Conditional Continuum Limit of 4d $SU(N_c)$ Yang–Mills Theory via Two-Layer Architecture, RG–Cauchy Uniqueness, and Step-Scaling Confinement*

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Abstract

Building on the lattice results of Papers [E26I]–[E26IX] — which, per the series audit (all companions now at v2/v3), are *windowed and conditional* rather than unconditional — we give a conditional construction of a scaling-limit state for pure $SU(N_c)$ lattice Yang–Mills theory in four Euclidean dimensions, along dyadic lattice spacings $a_k = a_0 2^{-k}$. The construction proceeds via a two-layer architecture. *Layer 1 (Local fields):* for bounded gauge-invariant local observables, expectations converge — without extracting subsequences — to a unique limit; precompactness is trivial ($|\langle O\rangle_{a,L}| \le 1$), and uniqueness follows from a multiscale RG–Cauchy estimate (Assumption 3.5), the single hard analytic input of this layer: as already recorded in v2 (Remark 3.6, Appendix B), the naive asymptotic-freedom rate $g_k^2 \sim c/k$ is *not summable*, so summability is a genuine hypothesis, not a consequence of the chain. *Layer 2 (Confinement):* the physical string tension $\sigma_{\rm phys} > 0$ is established through step-scaling of Creutz ratios at fixed physical loop size, conditionally on Assumptions 4.4, 4.7 and 4.9. The limiting state inherits Osterwalder–Schrader positivity and admits Hilbert-space reconstruction; the mass gap is conditional on a uniform physical transfer-matrix gap (Assumption A.2) and strong continuity (Assumption 5.5). Version 3 retags the input layer to the audited chain: the uniform-LSI inputs are conditional on (H-P0)+(H-YGZ)+(H-SFI)+(H-ABS), the DLR-LSI/mass-gap route on (H-DOB-blk), and all lattice statements hold in the volume window $L_{\rm vol} \le e^{C/g^2+O(1)}$. A new window-compatibility lemma (Lemma 1.4) shows this window is *not* an obstruction to the continuum limit: along the 2-loop trajectory the required lattice size $L/a_k = 2^k L/a_0$ satisfies $\ln(L/a_k) \approx 0.69\,k$ while the audited window allows $\ln L_{\rm lat} \le 32.3/g_k^2 \approx 3.12\,k$ — a margin factor ≈ 4.5 at the series' representative arithmetic. Assumption A.2 is now cross-referenced to its conditional lattice supplier (2602.0054 v2: transfer-matrix gap under (H-DOB-blk)+(H-P0), in kernel form). All quantitative claims, and exact validations of both layers in a solvable $d = 2$ toy, are adjudicated in a companion numerical suite.

*Version 3. Versions 1–2 described their lattice inputs as "unconditional lattice closure inputs [E26IX]" and stated that "Papers [E26I]–[E26IX] established" the uniform functional inequalities. Following the quantitative audit of the input chain (ai.viXra:2602.0041 v3, 2602.0051–0057 all v2), those descriptions are corrected: the lattice layer is *windowed and conditional* on the open hypothesis set (H-P0), (H-YGZ), (H-DOB-blk), (H-SFI), (H-ABS), and the flow window/(H-XOVER). v3 also adds a window-compatibility lemma (Lemma 1.4) showing that the dyadic continuum trajectory stays inside the audited volume window with a quantified margin, fixes a labeling inconsistency (Theorem 1.2(iv) cited "Theorems 4.4, 4.7, 4.9", which are Assumptions), and updates the companion references to their audited versions. See the Note added in v3 (Section 9).

Contents

1 Introduction

1.1 The problem

The construction of a rigorous four-dimensional quantum field theory with non-abelian gauge group, and the proof of its mass gap, is a Clay Millennium Problem [9]. Lattice gauge theory provides a non-perturbative, gauge-invariant regularization; removing the cutoff ($a \to 0$) requires (1) uniform functional inequalities independent of the volume, (2) a compactness–uniqueness argument for the limiting state, and (3) preservation of OS positivity, confinement and the mass gap.

Correction (v3). v1–v2 stated that "Papers [E26I]–[E26IX] established item (1) on the lattice." Per the audit of the chain (2602.0041 v3, 2602.0051–0057 v2), item (1) is established *conditionally and in a volume window*: the uniform LSI is windowed ($L_{\mathrm{vol}} \leq e^{C/g^2+O(1)}$) and conditional on (H-P0)+(H-YGZ)+(H-SFI)+ (H-ABS); the DLR route to the transfer gap additionally consumes (H-DOB-blk). The present paper carries out items (2) and (3) *given* that conditional input layer; its own assumptions (3.5, 4.4, 4.7, 4.9, 5.5, A.2) come on top and are stated explicitly.

1.2 The two-layer architecture

Standard compactness arguments face a dual obstruction in gauge theories: the connection fields are gauge-variant, and the natural gauge-invariant observables (Wilson loops) are nonlocal. We resolve this through a bipartite structure. *Layer 1*: bounded gauge-invariant local observables (Definition 2.4); trivial precompactness of expectations; uniqueness by a quantitative RG–Cauchy estimate under successive RG steps, guaranteeing convergence without subsequences. *Layer 2*: Creutz ratios on Wilson loops with lattice dimensions $I(a) = \lfloor R/a \rfloor$, $J(a) = \lfloor T/a \rfloor$, so that physical size is fixed as $a \to 0$, yielding $\sigma_{\mathrm{phys}} > 0$ conditionally.

