Conditional Continuum Limit of 4d $\mathrm{SU}(N_c)$ Yang–Mills Theory via Two-Layer Architecture, RG–Cauchy Uniqueness, and Step-Scaling Confinement

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Abstract

Building on the lattice results established in Papers [E26I]–[E26IX], we give a conditional construction of a scaling-limit state for pure $\mathrm{SU}(N_c)$ lattice Yang–Mills theory in four Euclidean dimensions, along dyadic lattice spacings $a_k = a_0\, 2^{-k}$. The construction proceeds via a *two-layer architecture*:

Layer 1 (Local fields). For bounded gauge-invariant local observables (Wilson loops, normalized plaquette traces), expectations converge—without extracting subsequences—to a unique limit. Tightness is trivial (L^∞ bound plus Prokhorov); uniqueness follows from a multiscale *RG–Cauchy estimate* that bounds the change of local expectations under a single RG step. The extension to unbounded observables such as smeared curvature monomials, which require additive renormalization, is deferred to future work.

Layer 2 (Confinement). The physical string tension $\sigma_{\mathrm{phys}} > 0$ is established through step-scaling of Creutz ratios evaluated on Wilson loops whose physical dimensions $R \times T$ are held fixed as $a \to 0$.

The limiting state on bounded observables inherits Osterwalder–Schrader positivity from the lattice and admits a Hilbert-space reconstruction via reflection positivity. $SO(4)$ rotational invariance is expected in the continuum (the hypercubic breaking being $O(a^2)$, subject to the standard operator classification and construction of renormalized Schwinger functions). The mass gap is established conditionally via uniform exponential clustering of connected correlators—an input from a uniform physical transfer-matrix spectral gap (Assumption A.2)—and the reconstruction theorem. Nontriviality follows conditionally from an area law for Wilson loops.

Key dependencies on prior papers: uniform LSI inputs [E26I]–[E26IX]; Balaban multi-scale effective action [E26III]–[E26V]; DLR-LSI [E26VII]; unconditional lattice closure inputs [E26IX].

Contents

1 Introduction

1.1 The problem

The construction of a mathematically rigorous four-dimensional quantum field theory with a non-abelian gauge group—and the proof of its associated mass gap—remains a central challenge in mathematical physics, listed as a Clay Millennium Problem [9]. Lattice gauge theory provides a non-perturbative, gauge-invariant regularization, but removing the lattice cutoff ($a \to 0$) unconditionally requires:

1. uniform functional inequalities (log-Sobolev, spectral gap) independent of the volume;
2. a compactness–uniqueness argument for the limiting state on bounded gauge-invariant local observables;
3. preservation of the physically essential properties (Osterwalder–Schrader axioms, confinement, mass gap).

Papers 68–76 of this series established item (1) on the lattice. The present paper carries out items (2) and (3).

1.2 The two-layer architecture

Standard compactness arguments in constructive QFT face a dual obstruction in gauge theories. First, the fundamental connection fields A_μ are gauge-variant, and the natural orbit space is nonlinear. Second, the natural gauge-invariant observables—Wilson loops—are nonlocal, exponential functionals requiring perimeter renormalization.

We resolve this through a bipartite logical structure:

Layer 1

Local fields and RG–Cauchy uniqueness. We work with bounded gauge-invariant local observables (Definition 2.4). The uniform L^∞ bound $\|\mathcal{O}\|_\infty \le 1$ yields tightness via Prokhorov's theorem; the LSI provides the stronger quantitative control (exponential concentration) used in later layers. Uniqueness is obtained not by identifying the theory through a single observable (such as the string tension), but by proving a quantitative *Cauchy estimate* for the expectations of local observables at a fixed physical scale under successive RG steps. This guarantees convergence without subsequences.

Layer 2

Step-scaling and confinement. To capture macroscopic physics, we analyse Creutz ratios on Wilson loops with lattice dimensions $I(a) = \lfloor R/a \rfloor$, $J(a) = \lfloor T/a \rfloor$, so that their physical size remains fixed as $a \to 0$. The result is a strictly positive physical string tension $\sigma_{\text{phys}} > 0$.

Remark 1.1 (What is *not* used for uniqueness). We do *not* use the string tension to determine the theory. The argument "σ fixes the theory" would require proving that σ_{phys} uniquely determines all Schwinger functions—a statement that is not established and not needed. Uniqueness follows entirely from the RG–Cauchy estimate (Layer 1). Confinement is an *output* of the construction, not an input to uniqueness.

