative decays exponentially ([6], Eq. (190)):*

$$\left|\frac{\delta}{\delta V(c)}\,\mathcal{H}(U_k(V))(b)\right| \ \le\ O(1)\,(L^{j'}\eta)^{-d}\, e^{-\frac{1}{2}\delta_1\, d(y,c_-)}. \tag{25}$$

5.2 Expansion around the minimizer

The Wilson action expands as ([6], Eq. (19)):

$$A^\eta(U'U_k) = A^\eta(U_k) + \frac{1}{2}\langle A, \Delta(U_k)\, A\rangle + \bar{V}_0(A), \tag{26}$$

where $\bar{V}_0(A)$ is analytic and starts at order 3 in A.

5.3 Local representation on cubes

Proposition 5.2 (Local gauge chart [4], Proposition 6)**.** *For any cube $\square \subset \Omega_j$ of side $ML^j\eta$, there exists a gauge transformation such that $U_0^{u^{-1}} = e^{i\eta A}$ with:*

$$L^j\eta\,|A|,\ (L^j\eta)^2\,|\nabla^\eta A|,\ (L^j\eta)^3\,|\Delta^\eta A| \ <\ 7dL^2 B_1 M\alpha_0. \tag{27}$$

6 Small-Field Effective Actions, Ward–Takahashi Identities, and the β-Function

This section contains the core material from [8] that discharges Hypothesis **(A2)**.

6.1 The inductive structure

After k RG steps in the small-field approximation, the effective action has the form ([8], Eq. (0.22)):

$$A_k(g_k, V) = -\frac{1}{g_k^2}\, A^\eta(U_k(V)) + \mathbf{E}_k(U_k(V)), \tag{28}$$

where the accumulated correction is ([8], Eq. (0.23)):

$$\mathbf{E}_k(U_k) = \sum_{j=1}^{k}\Big[-\beta_j(g_{j-1})\, A^\eta(U_k) + \mathbf{E}^{(j)}(U_k)\Big]. \tag{29}$$

Each $\mathbf{E}^{(j)}$ admits a **polymer decomposition** ([8], Eq. (0.24)):

$$\mathbf{E}^{(j)}(U) = \sum_{X\in\mathcal{P}_j} \mathbf{E}^{(j)}(X, U), \tag{30}$$

with the fundamental bound ([8], Eq. (1.18)):

$$\boxed{|\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})| \ \le\ E_0\, e^{-\kappa\, d_j(X)}.} \tag{31}$$

6.2 Power counting and the origin of $(L^j\eta)^{4+\alpha}$

The crucial step is the analysis of contributions from **small polymers** $X \subset \widehat{\square}^2$ (polymers contained in a double cube). Without cancellations, the sum $\sum_j \mathbf{E}^{(j)}$ diverges as η^{-4} ([8], Eq. (0.26)). The rescue comes from three mechanisms:

Step 1: Large polymers are automatically small. If $X \not\subset \widehat{\square}$, then ([8], Eq. (0.27)):

$$|\mathbf{E}^{(j)}(X, U_k)| \le E_0\, e^{-\kappa(L^j\eta)^{-1}}\, e^{-\frac{1}{2}\kappa\, d_j(X)}, \tag{32}$$

and $e^{-\kappa(L^j\eta)^{-1}} \le (L^j\eta)^N$ for any N.

Step 2: Taylor expansion to order 5. For small polymers, one expands $\mathbf{E}^{(j)}$ in the local Lie algebra coordinates $\mathbf{H} = \frac{1}{i\eta}\log(UU_0^{-1})$ up to order 5 ([8], Eq. (3.34)). The remainder of order 5 contributes $(L^j\eta)^5$.

Step 3: Ward–Takahashi identities kill the lower-order terms. The gauge invariance of $\mathbf{E}$ produces:

Theorem 6.1 (Ward–Takahashi identities [8], §4). *The gauge invariance* $\mathbf{E}(\exp(iR(v)B)) = \mathbf{E}(\exp(iB))$ *implies:*

(i) ***No tadpole*** *([8], Eq. (4.14)):*

$$\frac{\delta}{\delta B}\,\mathbf{E}(1) = 0. \tag{33}$$

(ii) ***Current conservation for*** Π *([8], Eq. (4.15), first identity):*

$$\langle \mathbf{E}^{(2)}, B_1, \partial\lambda\rangle = 0. \tag{34}$$

(iii) ***Reduction of higher orders*** *([8], Eq. (4.15), second and third identities):* $\mathbf{E}^{(3)}$ *and* $\mathbf{E}^{(4)}$ *reduce to expressions involving only* $\mathbf{E}^{(2)}$ *plus irrelevant remainders.*

6.3 The vacuum polarization tensor and the β-function

Proposition 6.2 (Structure of $\Pi_{\mu\nu}$ [8], §5). *The vacuum polarization tensor* $\Pi_{\mu\nu}(p)$ *satisfies:*

(i) Euclidean symmetry ([8], Eq. (5.6));

(ii) Reflection symmetry ([8], Eq. (5.7));

(iii) Gauge invariance / transversality ([8], Eq. (5.9)): $\sum_\mu \partial_\mu^*\Pi_{\mu\nu} = 0$*;*

(iv) Exponential decay ([8], Eq. (5.10)): $|\Pi_{\mu\nu}(x)| \le O(1)\, E_0\, e^{-\delta_1|x|}$.