Remark 1.1 (What is not used for uniqueness)**.** *We do not use the string tension to determine the theory. Uniqueness follows entirely from the RG–Cauchy estimate (Layer 1); confinement is an output of the construction, not an input to uniqueness.*

1.3 Main results

Theorem 1.2 (Main theorem (conditional); v3: inputs retagged)**.** *Let $G = SU(N_c)$, $d = 4$. Fix a physical scale $\ell_0 > 0$ and tune $\beta(a) = 2N_c/g(a)^2$ along a weak-coupling trajectory consistent with 2-loop asymptotic freedom. Then, under the assumptions listed below* and given the conditional, windowed lattice input layer of §1.5*:*

(i) Existence and uniqueness (dyadic) *(Assumption 3.5). For every bounded gauge-invariant local observable $O \in \mathcal{A}_\ell$ and fixed physical side length $L > 0$, $\langle O \rangle_{a_k,L}$ converges as $k \to \infty$*

along $a_k = a_0/2^k$ to a unique limit. Unbounded observables (smeared curvature monomials) are deferred.

(ii) Reflection positivity and reconstruction. *The limiting state on bounded observables inherits reflection positivity and admits a Hilbert-space reconstruction (Section 5).*

(iii) Mass gap *(Assumptions A.2 and 5.5). For each fixed physical spatial size $L \geq L_*$, the reconstructed H_L satisfies $\inf(\mathrm{spec}(H_L) \setminus \{0\}) \geq m_0 > 0$, with m_0 independent of L.*

(iv) Confinement *(conditional; Theorem 4.11). Under* Assumptions *4.4, 4.7 and 4.9* (v1–v2 miscited these as "Theorems"; corrected), $\sigma_{\mathrm{phys}} > 0$.

(v) Nontriviality *(conditional on confinement). The theory is not a generalized free field.*

The key assumptions are: Assumption 3.5 (summable RG–Cauchy), Assumption A.2 (uniform transfer-matrix gap), Assumption 5.5 (strong continuity), Assumption 4.9 (uniform Creutz-ratio positivity), and Assumption 4.7 (well-defined Creutz ratios). Their status, and the status of the lattice input layer, is discussed in Section 8.

1.4 Scope and limitations

This paper is conditional and works exclusively with bounded local gauge-invariant observables. We do not construct renormalized local fields as operator-valued distributions, and we do not verify the full OS axiom set for distribution-valued Schwinger functions.

Remark 1.3 (Order of limits)**.** *The dyadic continuum limit $a_k \to 0$ is taken at fixed physical volume (a four-torus of side L). The OS reconstruction yields $H = H_L$ at that fixed L. Infinite-volume statements require an additional thermodynamic-limit input $L \to \infty$, not proved here and treated as conditional whenever invoked.*

1.5 The lattice input layer and window compatibility (new in v3)

Hypothesis (Imported open hypotheses (audited chain))**.** *The lattice inputs consumed by this paper carry, per their v2/v3 replacements:* (H-P0) *(power-law large-field penalty; 2602.0052 v2),* (H-YGZ) *(fiber LSI constant not degraded by Holley–Stroock; 2602.0053 v2),* (H-SFI) *(slow-field interface; 2602.0056 v2),* (H-ABS) *(strong-form absorption; 2602.0057 v2),* (H-DOB-blk) *(block Dobrushin condition, consumed by the DLR→transfer-gap route; 2602.0053/0054 v2), and the flow window $L_{\mathrm{vol}} \leq e^{C/g^2+O(1)}$ with crossover hypothesis* (H-XOVER) *(2602.0051 v2). None is proved in the series. Every appeal to "[E26I]–[E26IX]" below is to be read with this load.*

Lemma 1.4 (Window compatibility of the dyadic trajectory; new in v3)**.** *Along the 2-loop trajectory (5), the lattice size required at step k is $L/a_k = 2^k L/a_0$, i.e. $\ln(L/a_k) = k\ln 2 + \ln(L/a_0)$, while the audited volume window of the input chain allows $\ln L_{\mathrm{lat}} \leq c_w/g_k^2$ with $c_w \approx 32.3$ (2602.0051 v2 arithmetic) and $g_k^{-2} = g_0^{-2} + 2b_0 k \ln 2$. Since*

$$c_w \cdot 2b_0 \ln 2 \;\approx\; 32.3 \times 0.0966 \;\approx\; 3.12 \;>\; \ln 2 \;\approx\; 0.693,$$

the window bound grows ≈ 4.5 times faster than the requirement: the dyadic continuum trajectory remains inside the audited window for all k, with margin ratio $\to c_w \cdot 2b_0 \approx 4.5$ (companion suite, check 1). The window is therefore not *an obstruction to the continuum limit; it is inherited as a condition on the constants (c_w, b_0) only.*

2 Observable spaces and renormalization conditions

2.1 The lattice theory

On the four-torus $\Lambda_{a,L} = a(\mathbb{Z}/(L/a)\mathbb{Z})^4$ of physical volume L^4, the Wilson action is

$$S_\beta(U) = \beta \sum_{p\in\mathcal{P}(\Lambda_{a,L})} \Big(1 - \tfrac{1}{N_c}\operatorname{Re}\operatorname{Tr} U_p\Big), \qquad \beta = \frac{2N_c}{g^2}, \tag{1}$$

with measure $d\mu_{\beta,a,L} = Z^{-1}e^{-S_\beta}\prod_e dU_e$, dU_e normalized Haar on $G = SU(N_c)$.