1.3 Main results

Theorem 1.2 (Main theorem (conditional))**.** *Let $G = \mathrm{SU}(N_c)$, $d = 4$. Fix a physical scale $\ell_0 > 0$ and tune the bare coupling $\beta(a) = 2N_c/g(a)^2$ along a 2-loop asymptotic freedom trajectory. Then, under the assumptions listed below:*

(i) ***Existence and uniqueness (dyadic)*** *(Assumption 3.5). For every bounded gauge-invariant local observable $\mathcal{O} \in \mathfrak{A}_\ell$, the expectation $\langle \mathcal{O} \rangle_{a_k}$ converges as $k \to \infty$ along the*

dyadic sequence $a_k := a_0/2^k$ to a unique limit, independent of subsequences of $\{a_k\}$. The extension to unbounded observables (smeared curvature monomials) requires additional renormalization and is deferred to future work.

(ii) ***Reflection positivity and reconstruction.*** *The limiting state on the quasi-local algebra of bounded gauge-invariant observables inherits reflection positivity from the lattice and admits a Hilbert-space reconstruction (Section 5). Euclidean covariance and the cluster property are conditional properties of the infinite-volume continuum theory (Section 5).*

(iii) ***Mass gap*** *(Assumptions A.2 and 5.5). The Hamiltonian H reconstructed via OS satisfies* $\inf(\mathrm{spec}(H) \setminus \{0\}) \geq m_0 > 0$.

(iv) ***Confinement*** *(conditional; Theorem 4.11). Under Theorems 4.4, 4.7 and 4.9, the physical string tension satisfies $\sigma_{\mathrm{phys}} > 0$.*

(v) ***Nontriviality*** *(conditional on confinement). The theory is not a generalized free field.*

The key assumptions are: Assumption 3.5 (summable RG–Cauchy estimate), Assumption A.2 (uniform transfer-matrix gap), Assumption 5.5 (strong continuity of the reconstructed semigroup), Assumption 4.9 (uniform positive string-tension input formulated via Creutz ratios), and the technical Assumption 4.7 (nonvanishing Creutz denominators). Their status is discussed in Section 8.

1.4 Scope and limitations

This paper is conditional and works exclusively with bounded local gauge-invariant observables $\mathcal{O} \in \mathfrak{A}_\ell$ (Definition 2.4). We do not construct renormalized local fields such as $F_{\mu\nu}$ as operator-valued distributions, and we do not verify the full Osterwalder–Schrader axiom set for distribution-valued Schwinger functions. Statements about the mass gap and confinement are therefore proved conditionally under explicit analytic inputs (Assumptions A.2 and 4.9).

2 Observable spaces and renormalization conditions

2.1 The lattice theory

We work on the torus $\Lambda_a = a(\mathbb{Z}/N_a\mathbb{Z})^4$ with physical volume $V = (N_a \cdot a)^4$, equipped with the Wilson action

$$S_\beta(U) = \beta \sum_{p \in \mathcal{P}(\Lambda_a)} \Big(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{Tr}\, U_p\Big), \qquad \beta = \frac{2N_c}{g^2}. \tag{1}$$

The lattice measure is $d\mu_{\beta,a}(U) = Z_{\beta,a}^{-1} e^{-S_\beta(U)} \prod_e dU_e$ with dU_e the normalised Haar measure on $G = \mathrm{SU}(N_c)$.

2.2 Smeared curvature observables

Definition 2.1 (Lattice curvature)**.** For a plaquette p in the $(\mu\nu)$-plane at lattice site x, define

$$F_{\mu\nu}^{(a)}(x) := \frac{1}{a^2}\Big(U_p(x) - U_p(x)^\dagger\Big)^{\text{traceless}} \in \mathfrak{su}(N_c). \tag{2}$$

Definition 2.2 (Smeared curvature monomial)**.** For test functions $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^4; \mathbb{R})$ and multi-indices (μ_i, ν_i), define

$$\mathcal{O}_a[\varphi_1, \ldots, \varphi_n] := \prod_{i=1}^{n} \Big(a^4 \sum_{x \in \Lambda_a} \varphi_i(x)\, \mathrm{tr}(F_{\mu_i\nu_i}^{(a)}(x)^2)\Big). \tag{3}$$

More generally, we allow products of gauge-invariant polynomials in $F_{\mu\nu}^{(a)}$ smeared against test functions.

Definition 2.3 (Lattice Schwinger functions (formal; not constructed here))**.** Define formally

$$G_n^{(a)}(\varphi_1, \dots, \varphi_n) := \langle \mathcal{O}_a[\varphi_1, \dots, \varphi_n] \rangle_{\mu_{\beta(a),a}}. \tag{4}$$

Since $\mathcal{O}_a[\varphi_1, \dots, \varphi_n]$ is unbounded ($\|F_{\mu\nu}^{(a)}\|_\infty \sim a^{-2}$), we do *not* claim convergence of $G_n^{(a)}$ as $a \downarrow 0$ in this paper. All convergence results are restricted to bounded local observables $\mathcal{O} \in \mathfrak{A}_\ell$ (Definition 2.4).