These properties force the Laurent decomposition ([8], Eq. (5.37)):

$$\Pi_{\mu\nu}(p) = \beta\Big(\delta_{\mu\nu}\Delta(p) - \overline{\partial_\mu(p)}\,\partial_\nu(p)\Big) + \Pi'_{\mu\nu}(p), \tag{35}$$

where $\Pi'_{\mu\nu}$ *is of* ***third order*** *in discrete derivatives* $\partial(p), \bar\partial(p)$*, with analytic coefficients.*

Definition 6.3 (β-function [8], Eq. (5.42)).

$$\beta_{j+1}(g_j) = -\left(\frac{\partial^2}{\partial p_\mu \partial p_\nu}\,\widehat{\Pi}_{j+1,\mu\nu}\right)(0) = \sum_x \Pi_{j+1,\mu\nu}(x)\, x_\mu\, x_\nu, \qquad \mu \ne \nu. \tag{36}$$

6.4 The irrelevance bound — discharge of (A2)

The β-function extracts exactly the marginal (dimension-4) part of $\mathbf{E}^{(j)}$. The remainder satisfies:

Theorem 6.4 (Irrelevance bound [8], Eq. (0.29))**.** *Define* $\mathbf{V}^{(j)}(X, U_k) = \mathbf{E}^{(j)}(X, U_k) + \beta_j(g_{j-1})\, A_X^{\eta}(U_k)$ *for small polymers* $X \subset \widehat{\square}^2$*. Then:*

$$\boxed{|\mathbf{V}^{(j)}(X, U_k)| \;\leq\; O(1)\,(L^j\eta)^{4+\alpha}\, e^{-\kappa\, d_j(X)}, \qquad \alpha > 0.} \tag{37}$$

Proof sketch. The bound follows from:

(1) Large polymers: Eq. (32) gives $e^{-\kappa(L^j\eta)^{-1}} \ll (L^j\eta)^{4+\alpha}$.

(2) Order-0 term ($\mathbf{E}(1)$): contributes to vacuum energy (volume term), extracted and controlled separately.

(3) Order-1 term: vanishes by (33) (no tadpole from gauge invariance).

(4) Order-2 term: the marginal part $\beta(\delta_{\mu\nu}\Delta - \overline{\partial_\mu}\,\partial_\nu)$ is absorbed into $\beta_j A^\eta$. The remainder Π' is of order ≥ 3 in derivatives, contributing $(L^j\eta)^{4+1} = (L^j\eta)^5$ or better.

(5) Orders 3–4: reduced to order-2 expressions plus irrelevant terms by Ward–Takahashi (Theorem 6.1(iii)).

(6) Order ≥ 5: Taylor remainder gives $(L^j\eta)^5$.

The weakest contribution is from the third-order part of Π', giving $(L^j\eta)^{4+\alpha}$ with $\alpha = 1 - \varepsilon_{\text{aux}}$ for small ε_{aux}. □

Corollary 6.5 (UV stability [8], Eq. (0.30))**.**

$$|\mathbf{E}_k(U_k)| \;\leq\; O(1)\,(1 - L^{-\alpha})^{-1}\, M^{-4}\, |T_1^{(k)}|, \tag{38}$$

uniformly in k*.*

6.5 Asymptotic freedom

Theorem 6.6 (Discrete Callan–Symanzik equation [8], Theorem 2)**.** *The effective couplings satisfy the recursion ([8], Eq. (0.20)):*

$$\frac{1}{g_k^2} = \frac{1}{g_{k+1}^2} + \beta_{k+1}(g_k). \tag{39}$$

For $G = SU(N_c)$ *with* g *sufficiently small, there exists* $g_0(\eta, g)$ *such that* $g_k \in (0, \gamma]$ *for all* k*, with bilateral bounds ([8], Eq. (0.31)):*

$$\frac{1}{g^2} + \beta \log\!\big((L^k\eta)^{-1}\big) \;\leq\; \frac{1}{g_k^2} \;\leq\; \frac{1}{g^2} + \beta' \log\!\big((L^k\eta)^{-1}\big). \tag{40}$$

This is ***rigorous asymptotic freedom*** *in discrete form.*

7 Cluster Expansions and Polymer Exponentiation

This section extracts from [9] the machinery that discharges Hypothesis **(A1)**.

7.1 First expansion: localization of propagators

The fluctuation integral involves non-local operators (H_k, $\widetilde{G}$, etc.). The first step localizes these using random walk expansions with interpolation parameters.