2.2 Smeared curvature observables

Definition 2.1 (Lattice curvature)**.** *For a plaquette p in the $(\mu\nu)$-plane at site x,*

$$F^{(a)}_{\mu\nu}(x) := \frac{1}{a^2}\big(U_p(x) - U_p(x)^\dagger\big)^{\text{traceless}} \in \mathfrak{su}(N_c). \tag{2}$$

Definition 2.2 (Smeared curvature monomial)**.** *For $\varphi_1,\dots,\varphi_n \in \mathcal{S}(\mathbb{R}^4;\mathbb{R})$,*

$$O_a[\varphi_1,\dots,\varphi_n] := \prod_{i=1}^{n}\Big(a^4 \sum_{x\in\Lambda_{a,L}} \varphi_i(x)\operatorname{tr} F^{(a)}_{\mu_i\nu_i}(x)^2\Big). \tag{3}$$

Definition 2.3 (Lattice Schwinger functions (formal; not constructed here))**.**

$$G^{(a)}_n(\varphi_1,\dots,\varphi_n) := \big\langle O_a[\varphi_1,\dots,\varphi_n]\big\rangle_{\mu_{\beta(a),a,L}}. \tag{4}$$

Since O_a is unbounded ($\|F^{(a)}\|_\infty \sim a^{-2}$), no convergence is claimed for $G^{(a)}_n$ in this paper.

2.3 Local observable algebra at fixed physical scale

Definition 2.4 (Bounded local gauge-invariant observables)**.** *Fix $\ell > 0$. $\mathcal{A}_\ell$ is the class of gauge-invariant lattice observables that (a) are bounded measurable functions of finitely many links with $\|O\|_\infty \le 1$; (b) are supported in physical regions of diameter $\le \ell$; (c) are stable under normalized products and convex combinations. Typical elements: $\frac{1}{N_c}\operatorname{Re}\operatorname{Tr} U_p$, traces of closed paths of diameter $\le \ell$, and smooth bounded functions thereof.*

Remark 2.5 (Why bounded observables)**.** $\|F^{(a)}\|_\infty \sim a^{-2}$*; uniform moment bounds for curvature monomials would require vacuum subtraction and wave-function renormalization. Restricting to $\|O\|_\infty \le 1$ gives uniform control without subtractions.*

Remark 2.6 (Two classes of observables)**.** *The $G^{(a)}_n$ involve unbounded fields; $\mathcal{A}_\ell$ restricts to bounded observables with automatic uniform control (see Remark 2.7).*

Remark 2.7 (Relation to smeared curvature observables)**.** *Layer 1 applies in two modes:* bounded mode *(convergence from Assumption 3.5 alone) and* unbounded mode *(requires additive and multiplicative renormalization; deferred, no claims here).*

2.4 Renormalization trajectory

The bare coupling is tuned as

$$\beta(a) = \frac{2N_c}{g(a)^2}, \qquad g(a)^{-2} = 2b_0 \ln(a_0/a) + O(\ln\ln(a_0/a)), \tag{5}$$

$b_0 = 11N_c/(48\pi^2)$, with a_0 chosen so that $g(a_0) = g_0 \le \gamma_0$.

Definition 2.8 (Master threshold, recalled from [E26IX]). *$\gamma_0 > 0$ is the largest value satisfying conditions (T1)–(T5) of [E26IX, Definition 4.11].* (v3) *Per [E26IX]'s v2 replacement (2602.0051 v2), the simultaneous validity so ensured is* windowed *and carries the imported hypotheses of §1.5; by Lemma 1.4 the window itself is compatible with the dyadic trajectory.*

3 Layer 1: Existence and uniqueness of local fields

3.1 Tightness

Proposition 3.1 (Tightness for bounded observables). *Fix $L > 0$. For every $O \in \mathcal{A}_\ell$, $|\langle O\rangle_{a,L}| \le 1$ for all $a \in (0, a_0]$ with $L/a \in \mathbb{N}$; more generally*

$$\big|\langle O_1 \cdots O_n\rangle_{a,L}\big| \;\le\; \prod_{i=1}^{n} \|O_i\|_\infty. \tag{6}$$

Remark 3.2 (Curvature monomials are not covered). *$\|O_a[\varphi]\|_\infty$ diverges as $a \to 0$; uniform moment bounds would require renormalization, not carried out here.*

Corollary 3.3 (Tightness and subsequential convergence). *For $O \in \mathcal{A}_\ell$ the family $\{\langle O\rangle_{a,L}\}_a$ lies in $[-1, 1]$, hence is trivially precompact. For the random-field upgrade, define*

$$\Phi_a(\varphi) := a^4 \sum_{x\in\Lambda_{a,L}} \varphi(x)\, \frac{1}{6} \sum_{1\le\mu<\nu\le 4} \tfrac{1}{N_c} \operatorname{Re}\operatorname{Tr} U_{p_{\mu\nu}(x)}; \tag{7}$$

$|\Phi_a(\varphi)| \le L^4\|\varphi\|_\infty$ deterministically, and Mitoma's theorem [11] gives tightness of the laws in $\mathcal{S}'(\mathbb{R}^4)$.

Remark 3.4 (Mitoma requires random variables). *Mitoma's theorem concerns laws of $\mathcal{S}'$-valued random variables; it cannot be applied to deterministic expectation families. Method 1 (trivial precompactness) suffices for this paper.*

3.2 RG–Cauchy estimate

Assumption 3.5 (RG–Cauchy along the trajectory). *Along $a \mapsto \beta(a)$ there exists $\varepsilon : (0, a_0] \to (0, \infty)$ with $\varepsilon(a) \downarrow 0$ and $\sum_k \varepsilon(a_0 2^{-k}) < \infty$, such that for every $L > 0$, $\ell > 0$, $O \in \mathcal{A}_\ell$,*

$$\big|\langle O\rangle_{a,L} - \langle O\rangle_{a/2,L}\big| \;\le\; \varepsilon(a), \qquad a \in (0, a_0],\ L/a \in \mathbb{N}. \tag{8}$$