2.3 Local observable algebra at fixed physical scale

Definition 2.4 (Bounded local gauge-invariant observables)**.** Fix a physical scale $\ell > 0$. Define $\mathfrak{A}_\ell$ to be the class of gauge-invariant lattice observables $\mathcal{O}_a$ that:

(a) are *bounded* measurable functions of finitely many link variables, with $\|\mathcal{O}_a\|_\infty \le 1$;

(b) are supported in physical regions of diameter $\le \ell$;

(c) are stable under products (with re-normalization $\mathcal{O}_1 \cdot \mathcal{O}_2 / \|\mathcal{O}_1 \cdot \mathcal{O}_2\|_\infty$) and convex combinations.

Typical elements include: normalized characters $\frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\,U_p$, traces of products of links around closed paths of diameter $\le \ell$, and smooth bounded functions thereof.

Remark 2.5 (Why bounded observables)*.* The smeared curvature monomials of Definition 2.2 involve $F_{\mu\nu}^{(a)}(x) \sim a^{-2}$(plaquette deviation), which satisfies $\|F^{(a)}\|_\infty \sim a^{-2}$ (by compactness of G). Uniform moment bounds along $a \downarrow 0$ for such observables would require nontrivial renormalization (vacuum subtraction, wave-function renormalization). By restricting $\mathfrak{A}_\ell$ to bounded observables with $\|\mathcal{O}\|_\infty \le 1$, we obtain uniform control without such subtractions. The smeared curvature monomials $\mathcal{O}_a[\varphi_1, \dots, \varphi_n]$ are *not* elements of $\mathfrak{A}_\ell$ under this definition; their continuum limits require additional renormalization deferred to future work.

Remark 2.6 (Two classes of observables)*.* The lattice Schwinger functions $G_n^{(a)}$ of Definition 2.3 involve the unbounded curvature fields $F_{\mu\nu}^{(a)}$. The observable class $\mathfrak{A}_\ell$ (Definition 2.4) restricts to bounded observables, for which uniform control is automatic. The interplay between these two classes is discussed in Remark 2.7.

Remark 2.7 (Relation to smeared curvature observables)*.* The Schwinger functions $G_n^{(a)}$ of Definition 2.3 involve unbounded observables. Layer 1 of the construction applies in two modes:

1. *Bounded mode:* For $\mathcal{O} \in \mathfrak{A}_\ell$ (e.g. Wilson loops, normalized plaquette traces), convergence follows from Assumption 3.5 alone, with no moment bound issues.
2. *Unbounded mode:* Smeared curvature monomials $\mathcal{O}_a[\varphi_1, \dots, \varphi_n]$ are *not* elements of $\mathfrak{A}_\ell$ (Remark 2.5). Their continuum limit requires additive and multiplicative renormalization beyond the scope of this paper. We do not claim convergence results for them here.

2.4 Renormalization trajectory

The bare coupling is tuned as

$$\beta(a) = \frac{2N_c}{g(a)^2}, \qquad g(a)^{-2} = 2b_0 \ln(a_0/a) + O(\ln\ln(a_0/a)), \tag{5}$$

with $b_0 = 11N_c/(48\pi^2)$ the one-loop coefficient, following the 2-loop asymptotic freedom formula. The reference scale a_0 is chosen so that $g(a_0) = g_0 \le \gamma_0$, where γ_0 is the master threshold of [E26IX] (Definition 2.8).

Definition 2.8 (Master threshold, recalled from [E26IX])**.** $\gamma_0 > 0$ is the largest value satisfying conditions (T1)–(T5) of [E26IX, Definition 4.11], ensuring simultaneous validity of: the Balaban one-step RG map, the coupling bootstrap, the small-field coercivity, the large-field suppression, and the weak-dependence criterion for DLR-LSI.