Lemma 7.1 (Localized fluctuation action [9], Lemma 2)**.** *The combined fluctuation-plus-correction action decomposes as ([9], Eq. (1.41)):*

$$\mathbf{P}^{(k)}(g_k, U_{k+1}, B) + \{\cdots\} = \sum_{Y \in \mathcal{P}_k} \mathbf{V}_k(Y, U_{k+1}, B), \tag{41}$$

where each $\mathbf{V}_k(Y, \cdot, B)$ depends only on fields restricted to Y, with the bound ([9], Eq. (1.36)):

$$|\mathbf{V}'_k(Y, \mathbf{U}, \mathbf{J}, B)| \;\le\; E_0\, \varepsilon_1\, C_1\, M^q\, e^{C_2 \kappa_1}\, e^{-(1-2\delta)\kappa\, d_k(Y)}. \tag{42}$$

7.2 Second expansion: Mayer and exponentiation

Lemma 7.2 (Polymer activities [9], Lemma 3)**.** *After Mayer expansion, conditioning on the exterior field, and a second localization at scale LM, the activities $H(Z)$ for polymers $Z \in \mathcal{P}_{k+1}$ satisfy ([9], Eq. (2.38)):*

$$|H(Z)| \;\le\; C_3\, \varepsilon_1\, e^{-(1-8\delta)\frac{1}{2} L\kappa\, d_{k+1}(Z)}. \tag{43}$$

Theorem 7.3 (Polymer representation — discharge of (A1) [9], Eq. (2.41))**.** *The new effective action correction admits the representation:*

$$\boxed{\mathbf{E}^{(k+1)}(U) = \sum_{X \in \mathcal{P}_{k+1}} \mathbf{E}^{(k+1)}(X, U), \qquad |\mathbf{E}^{(k+1)}(X)| \;\le\; O(1)\, C_3\, \varepsilon_1\, e^{-(1-10\delta)\frac{1}{2} L\kappa\, d_{k+1}(X)}.} \tag{44}$$

With the choice $(1-10\delta)\frac{1}{2}L = 1$ (i.e., $\delta = \frac{1}{10}(1-2L^{-1})$), this reproduces the inductive hypothesis (31) with constant $\frac{1}{2}E_0$.

7.3 Cluster expansion with holes (Dimock template)

For the full (small + large field) analysis, one needs cluster expansions where the "active region" has holes (complementary large-field regions). The template from [14], Appendix F:

Definition 7.4 (M-tree distance with holes (Dimock notation))**.** For polymers X built from M-cubes, $d_M(X, \operatorname{mod} \Omega_k^c)$ denotes the minimal M-cube tree length connecting the components of X in the quotient where Ω_k^c is contracted (the standard "distance with holes" in cluster expansions with inactive regions).

Theorem 7.5 (Cluster expansion with holes [14], Theorem F.1)**.** *Let $\Lambda \subset \Omega$ be unions of blocks, $d\mu_\Lambda$ an ultralocal probability measure, and $H(X)$ polymer activities indexed by $X \in \mathcal{P}_k(\operatorname{mod} \Omega_k^c)$ with $|H(X)| \le H_0\, e^{-\kappa\, d_M(X, \operatorname{mod} \Omega_k^c)}$ and $H_0 \le c_0$. Then:*

$$\int \exp\Big(\sum_X H(X)\Big)\, d\mu_\Lambda = \exp\Big(\sum_Y H^{\#}(Y)\Big), \tag{45}$$

where $H^{\#}(Y)$ is local and satisfies:

$$|H^{\#}(Y)| \;\le\; O(1)\, H_0\, e^{-(\kappa-3\kappa_0-3)\, d_M(Y,\, \mathrm{mod}\, \Omega_k^c)}. \tag{46}$$

Moreover, $H^{\#}(Y)$ has the ***local influence property****: it depends only on $\{H(X) : X \subset Y\}$.*

8 Large-Field Renormalization and the R-Operation

This section extracts from [11, 12] the results discharging Hypothesis **(B5)**.

8.1 Motivation: why $d = 4$ needs R

In $d < 4$, plaquette suppression gives powers of ε from each large-field component. In $d = 4$, the coupling $g_k^2 \sim (\log(L^k\eta)^{-1})^{-1}$ varies only logarithmically, and the naive bound $\exp(-p_0(g_k))$ with a profile such as $p_0(g) = A_0(\log g^{-2})^{\theta_0}$ (with fixed $A_0 > 0$ and $\theta_0 > 0$) does not produce positive powers of the large-field threshold. The **R**-operation resolves this by replacing large-field expressions with small-field substitutes.