Remark 3.6 (Summability is genuinely stronger than the naive AF estimate). *A naive asymptotic-freedom estimate gives one-step changes $\lesssim g_k^2 \sim c/k$:* not summable. *Assumption 3.5 postulates an improved, summable control; 2-loop asymptotic freedom does not by itself imply it.* (v3) *Adjudicated numerically: the partial sums $\sum_{k\le N} g_k^2$ grow like $(2b_0 \ln 2)^{-1} \ln N$ without bound (suite check 2), while both sufficient rates of Remark 3.7 are summable. In the exact $d = 2$ toy the true one-step changes decay* geometrically *($\sim 4^{-k}$; suite check 3), illustrating what a correct mechanism must deliver in $d = 4$.*

Remark 3.7 (Examples of summable rates; status)**.** *Sufficient:* $\varepsilon(a) \le C(a/\ell)^\eta$ *(*$\eta > 0$*), or* $\varepsilon(a) \le C/[\ln(a_0/a)]^{1+\delta}$ *(*$\delta > 0$*). We do not derive such a rate; Assumption 3.5 is the single hard analytic input of Layer 1 (see Appendix B).*

3.3 Uniqueness of local Schwinger functions

Proposition 3.8 (Dyadic convergence of local expectations)**.** *Under Assumption 3.5, for each* $O \in \mathcal{A}_\ell$ *the dyadic limit*

$$\lim_{k\to\infty} \langle O \rangle_{a_k,L}, \qquad a_k := a_0/2^k, \tag{9}$$

exists: for $m > n$*, telescoping gives*

$$\big|\langle O \rangle_{a_m,L} - \langle O \rangle_{a_n,L}\big| \;\le\; \sum_{k=n}^{m-1} \varepsilon(a_k) \;\xrightarrow[n\to\infty]{}\; 0. \tag{10}$$

Theorem 3.9 (Existence and uniqueness of continuum local expectations (dyadic))**.** *Assume Assumption 3.5. Then for every* $L > 0$*,* $\ell > 0$*,* $O \in \mathcal{A}_\ell$*,*

$$\omega_L(O) \;:=\; \lim_{k\to\infty} \langle O \rangle_{a_k,L} \tag{11}$$

exists and is independent of subsequences of $\{a_k\}$*. (Convergence along general* $a \to 0$ *would require a Cauchy estimate for non-halving steps, not provided by Assumption 3.5.)*

Remark 3.10 (A natural C^*-completion)**.** *Each* $\mathcal{A}_\ell$ *carries the* L^∞ *norm; the completion of* $\bigcup_\ell \mathcal{A}_\ell$ *gives a quasi-local* C^**-algebra on which*

$$\omega_L(O) = \lim_k \langle O \rangle_{a_k,L} \tag{12}$$

defines a state (positivity and normalization pass to the limit).

Remark 3.11 (Extension to smeared curvature monomials)**.** *Deferred; their* L^∞ *norms diverge as* a^{-2n}*.*

4 Layer 2: Step-scaling confinement

4.1 Wilson loops and Creutz ratios at fixed physical scale

Definition 4.1 (Wilson loops in lattice and physical units)**.** *For* $I, J \in \mathbb{N}$*,*

$$W_a(I,J) := \Big\langle \tfrac{1}{N_c}\,\mathrm{Re}\,\mathrm{Tr} \prod_{e\in\partial(I\times J)} U_e \Big\rangle_{a,L}, \tag{13}$$

and at fixed physical size $R \times T$*,*

$$W_a(R,T) := W_a\big(\lfloor R/a \rfloor, \lfloor T/a \rfloor\big). \tag{14}$$

Remark 4.2 (Resolution of the fixed-I, J error)**.** *A previous draft incorrectly held* I, J *fixed as* $a \to 0$ *(shrinking physical loops). All Layer-2 statements use fixed physical* R, T*.*

Definition 4.3 (Creutz ratio (finite and infinite volume))**.**

$$\chi_L^{(a)}(R,T) := -\ln \frac{W_a(R,T)\, W_a(R-a,T-a)}{W_a(R-a,T)\, W_a(R,T-a)} \Big/ a^2. \tag{15}$$

Assumption 4.4 (Thermodynamic limit for Creutz ratios (conditional))**.** *For each a along the trajectory and each fixed R,T, the limit*

$$\chi_\infty^{(a)}(R,T) := \lim_{L\to\infty} \chi_L^{(a)}(R,T) \tag{16}$$

exists, and the dyadic limit $\lim_k \chi_\infty^{(a_k)}(R,T)$ exists.

Definition 4.5 (Physical string tension (infinite volume; conditional))**.** *Assume Assumption 4.4. Set*

$$\chi_\infty(R,T) := \lim_{k\to\infty} \chi_\infty^{(a_k)}(R,T), \tag{17}$$

$$\sigma_{\text{phys}} := \liminf_{\substack{R,T\to\infty \\ R/T \text{ fixed}}} \chi_\infty(R,T). \tag{18}$$

Proposition 4.6 (Existence of the infinite-volume dyadic Creutz-ratio limit)**.** *Assume Assumptions 4.4 and 4.7. Then $\chi_\infty(R,T)$ exists for each fixed $R,T>0$.*

Assumption 4.7 (Creutz ratio well-defined in the thermodynamic limit (conditional))**.** *The Wilson-loop expectations entering* (15) *are strictly positive in the thermodynamic limit (nonvanishing denominators).*

Remark 4.8 (Mass gap does not imply confinement)**.** *Exponential clustering implies perimeter-law decay for large loops but does not by itself produce an area law; confinement requires the independent input Assumption 4.9.*

Assumption 4.9 (Uniform positive string tension input via Creutz ratios (infinite volume))**.** *Assume Assumption 4.4. There exist $\sigma_0 > 0$ and $R_*, T_* > 0$ such that for all sufficiently small a along the trajectory and all $R \geq R_*$, $T \geq T_*$,*

$$\chi_\infty^{(a)}(R,T) \;\geq\; \sigma_0. \tag{19}$$

Remark 4.10 (Status of Assumption 4.9)**.** *A uniform infrared positivity input. In strong coupling, area law and positivity of Creutz ratios are classical [15]; in weak coupling the input is supported by Monte Carlo and large-N_c evidence but remains open.* (v3) *In the exact $d=2$ toy the input holds with $\sigma_0 = \sigma_{\text{phys}}$ exactly (suite check 4).*

Theorem 4.11 (Confinement (conditional))**.** *Assume Assumptions 4.4, 4.9 and 4.7. Then*

$$\sigma_{\text{phys}} > 0 : \tag{20}$$

by Proposition 4.6 the continuum Creutz ratio exists; by Assumption 4.9 and the dyadic limit,

$$\sigma_{\text{phys}} = \liminf_{\substack{R,T\to\infty \\ R/T \text{ fixed}}} \chi_\infty(R,T) \;\geq\; \sigma_0 \;>\; 0. \tag{21}$$

5 Reflection positivity and reconstruction (bounded observables)

5.1 OS-Positivity

Lemma 5.1 (Osterwalder–Schrader positivity)**.** *For each $a > 0$ the lattice Yang–Mills measure satisfies reflection positivity with respect to any lattice hyperplane [14]:*

$$\big\langle (\Theta A)^* A \big\rangle_{\mu_{\beta(a),a,L}} \;\geq\; 0 \tag{22}$$

for functionals A supported in $\{x_0 > 0\}$. If expectations converge along dyadic scales (Theorem 3.9), the limiting state inherits reflection positivity: for $O \in \mathcal{A}_\ell$ supported in $\{x_0 > \delta\}$, $(\Theta O)^* O$ is again a bounded gauge-invariant observable and $\omega_L((\Theta O)^* O) = \lim_k \langle (\Theta O)^* O \rangle_{a_k,L} \geq 0$; general finite families pass to the limit the same way.

Remark 5.2 (Subtlety for unbounded observables)**.** *For smeared curvature monomials, OS positivity of the limit would first require convergence of renormalized Schwinger functions; not treated here.*

5.2 Euclidean covariance: SO(4) rotational invariance

Proposition 5.3 (SO(4) invariance (conditional))**.** *The lattice action is hypercubic-invariant; SO(4)-breaking operators in the effective action have engineering dimension ≥ 6, hence are suppressed by $O(a^2)$. At the level of $\mathcal{A}_\ell$, hypercubic symmetry is exact at each a and inherited by the dyadic limit; full SO(4) covariance concerns renormalized Schwinger functions and is deferred.*

Remark 5.4 (Precise citation needed)**.** *Excluding relevant SO(4)-breaking mixing uses the classification of gauge-invariant operators under the hypercubic group [17] and the multiscale control of Papers [E26VIII]–[E26IX].* (v3) *The latter is windowed/conditional per §1.5.*

5.3 Translation invariance and cluster property

In finite volume, torus translation invariance is inherited by the dyadic limit. Writing $\omega := \lim_{L\to\infty} \omega_L$ (conditional thermodynamic limit), the cluster property

$$\omega(O_1 \tau_R O_2) \;\xrightarrow[|R|\to\infty]{}\; \omega(O_1)\,\omega(O_2) \tag{23}$$

is conditional, following from exponential clustering (Lemma 6.1) under Assumption A.2.

5.4 Reconstruction

Assumption 5.5 (Strong continuity of Euclidean time translations)**.** *The semigroup $(T_L(t))_{t\geq 0}$ induced on the reconstructed Hilbert space is strongly continuous.*

Corollary 5.6 (Hilbert-space reconstruction)**.** *Assume ω_L is translation invariant and reflection positive (Lemma 5.1). With $\mathcal{A}_+$ the subalgebra supported in $\{x_0 > 0\}$ and*

$$\langle A, B \rangle_{\mathrm{RP}} := \omega_L\big((\Theta A)^* B\big), \tag{24}$$

quotienting nulls and completing yields a separable $\mathcal{H}_L$ with cyclic Ω_L; time translations induce a contraction semigroup; and under Assumption 5.5 there is a self-adjoint $H_L \geq 0$ with $T_L(t) = e^{-tH_L}$, $H_L \Omega_L = 0$ [12,13,8].

Remark 5.7 (Dependence of H on the observable algebra)**.** *$\mathcal{H}_L$ depends on the input algebra; it may be a proper subspace of the space from a curvature-field reconstruction. Theorem 6.2 bounds the gap of H_L on this space.*

6 Mass gap

6.1 Uniform exponential clustering

Lemma 6.1 (Uniform exponential clustering)**.** *Under Assumption A.2 there exists $m_0 > 0$ such that for all $L \geq L_*$, all $a \in (0, a_0]$ $(L/a \in \mathbb{N})$, and all $O_1, O_2 \in \mathcal{A}_\ell$ at physical separation d:*

$$\left|\langle O_1 O_2\rangle_{a,L} - \langle O_1\rangle_{a,L}\langle O_2\rangle_{a,L}\right| \;\leq\; C_{O_1,O_2} e^{-m_0 d}. \tag{25}$$

(Details in Appendix A.)

6.2 From clustering to mass gap

Theorem 6.2 (Mass gap (conditional))**.** *Assume Assumptions 3.5, A.2 and 5.5. Then for every fixed $L \geq L_*$,*

$$\inf\bigl(\operatorname{spec}(H_L) \setminus \{0\}\bigr) \;\geq\; m_0 \;>\; 0, \tag{26}$$

with m_0 independent of L: the lattice clustering bound passes to the limit,

$$\left|\omega_L(O_1 O_2) - \omega_L(O_1)\omega_L(O_2)\right| \;\leq\; C e^{-m_0 d}, \tag{27}$$

and for $A \in \mathcal{A}_+$ with $\omega_L(A) = 0$,

$$\left|\langle [A], T_L(t)[A]\rangle\right| \;\leq\; C_A e^{-m_0 t}, \tag{28}$$

so the spectral measure of $[A]$ is supported in $[m_0, \infty)$ by the Laplace-transform argument (suite check 6 verifies this mechanism exactly on a finite transfer matrix).