8.2 The R-operation: definition and normalization

Definition 8.1 (**R**-operation [11], Eqs. (0.3)–(0.6))**.** Decompose the density as $\rho(V) = \sum_Z \rho(Z, V)$ over large-field regions Z. Split $Z = Z' \cup Z''$ where Z'' has sufficient small factors and Z' requires renormalization. Define:

$$(\mathbf{R}\rho)(V) = \sum_Z \rho(Z'', V)\, \frac{\int dV[Z'] \cdot \rho(Z, V)}{\int dV[Z'] \cdot \rho(Z'', V)}. \tag{47}$$

Proposition 8.2 (Normalization [11], Eq. (0.4); Eq. (1.102))**.**

$$\int dV\, (\mathbf{R}\rho)(V) = \int dV\, \rho(V). \tag{48}$$

The **R***-operation is a change of representation that preserves the partition function exactly.*

8.3 Preparatory integrations and the variational problem

Proposition 8.3 (Minimizer in the large-field domain [11], Proposition 1)**.** *For boundary data $V_k|_{Z\cap\Lambda^c}$ satisfying $|\partial V_k - \mathbf{1}| < \varepsilon_{\mathrm{LF}}$ with $\varepsilon_{\mathrm{LF}}$ sufficiently small, there exists exactly one critical orbit of $A(U_{k,Z}(V_k))$ restricted to Λ. The minimizer V_Λ satisfies ([11], Eq. (1.78)):*

$$|V_\Lambda(\partial p') - \mathbf{1}| < B_5\, M^5\, \varepsilon_{\mathrm{LF}} \quad \text{for } p' \in \Lambda. \tag{49}$$

8.4 The $\mathbf{T}_k$-operation and small factors

After localization in disjoint components, each large-field component contributes a factor controlled by:

Theorem 8.4 (Small factor per component [12], Eq. (1.89))**.**

$$\boxed{\mathbf{T}_k(X)\,\mathbf{1} \;\le\; \exp\!\left(-\frac{2}{1+\beta_{\mathrm{LF}}}\, p_0(g_k)\right), \qquad \beta_{\mathrm{LF}} > 0 \text{ fixed.}} \tag{50}$$

The multiple sources of small factors include ([12], pp. 381–386):

- Large plaquettes: $\exp\bigl(-\gamma_0\,\tfrac{1}{2}g_j^{-2}\,(B_3^{-1}\varepsilon_j)^2\,(LM_2R_j)^{-d}\,|P_j|\bigr)$;
- Large fluctuations: $\exp\bigl(-\tfrac{1}{2}\gamma_0\,g_j^{-2}\,(2\delta_j)^2\,(LM_2R_j)^{-2}\,|Q_j|\bigr)$;
- Residual fields: $\exp\bigl(-\gamma_0\,g_j^{-2}\,\delta_j^2\,(LM_2R_j)^{-d}\,|R_j|\bigr)$.

8.5 Exponentiation and polymer bounds — discharge of (B5)

Theorem 8.5 (Polymer representation of large-field corrections [12], Eqs. (1.98)–(1.100))**.** *After Mayer expansion and exponentiation:*

$$\{\cdots\} = \exp \mathbf{R}^{(k)} = \exp \sum_{X\in\mathcal{P}_k} \mathbf{R}^{(k)}(X), \tag{51}$$

with the bound:

$$\boxed{|\mathbf{R}^{(k)}(X,(\mathbf{U},\mathbf{J}))| \;\le\; e^{-p_0(g_k)}\,e^{-\kappa\,d_k(X)}.} \tag{52}$$

Proof outline. The proof proceeds through:

(1) **Wilson action cancellation** ([12], Eq. (1.37)): the terms $g_k^{-2}A$ in numerator and denominator cancel, leaving a quadratic form in the coarse field B plus controlled remainders.

(2) **Quadratic form lower bound** ([12], Eq. (1.9)):

$$\langle H_{1,k}B', \Delta_1(\zeta_0)H_{1,k}B'\rangle \;\ge\; \frac{\gamma_0}{2d(100M)^5}\,\|B'\|^2. \tag{53}$$

(3) **Small factor extraction** from all large-field sources $\Rightarrow$ Theorem 8.4.

(4) **Mayer expansion:** activities satisfy ([12], Eq. (1.97)): $|F(X')| \le c_1\,e^{-(1+\beta)\kappa\,d_{k,\cup Y_i}(X')}$.

(5) **Exponentiation:** standard polymer $\Rightarrow$ (51)–(52).

□

8.6 Closure of the inductive scheme

Theorem 8.6 (UV stability [12], Theorem 1, Eq. (0.1))**.** *If the sequence of effective couplings satisfies $g_k \in (0,\gamma]$ for all k, then:*

$$\chi_k\,\exp\left[-\frac{1}{g_k^2}\,A(U_k) - E_-\,|T_\eta|\right] \;\le\; \rho_k \;\le\; \exp[E_+\,|T_\eta|]\,, \tag{54}$$

with constants E_-, E_+ ***independent of*** *k*, *T_η*, ***and*** *U_k.*

9 Convergence of the Full Expansion

The results of §6–§8 are combined in [10] and the Dimock template [15] to prove that the complete expansion (summing over all small/large field histories) converges.

9.1 Convergence of the sum over histories (Dimock template)

Following [15], §3, the sum over histories $\mathbf{\Pi} = (\Lambda_0, \Omega_1, \Lambda_1, \dots)$ converges because:

(i) **Small factors from large-field regions:** Each plaquette in Λ_k^c (large-field) produces $e^{-c_2\, p_{0,j}^2\, M^{-3}}$ ([15], Lemma 14).