7 Nontriviality

Theorem 7.1 (Nontriviality (conditional))**.** *Under the hypotheses of Theorem 4.11 and the area-law input,*

$$\sigma_{\text{phys}} > 0 \tag{29}$$

implies the continuum theory is not a generalized free field: a Gaussian (generalized free) state cannot produce area-law decay of Wilson loops at all scales, since its truncated correlators factorize and yield perimeter-law behavior for large loops.

8 Discussion

8.1 Honest assessment of the logical structure

Result	Mechanism	Status (v3)
Existence of continuum exp.	telescoping + RG–Cauchy	cond. Assumption 3.5
Tightness / compactness	$\lvert\langle O\rangle\rvert \leq 1$; Mitoma	proved (bounded obs.)
OS-positivity	lattice RP + convergence	proved (Lemma 5.1)
SO(4) invariance	$O(a^2)$ breaking	cond. (deferred G_n)
Mass gap	transfer gap + OS	cond. A.2 + 5.5
$\sigma_{\text{phys}} > 0$	Creutz step-scaling	cond. 4.4 + 4.7 + 4.9
Nontriviality	area vs. perimeter	cond. on confinement
Lattice input layer	*chain 0041–0057*	*windowed; (H-P0)(H-YGZ) (H-SFI)(H-ABS)(H-DOB-blk)*
Window compatibility	Lemma 1.4	**verified** (margin 4.5×)

Assumption 3.5 is mappable to the multiscale outputs of [E26VIII]–[E26IX] only at the non-summable $O(g_k^2)$ level (Appendix B); upgrading to a summable rate is the single hardest analytical step of Layer 1.

8.2 What this paper does not do

(1) We do not prove the RG–Cauchy estimate from first principles (naive AF rates are not summable: Remark 3.6). (2) The transfer-matrix gap (Assumption A.2) is postulated; the LSI chain controls a Glauber-dynamics gap, a different operator (Remark A.1); its conditional lattice supplier is 2602.0054 v2 (see Remark A.8). (3) The Creutz positivity input (Assumption 4.9) is postulated. (4) The thermodynamic limit $L \to \infty$ is not proved. (5) We work exclusively with bounded local observables. (v3) (6) The lattice input layer itself is windowed and conditional (§1.5); this paper does not discharge any of the six open hypotheses.

8.3 Comparison with previous work

The two-layer separation (bounded-observable uniqueness vs. physical string tension) avoids both the gauge-variance obstruction and the Wilson-loop renormalization problem at the price of conditionality; cf. [8,14,15] for the classical lattice inputs used.

9 Note added in v3

Versions 1–2 described the lattice inputs as unconditional. Following the audit of the chain (2602.0041 v3; 2602.0051–0057 v2), v3 records:

- The phrase "unconditional lattice closure inputs [E26IX]" (abstract) and the sentence "Papers [E26I]–[E26IX] established item (1)" (§1.1) are corrected: the lattice layer is windowed ($L_{\mathrm{vol}} \le e^{C/g^2+O(1)}$) and conditional on (H-P0), (H-YGZ), (H-SFI), (H-ABS), and — for the DLR→transfer route — (H-DOB-blk) (§1.5).

- New Lemma 1.4: the dyadic continuum trajectory is *compatible* with the audited window, with margin $c_w \cdot 2b_0 \approx 4.5$ (suite check 1). The window constrains constants, not the construction.

- Assumption A.2 is cross-referenced to its conditional lattice supplier: 2602.0054 v2 proves a transfer-matrix gap conditionally on (H-DOB-blk)+(H-P0), in the window, with correlation identities in kernel/symmetrized form (its Remark 2.8); this paper's Appendix A uses only the spectral gap, which that correction leaves unchanged (Remark A.8).

- Labeling fix: Theorem 1.2(iv) cited "Theorems 4.4, 4.7 and 4.9"; these are *Assumptions* 4.4, 4.7, 4.9. Corrected here.

- Reference hygiene: [22] cited 2602.0055 under its withdrawn v1 title ("Unconditional uniform LSI…"); [24] cited 2602.0053 as "…and unconditional mass gap"; [20] cited 2602.0057 under a pre-2602 title. All companion references now carry their audited titles and versions. The Remark 3.6/B.1 non-summability discussion of v2 — which anticipated the audit spirit — is retained verbatim and now validated numerically (suite check 2).

- What is validated end-to-end in the exact $d = 2$ toy (suite checks 3–7): the full Layer-1 mechanism (dyadic Cauchy with geometric rate, unique limit independent of the anchor a_0), the full Layer-2 pipeline (Creutz ratios $\to \sigma_{\mathrm{phys}}$ exactly, area law exact), lattice RP via PSD Gram matrices, and the clustering→spectral-gap Laplace mechanism on finite transfer matrices.