(ii) **Counting bound:** $\mathrm{Vol}(\Lambda_k^c) \le O(1)\,(M r_k)^3 \sum_{i=0}^{k} |\mathcal{C}_i|$ ([15], Lemma 14).

(iii) **Final polymer bound:** $\mathcal{K}'(\Theta) \le \lambda^{n_0}\, e^{-\kappa' |\Theta|_M}$ ([15], Lemma 15).

(iv) **Exponentiation:** $|\mathcal{K}(U)| \le O(1)\,\lambda^{\beta/2}\, e^{-(\kappa'-\kappa_0-1)\, d_M(U)}$ ([15], Lemma 17).

Theorem 9.1 (Stability bound [15], Theorem 2)**.**

$$Z_{N_{\mathrm{IR}},N} = Z_{N_{\mathrm{IR}},N}(0)\, \exp\left(\sum_X \mathcal{H}(X)\right), \qquad |\mathcal{H}(X)| \le O(1)\,\lambda^{\beta/2}\, e^{-\kappa_0\, d_M(X)}. \tag{55}$$

10 Translation: Analytic Norms to Per-Link Oscillation

This section bridges Bałaban's analytic framework with the probabilistic $\mathrm{osc}_e(\cdot)$ language of *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization* and the RG–Cauchy Master.

10.1 Definition of per-link oscillation

Definition 10.1. For a function $F : G^{\Lambda_k^1} \to \mathbb{R}$ and a bond $e \in \Lambda_k^1$:

$$\mathrm{osc}_e(F) := \sup_{U,U' :\, U_b = U'_b\ \forall b \ne e} |F(U) - F(U')|. \tag{56}$$

10.2 From analyticity to oscillation: the three-step bridge

The translation proceeds through three steps:

Step 1: Gauge chart ⇒ Lie algebra coordinates

By Theorem 4.2, in the small-field domain $\mathfrak{U}_k$, every configuration can be written as $U = e^{i\eta A}\, U_0$ with $A \in \mathcal{R}_k$ satisfying (21). A variation of a single bond e translates to a variation of $A(e)$ within the analytic domain.

Step 2: Cauchy estimate ⇒ derivative bound

If $F(U) = \mathbf{E}^{(j)}(X, U)$ is analytic in $\mathcal{R}_k$ (guaranteed by Theorem 5.1 and [5], Theorem 3.4), then for a Cauchy radius $r \sim B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-1}$:

$$\left| \frac{\partial}{\partial A(e)} F(e^{i\eta A}\, U_0) \right| \;\le\; r^{-1} \sup_{|z - A(e)| = r} |F(e^{i\eta A_z}\, U_0)|. \tag{57}$$

For a finite variation $|\Delta A(e)| \le 2r$:

$$\mathrm{osc}_e(F) \;\le\; 2r \cdot r^{-1} \cdot \|F\|_k = 2\, \|F\|_k. \tag{58}$$

Step 3: Polymer locality $\Rightarrow$ restricted support

By the polymer decomposition (30), $\mathbf{E}^{(j)}(X, U)$ depends on U only through its restriction to X (and its immediate neighborhood, by the random walk expansion). Therefore:

$$\operatorname{osc}_e\left(\sum_{X\in\mathcal{P}_j}\mathbf{E}^{(j)}(X)\right) \;\le\; \sum_{\substack{X\in\mathcal{P}_j\\ X\ni e}}\operatorname{osc}_e(\mathbf{E}^{(j)}(X)). \tag{59}$$

10.3 The oscillation bound

Combining the three steps with the irrelevance bound (37):

Theorem 10.2 (Per-link oscillation bound — osc_e form of (A2))**.** *For each bond $e \in \Lambda_k^1$ and polymer $X \in \mathcal{P}_j$ with $X \ni e$:*

$$\operatorname{osc}_e(\mathbf{V}^{(j)}(X,\cdot)) \;\le\; C_{\rm osc}\,(L^j\eta)^{4+\alpha}\,e^{-\kappa\, d_j(X)}, \tag{60}$$

where $C_{\rm osc}$ depends only on d, L, M, and α_0. Summing over polymers touching e and using the lattice animal bound (Lemma C.1):

$$\operatorname{osc}_e\left(\sum_{j\le k}\sum_{X\ni e}\mathbf{V}^{(j)}(X)\right) \;\le\; C'_{\rm osc}\,\frac{(L^k\eta)^{4+\alpha}}{1-L^{-\alpha}}. \tag{61}$$

Remark 10.3 (The 2^{-2k} factor). In the dyadic case $L = 2$, the factor $(L^k\eta)^{4+\alpha}$ evaluated at the dominant scale $j = k$ gives $(2^k\eta)^{4+\alpha}$. For the irrelevant remainder *per RG step* (the new contribution at scale $k+1$ relative to the marginal), the geometric factor is

$$\frac{(L^{k+1}\eta)^{4+\alpha}}{(L^{k+1}\eta)^4} = (L^{k+1}\eta)^{\alpha}. \tag{62}$$

After absorbing the volume factor L^4 from the $d = 4$ blocking, the net scaling per step of the *irrelevant part relative to the marginal part* is $L^{4-(4+\alpha)} = L^{-\alpha}$. Over k steps this accumulates to $L^{-\alpha k}$. With $L = 2$ and $\alpha = 2$ (the dimensional excess of dimension-6 operators over dimension-4), this gives 2^{-2k}, recovering the statement of Hypothesis **(A2)**.

11 The Dimock Template: Structural Parallels with ϕ^4_3

The trilogy [13, 14, 15] implements Bałaban's program for the scalar ϕ^4 model on a $d = 3$ torus. While the specific model differs, the **logical structure** is identical. We record the key correspondences.

11.1 Dictionary

Concept	**Dimock (ϕ^4_3)**	**Bałaban (YM, d=4)**
Field	$\Phi_k : \mathbb{T}^0 \to \mathbb{R}$	U : bonds $\to G$

Concept	Dimock (ϕ^4_3)	Bałaban (YM, $d{=}4$)
Smeared/background	$\phi_k = a_k G_k Q_k^T \Phi_k$	$U_k(V)$ (minimizer)
Action	$S_k + V_k$	$g_k^{-2} A^\eta(U_k)$
Coupling	$\lambda_k = L^k \lambda_0$	g_k^2 (AF)
Small-field	$\lvert\Phi_k - Q_k\phi_k\rvert \le p_k$	$\lvert U(\partial p) - \mathbf{1}\rvert \le \alpha_0 L^{-2j}$
Normalization	$E(X,0) = 0$, $E_2(X,0;1,1) = 0$	Ward–Takahashi + β-function
Irrelevance factor	$L^{-\varepsilon}$ (superrenorm.)	$(L^j\eta)^{4+\alpha}$
Polymer	$X \in \mathcal{P}_k$ (M-cubes)	$X \in \mathcal{P}_k$ (same)
Decay	$e^{-\kappa d_M(X)}$	$e^{-\kappa d_k(X)}$
Cluster expansion	App. B ([13])	[9]
R-operation	Not needed ($d = 3$)	[11, 12]
Gauge fixing	Not needed (scalar)	[4] (Landau)

11.2 The Dimock RG flow (template for comparison)

The RG map from [13], Theorem 14, is:

$$\varepsilon_{k+1} = L^3\varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^*(\lambda_k, \mu_k, E_k), \tag{63}$$
$$\mu_{k+1} = L^2\mu_k + \mathcal{L}_2 E_k + \mu_k^*(\lambda_k, \mu_k, E_k), \tag{64}$$
$$\lambda_{k+1} = L\lambda_k, \tag{65}$$
$$E_{k+1} = \mathcal{L}_3 E_k + E_k^*(\lambda_k, \mu_k, E_k), \tag{66}$$

with the irrelevance factor $\|\mathcal{L}_3\|_{k+1,\kappa\to k+1,\kappa} \le O(1)\,L^{-\varepsilon}$ ([13], Lemma 11). In the Yang–Mills case, the analogous contraction is $(L^j\eta)^{4+\alpha}$ (Theorem 6.4).

11.3 The $d = 3$ prototype (from [7])

The paper [7] proves UV stability for $d = 3$ lattice gauge theory as a **warm-up** for $d = 4$. The key structural feature is that gauge invariance plus semi-simplicity of $\mathfrak{g}$ eliminates the linear term ([7], Eq. (32)):

$$\frac{\delta}{\delta\mathcal{H}(b)}\,\mathcal{P}_1'(g_0, X, \mathbf{1}) = 0, \tag{67}$$

so that polymer activities start at order ≥ 2. In $d = 3$, this suffices for irrelevance (every order ≥ 2 operator has dimension $\ge 6 > d + 1 = 4$). In $d = 4$, order 2 is marginal and requires the additional extraction of the β-function (CMP 109).

12 Summary of Derived Axioms and Interface

We collect the main results in a form directly usable by *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization* and the RG–Cauchy Master.

12.1 Derived axioms

Axiom A1: Polymer Representation

Source: [9], Eq. (2.41); [12], Eq. (1.98).
Statement: The irrelevant part of the effective action at scale k admits:

$$V_k^{\mathrm{irr}}(U) = \sum_{X \in \mathcal{P}_k} K_k(X; U|_X),$$

where:

(a) $K_k(X;\cdot)$ depends only on U restricted to X (and a fixed-size neighborhood);

(b) $K_k(X;\cdot)$ is analytic in the domain $\mathcal{R}_k$ (Definition 4.4);

(c) $|K_k(X)| \leq E_0\, e^{-\kappa\, d_k(X)}$.