All numerical adjudications are reproduced by the deterministic companion suite `verify_2602_0063.py` (7 checks) at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, directory `verification/2602-0`

A Transfer matrix and uniform clustering

A.1 Setup

$\Lambda_{a,L} = a(\mathbb{Z}/N_a\mathbb{Z})^3 \times a(\mathbb{Z}/M_a\mathbb{Z})$; the transfer matrix $\widehat{T}_a$ acts on $L^2(\mu_{\beta(a),\Sigma_a})$ over a spatial slice Σ_a; we write

$$\widehat{T}_a \;=\; e^{-aH_a} \quad \text{(spectral definition on gauge-invariant states).}$$

A.2 Mass gap: transfer-matrix spectral gap

Remark A.1 (LSI vs. transfer-matrix gap)**.** *The LSI of [E26VII, E26IX] controls the spectral gap of a Glauber-type dynamics — a different operator from $\widehat{T}_a$. An LSI bound does not imply a transfer-matrix gap bound; hence the separate assumption.*

Assumption A.2 (Uniform physical transfer-matrix gap along the trajectory)**.** *There exist $m_0 > 0$, $L_* > 0$ such that for every $L \geq L_*$ and all sufficiently small a ($L/a \in \mathbb{N}$), the two largest eigenvalues $\lambda_0 > \lambda_1 \geq 0$ of $\widehat{T}_a$ on gauge-invariant states satisfy*

$$m_{\mathrm{phys}}(a, L) := -\frac{1}{a} \ln\Big(\frac{\lambda_1(a, L)}{\lambda_0(a, L)}\Big) \;\geq\; m_0. \tag{30}$$

At fixed finite L a gap exists by compactness; the content is uniformity as $L \to \infty$ and $a \to 0$.

Lemma A.3 (Uniform exponential clustering from the transfer gap)**.** *Under Assumption A.2, for all $L \geq L_*$, $a \in (0, a_0]$, and $O_1, O_2 \in \mathcal{A}_\ell$ separated by Euclidean-time distance d,*

$$\big|\langle O_1 O_2\rangle_{a,L} - \langle O_1\rangle_{a,L}\langle O_2\rangle_{a,L}\big| \;\leq\; Ce^{-m_0 d}, \tag{31}$$

by the spectral expansion of time-separated connected correlators in eigenvalues of $\widehat{T}_a$; general Euclidean separations use a chessboard estimate [8]. In physical units,

$$e^{-\Delta_a(L)\tau} = e^{-(\Delta_a(L)/a)d} = e^{-m_{\mathrm{phys}}(a,L)d}, \tag{32}$$

$$m_{\mathrm{phys}}(a, L) = \Delta_a(L)/a, \qquad \Delta_a(L) = -\ln(\lambda_1/\lambda_0). \tag{33}$$

Remark A.4 (Finite-volume gap and the role of Assumption A.2)**.** *At fixed (a, L), $\widehat{T}_a$ is compact and positivity-improving, hence gapped [14,8]; the assumption is the* uniform *lower bound along the trajectory.*

Remark A.5 (No circularity)**.** *The mass gap and confinement are logically independent, each conditional on its own input (Assumptions A.2 and 4.9).*

Remark A.6 (LSI concentration vs. spatial clustering vs. transfer gap)**.** *The DLR-LSI constant controls a Markov-generator gap and Herbst concentration; neither is a statement about Euclidean-time decay. Assumption A.2 is an independent input.*

Remark A.7 (Heuristic: $m_{\rm phys}$ vs. $\sigma_{\rm phys}$ (not used))**.** $m_{\rm phys} \geq c\sqrt{\sigma_{\rm phys}}$ *on the lattice [15] motivates but is not used in any proof.*

Remark A.8 (Conditional lattice supplier of Assumption A.2; new in v3)**.** *The companion 2602.0054 v2 establishes precisely a transfer-matrix spectral gap for the Wilson theory,* windowed *and* conditional on (H-DOB-blk)+(H-P0)*, with the correlation identities stated in kernel/symmetrized form (F-TN, its Remark 2.8). The present appendix consumes only the spectral gap* $\Delta_a(L)$*, which is invariant under that normalization correction (similarity of* $D^{-1}K$ *and* $D^{-1/2}KD^{-1/2}$*). Assumption A.2 is therefore not free- floating: it is the* $L \to \infty$*,* $a \to 0$ *uniformity upgrade of a conditionally established lattice result.*

B Bridge lemma: mapping Assumption 3.5 to [E26VIII]–[E26IX]

B.1 The one-step difference

In Balaban's framework the effective action after k steps is

$$S_k^{\rm eff} = \beta_k A(U_k) + S_k^{\rm loc,irr} + \sum_X R^{(k)}(X) + \sum_X B^{(k)}(X), \tag{34}$$

and the one-step change of a local expectation collects the running coupling shift, the polymer-activity change, and the boundary-term change.

B.2 Bounding each contribution

Running coupling: by [E26IX, Theorem 4.3] (= 2602.0051 v2; the UV direction $a \to a/2$ has the *correct* sign here: $g_{k+1}^{-2} - g_k^{-2} = 2b_0 \ln 2 + O(g_k^2)$), the shift contributes $C_O g_k^2$. Polymer activities (IC4) and boundary terms (IC5) contribute $C_O e^{-c\,p_0(g_k)}$ each. (v3) IC4/IC5 are outputs of the audited chain and are themselves conditional on (B1)–(B4)/(H-P0) and windowed (2602.0055/0056 v2); the bridge inherits that load.

B.3 Summability over scales

The total one-step change is

$$\big|\langle O\rangle_{a_k,L} - \langle O\rangle_{a_{k+1},L}\big| \;\leq\; C_O g_k^2 + e^{-c' p_0(g_k)} \;\leq\; C_O' g_k^2, \tag{35}$$

and since $g_k^{-2} = g_0^{-2} + 2b_0 k \ln 2 + O(1)$, $\sum_{k\leq N} g_k^2 \sim (2b_0 \ln 2)^{-1} \ln N \to \infty$: *the bridge bound is not summable* (suite check 2 reproduces the logarithmic growth, and shows the $e^{-c' p_0}$ term cannot rescue it: $p_0(g_k)$ grows only logarithmically in k). This is the content of Remarks 3.6 and B.1.

Remark B.1 (Logarithmic divergence of the naive bridge bound)**.** *In* $d = 4$ *along dyadic scales, the naive bound* (35) *diverges logarithmically upon summation. It therefore does not imply an RG–Cauchy estimate of the form postulated in Assumption 3.5.*

Remark B.2 (Strengthening to power law)**.** *To obtain* $\varepsilon(a) \leq C(a/\ell)^\eta$ *one must exploit cancellations (e.g. large-field penalties with power-law* p_0*, i.e. (H-P0), or contour-improved one-step maps) rather than the perturbative* g_k^2 *bound. This upgrade is the missing analytic step; we defer it.*

C Notational concordance with [E26IX] and the audited chain

This paper	Chain	Meaning
$\beta(a), g(a)$	β, g_0	bare couplings
γ_0	γ_0	master threshold (windowed per 0051 v2)
$\alpha_*, \alpha_{\mathrm{DLR}}$	id.	LSI / DLR-LSI constants (conditional)
$\Delta_{\mathrm{TM}}(a, L)$	0054 v2	transfer gap (kernel form; (H-DOB-blk)+(H-P0))
$m_{\mathrm{phys}} = \Delta_{\mathrm{TM}}/a$	—	physical mass gap
σ_{phys}	—	physical string tension
ω_L	—	continuum limiting state at fixed L
open set	§1.5	(H-P0)(H-YGZ)(H-DOB-blk)(H-SFI)(H-ABS), window

References

[1] T. Balaban, *Averaging operations for lattice gauge theories*, Comm. Math. Phys. **98** (1985), 17–51.

[2] T. Balaban, *Renormalization group approach to lattice gauge field theories. I*, Comm. Math. Phys. **109** (1987), 249–301.

[3] T. Balaban, *Large field renormalization. I*, Comm. Math. Phys. **116** (1988), 215–254.

[4] T. Balaban, *Large field renormalization. II*, Comm. Math. Phys. **119** (1988), 243–285.

[5] T. Balaban, *Large field renormalization. I. The basic step of the R operation*, Comm. Math. Phys. **122** (1989), 175–202.

[6] T. Balaban, *Large field renormalization. II. Localization, exponentiation, and bounds for the R operation*, Comm. Math. Phys. **122** (1989), 355–392.

[7] D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.

[8] J. Glimm, A. Jaffe, *Quantum Physics*, 2nd ed., Springer, 1987.

[9] A. Jaffe, E. Witten, *Quantum Yang–Mills theory*, Clay Millennium Prize Problems, 2000.

[10] M. Lüscher, P. Weisz, *String excitation energies in SU(N) gauge theories*, JHEP **07** (2004), 014.

[11] I. Mitoma, *Tightness of probabilities on $C([0,1];\mathcal{S}')$ and $D([0,1];\mathcal{S}')$*, Ann. Probab. **11** (1983), 989–999.

[12] K. Osterwalder, R. Schrader, *Axioms for Euclidean Green's functions*, Comm. Math. Phys. **31** (1973), 83–112.

[13] K. Osterwalder, R. Schrader, *Axioms for Euclidean Green's functions. II*, Comm. Math. Phys. **42** (1975), 281–305.

[14] K. Osterwalder, E. Seiler, *Gauge field theories on a lattice*, Ann. Physics **110** (1978), 440–471.

[15] E. Seiler, *Gauge Theories as a Problem of Constructive QFT and Statistical Mechanics*, LNP 159, Springer, 1982.

[16] D. Stroock, B. Zegarlinski, *The logarithmic Sobolev inequality for continuous spin systems on a lattice*, J. Funct. Anal. **104** (1992), 299–326.

[17] P. Weisz, *Continuum limit improved lattice action for pure Yang–Mills theory (I)*, Nucl. Phys. B **212** (1983), 1–17.

[18] L. Eriksson, *Uniform log-Sobolev inequality for lattice Yang–Mills via multiscale renormalization and entropy telescoping*, ai.viXra:2602.0041 (v3), 2026. [E26I]

[19] L. Eriksson, *Synthetic Ricci curvature and conditional log-Sobolev inequalities for lattice gauge theories on the orbit space*, ai.viXra:2602.0046, 2026. [E26II]

[20] L. Eriksson, *Integrated cross-scale derivative bounds for Wilson lattice gauge theory: closing the log-Sobolev gap — a conditional and windowed closure*, ai.viXra:2602.0057 (v2), 2026. [E26III] (v1–v2 of the present paper cited this entry under a pre-2602 title; corrected.)

[21] L. Eriksson, *Large-field suppression for lattice gauge theories: from Balaban's renormalization group to conditional concentration — a conditional and windowed verification*, ai.viXra:2602.0056 (v2), 2026. [E26IV]

[22] L. Eriksson, *Residual derivative bounds and windowed uniform log-Sobolev inequality for $SU(N_c)$ lattice Yang–Mills at weak coupling*, ai.viXra:2602.0055 (v2), 2026. [E26V] (formerly cited under the withdrawn v1 title "Unconditional uniform LSI…".)

[23] L. Eriksson, *From uniform log-Sobolev inequality to mass gap for lattice Yang–Mills at weak coupling: a conditional and windowed assembly*, ai.viXra:2602.0054 (v2), 2026. [E26VI]

[24] L. Eriksson, *DLR-uniform log-Sobolev inequality for lattice Yang–Mills: a conditional and windowed reduction*, ai.viXra:2602.0053 (v2), 2026. [E26VII] (formerly cited as "…and unconditional mass gap"; corrected.)

[25] L. Eriksson, *Interface lemmas for the multiscale proof of the lattice Yang–Mills mass gap*, ai.viXra:2602.0052 (v2), 2026. [E26VIII]

[26] L. Eriksson, *[Series erratum and windowed multiscale flow]*, ai.viXra:2602.0051 (v2), 2026. [E26IX] Master threshold, corrected AF flow, window $k^*(\beta)$, (H-XOVER).