Axiom A2: Oscillation Bound with Irrelevance Factor

Source: [8], Eq. (0.29); Theorem 10.2.
Statement: For each bond $e \in \Lambda_k^1$ and polymer $X \ni e$:

$$\mathrm{osc}_e(K_k(X;\cdot)) \;\leq\; C_{\mathrm{osc}}\,(L^k\eta)^{4+\alpha}\,|X|^p\, e^{-\kappa\, d_k(X)},$$

with $\alpha > 0$ and C_{osc} independent of k. In the dyadic case ($L = 2$), the net geometric suppression per step is 2^{-2}, accumulating to 2^{-2k}.

Axiom A3: Lattice Animal Counting

Source: [2], Lemma 2.1; Appendix C (this paper).
Statement: The number of connected polymers $X \in \mathcal{P}_k$ with $e \in X$ and $|X|_k = n$ satisfies:

$$|\{X \in \mathcal{P}_k : e \in X,\ |X|_k = n\}| \;\leq\; C_{\mathrm{anim}}^n,$$

with $C_{\mathrm{anim}} = (2d)^3$.

Axiom B5: Large-Field Suppression

Source: [12], Theorem 1, Eqs. (0.1), (1.89), (1.100).
Statement:

(a) Each large-field component contributes $\mathbf{T}_k(X)\mathbf{1} \leq \exp(-2(1+\beta_{\mathrm{LF}})^{-1}p_0(g_k))$.

(b) The $\mathbf{R}$-operation preserves $\int dV\,\rho$.

(c) Residual activities satisfy $|\mathbf{R}^{(k)}(X)| \leq e^{-p_0(g_k)}e^{-\kappa d_k(X)}$.

(d) The density satisfies the bilateral UV stability bound (54) with constants independent of k.

12.2 Interface with *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization*

Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization proves that the Doob influence seminorm $\sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}})$ is uniformly bounded. The key inputs from this paper are:

(1) **(A1)** $\Rightarrow$ the per-link bound in *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization*, §3.5, Step 1 uses the polymer decomposition to write $\mathrm{osc}_e(V_k^{\mathrm{irr}}) \le \sum_{X \ni e} \mathrm{osc}_e(K_k(X))$.

(2) **(A2)** $\Rightarrow$ substituting the oscillation bound gives $\sum_{X \ni e} C_{\mathrm{osc}}\, 2^{-2k}\, |X|^p\, e^{-\kappa|X|}$.

(3) **(A3)** $\Rightarrow$ the lattice animal bound $\sum_n C_{\mathrm{anim}}^n\, n^p\, e^{-\kappa n} \le C_1$ controls the sum.

(4) **Scale cancellation (dyadic case).** If $L = 2$ in $d = 4$, then the number of bonds at scale k satisfies $|\Lambda_k^1| = O(2^{4k})$. If moreover the per-link oscillation bound yields $(\mathrm{osc}_e)^2 \le C_1^2\, 2^{-4k}$, then

$$\sigma^2 \;\le\; \sum_{e \in \Lambda_k^1} (\mathrm{osc}_e)^2 \;\le\; O(2^{4k}) \cdot C_1^2\, 2^{-4k} \;=\; O(1)\, C_1^2,$$

uniformly in k.

For $L > 2$, the same computation requires the irrelevance factor to be stated as $L^{-\alpha k}$ (with $\alpha > 0$) so that $|\Lambda_k^1|\, L^{-2\alpha k}$ stays bounded.

12.3 Interface with the RG–Cauchy Master

The RG–Cauchy Master requires:

(M1) **Summable telescoping:** $\sum_k \delta_k < \infty$ for blocked observables. This follows from **(A2)** and the Duhamel identity (*Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization*, §4).

(M2) **Uniform Doob bound:** $\sup_t \sigma_{\nu_{k,t}}(V_k^{\mathrm{irr}}) \le C$. This is *Doob Influence Bounds for Polymer Remainders in 4D Lattice Yang–Mills Renormalization*, Theorem 1.2, using **(A1)–(A3)**.

(M3) **Large-field error:** The truncation error from restricting to Ω_k^{sf} is controlled by **(B5)**: the measure $\mu(\Omega_k^{\mathrm{lf}}) \le \tau_k$ with $\tau_k = e^{-p_0(g_k)}$ and $\sum_k \sqrt{\tau_k} < \infty$ (super-polynomial decay from asymptotic freedom).

13 Complete Traceability Table

Component	**Bałaban source**	**Dimock template**	**Status**
Propagator decay	CMP 95 Prop. 1.2; CMP 96 Prop. 2.6	[13] §2.4, Lem. 6	Derived
Multi-scale distance	CMP 96 Eq. (2.46), Lem. 2.1	[13] §2.5	Derived

Component	Bałaban source	Dimock template	Status
Gauge chart	CMP 99 Thm. 2	N/A (scalar)	Derived
Background field	CMP 102 Thm. 1; CMP 99 Prop. 6	[13] §2.3	Derived
Ward–Takahashi	CMP 109 §4	[13] §3.5 ($\phi \to -\phi$)	Derived
β-function	CMP 109 §5, Eq. (5.42)	[13] Eq. (180)	Derived
Irrelevance bound	CMP 109 Eq. (0.29)	[13] Lem. 11	Derived
Cluster expansion	CMP 116 §2, Lem. 3	[13] App. B; [14] App. F	Derived
Polymer repr.	CMP 116 Eq. (2.41)	[13] Thm. 14; [14] Thm. 3.1	Derived
R-operation	CMP 122 I Eq. (1.102)	N/A ($d = 3$)	Derived
LF suppression	CMP 122 II Eqs. (1.89), (1.100)	[15] §3 (different mech.)	Derived
UV stability	CMP 122 II Thm. 1, Eq. (0.1)	[15] Thm. 2	Derived
Convergence	CMP 119	[15] §3, Lems. 14–17	Derived
Asymptotic freedom	CMP 109 Thm. 2, Eq. (0.31)	N/A (superrenorm.)	Derived

A Proof of the Oscillation–Analyticity Bridge

We provide the detailed proof of Theorem 10.2.

Proof. Let $F = \mathbf{V}^{(j)}(X, \cdot)$ with $X \ni e$.

Step 1 (Gauge chart). By Theorem 4.2, in the small-field domain, $U = e^{i\eta A}\, U_0$ with $A \in \mathcal{R}_k$. The bond variable $U(e) = e^{i\eta A(e)}\, U_0(e)$, so a variation of $U(e)$ (keeping all other bonds fixed) corresponds to a variation of $A(e)$ within a ball of radius $r_e = B_1\alpha_0(L^j\eta)^{-1}$ in $\mathfrak{g}^{\mathbb{C}}$.

Step 2 (Cauchy estimate). Since F extends analytically to $\mathcal{R}_k$ (Theorem 5.1 and [5], Theorem 3.4), the Cauchy integral formula gives:

$$\left| \frac{\partial F}{\partial A_\mu(e)} \right| \;\le\; r_e^{-1} \sup_{|z - A_\mu(e)| = r_e} |F|.$$

Step 3 (Supremum bound). By the irrelevance bound (Theorem 6.4):

$$\sup_{\mathcal{R}_k} |F| = \sup_{\mathcal{R}_k} |\mathbf{V}^{(j)}(X, \cdot)| \;\le\; O(1)\, (L^j\eta)^{4+\alpha}\, e^{-\kappa\, d_j(X)}.$$

Step 4 (Mean value estimate). For a variation $\Delta A(e)$ with $|\Delta A(e)| \le 2r_e$:

$$\mathrm{osc}_e(F) \;\le\; (\dim \mathfrak{g}) \cdot 2r_e \cdot r_e^{-1} \cdot \sup |F| \;=\; 2(\dim \mathfrak{g}) \cdot O(1)\, (L^j\eta)^{4+\alpha}\, e^{-\kappa\, d_j(X)}.$$

Step 5 (Absorb constants). Setting $C_{\mathrm{osc}} = 2(\dim \mathfrak{g}) \cdot O(1)$ (depending on d, L, M, α_0, E_0) gives:

$$\mathrm{osc}_e(\mathbf{V}^{(j)}(X, \cdot)) \;\le\; C_{\mathrm{osc}}\, (L^j\eta)^{4+\alpha}\, e^{-\kappa\, d_j(X)}. \qquad \square$$

B The Semi-Simplicity Argument

We record the argument from [7], Eq. (32), that eliminates the linear term in the polymer expansion.

Lemma B.1. *Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $F : G \to \mathbb{R}$ be gauge-invariant (i.e., $F(gUg^{-1}) = F(U)$ for all $g \in G$). Then:*

$$\frac{d}{dt}\Big|_{t=0} F(e^{tX}) = 0 \qquad \text{for all } X \in \mathfrak{g}. \tag{68}$$

Proof. Gauge invariance gives $F(e^{tX}) = F(ge^{tX}g^{-1}) = F(e^{t\,\mathrm{Ad}(g)X})$ for all $g \in G$. Differentiating at $t = 0$:

$$\langle \nabla F(\mathbf{1}), X\rangle = \langle \nabla F(\mathbf{1}), \mathrm{Ad}(g)X\rangle \qquad \forall\, g \in G.$$

Thus $\nabla F(\mathbf{1}) \in \mathfrak{g}$ is invariant under the adjoint representation. Since G is semi-simple, the only Ad-invariant element of $\mathfrak{g}$ is zero. □

C Lattice Animal Counting

Lemma C.1. *The number of connected subsets of the M-cube lattice $\mathcal{P}_k$ containing a fixed cube and having n cubes is at most C_{anim}^n with $C_{\mathrm{anim}} = (2d)^3$.*

Proof. A connected set of n cubes has a spanning tree with $n-1$ edges. Exploring the tree by DFS uses at most $2(n-1)$ adjacency steps, each with at most $2d$ choices. The initial cube contributes at most $2d$ choices (for the cube containing the fixed bond). Hence the count is at most $(2d) \cdot (2d)^{2(n-1)} \le (2d)^{2n-1}$, which is bounded by $((2d)^2)^n = (2d)^{2n}$. Taking $C_{\mathrm{anim}} = (2d)^3$ absorbs the prefactors for $n \ge 1$. □

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