3 Layer 1: Existence and uniqueness of local fields

3.1 Tightness

Proposition 3.1 (Tightness for bounded observables)**.** *For every $\mathcal{O} \in \mathfrak{A}_\ell$ with $\|\mathcal{O}\|_\infty \le 1$, the expectation $\langle \mathcal{O} \rangle_a$ satisfies $|\langle \mathcal{O} \rangle_a| \le 1$ for all $a \in (0, a_0]$. More generally, for every $n \ge 1$ and bounded observables $\mathcal{O}_1, \dots, \mathcal{O}_n \in \mathfrak{A}_\ell$,*

$$|\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_a| \le \prod_{i=1}^{n} \|\mathcal{O}_i\|_\infty \qquad \forall\, a \in (0, a_0]. \tag{6}$$

Proof. Immediate from $|\mathcal{O}_i| \le \|\mathcal{O}_i\|_\infty$ and positivity of the measure. □

Remark 3.2 (Curvature monomials are not covered). The smeared curvature monomials $\mathcal{O}_a[\varphi_1, \dots, \varphi_n]$ of Definition 2.2 involve $F^{(a)}_{\mu\nu}(x) \sim a^{-2}$, so their L^∞ norm diverges as $a \to 0$. Uniform moment bounds for these observables would require nontrivial renormalization (vacuum subtraction). This is not carried out in the present paper.

Corollary 3.3 (Tightness and subsequential convergence)**.** *For bounded observables $\mathcal{O} \in \mathfrak{A}_\ell$, the family $\{\langle \mathcal{O} \rangle_a\}_{a \in (0,a_0]}$ is precompact in $\mathbb{R}$ (trivially, since $|\langle \mathcal{O} \rangle_a| \le 1$). For the $\mathcal{S}'$-valued random field Φ_a constructed from bounded densities (see Method 2 below), the family of laws $\{\nu_a\}$ is tight in $\mathcal{S}'(\mathbb{R}^4)$.*

Proof. **Method 1 (Direct, for bounded observable expectations).** For any fixed $\mathcal{O} \in \mathfrak{A}_\ell$, the family $\{\langle \mathcal{O} \rangle_a\}_{a \in (0,a_0]}$ lies in $[-1, 1]$ by Proposition 3.1, hence is trivially precompact in $\mathbb{R}$. More generally, for n-point functionals $F^{(a)}_n(\mathcal{O}_1, \dots, \mathcal{O}_n) := \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_a$ with $\mathcal{O}_i \in \mathfrak{A}_\ell$, the bound (6) gives $|F^{(a)}_n| \le \prod_i \|\mathcal{O}_i\|_\infty$ uniformly in a, so each such family is precompact.

Method 2 (Mitoma, for random-field construction). To construct a continuum *random field* (not just expectation values), one defines for each a the $\mathcal{S}'(\mathbb{R}^4)$-valued random variable

$$\Phi_a(\varphi) := a^4 \sum_{x \in \Lambda_a} \varphi(x)\, \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{p(x)}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^4), \tag{7}$$

where $U_{p(x)}$ is a plaquette variable (a bounded, gauge-invariant density with $\|\cdot\|_\infty \le 1$). Let ν_a be the law of Φ_a under $\mu_{\beta(a),a}$. Since $|\Phi_a(\varphi)| \le \|\varphi\|_{L^1}$ deterministically, the family $\{\Phi_a(\varphi)\}_a$ is uniformly bounded (hence tight) in $\mathbb{R}$ for every $\varphi \in \mathcal{S}$. Mitoma's theorem [11] then gives tightness of $\{\nu_a\}$ in $\mathcal{S}'(\mathbb{R}^4)$, hence subsequential weak convergence.

For the results of this paper (convergence of expectations of bounded observables), Method 1 suffices. Method 2 constructs a continuum *random field* from bounded densities; it is relevant for the stronger statement that the lattice measures have a weak limit as probability measures on $\mathcal{S}'$. □

Remark 3.4 (Mitoma requires random variables, not deterministic functionals). Mitoma's theorem [11] concerns tightness of *probability measures* on $\mathcal{S}'$—i.e., laws of $\mathcal{S}'$-valued random variables. It cannot be applied directly to a parametrized family of deterministic expectations. In Method 1, the precompactness of bounded observable expectations follows trivially from $|\langle \mathcal{O} \rangle_a| \le 1$. Method 2 shows how Mitoma applies when one upgrades to the random-field setting with bounded densities.

3.2 RG–Cauchy estimate

Assumption 3.5 (RG–Cauchy along the trajectory)**.** Along the chosen continuum trajectory $a \mapsto \beta(a)$, there exists a function $\varepsilon\colon (0, a_0] \to (0, \infty)$ with $\varepsilon(a) \downarrow 0$ and $\sum_{k=0}^{\infty} \varepsilon(a_0/2^k) < \infty$, such that for every $\ell > 0$ and every $\mathcal{O} \in \mathfrak{A}_\ell$ (bounded, gauge-invariant, supported in a physical region of diameter $\leq \ell$),

$$|\langle \mathcal{O} \rangle_a - \langle \mathcal{O} \rangle_{a/2}| \leq \varepsilon(a), \qquad \forall\, a \in (0, a_0], \tag{8}$$

where $\langle \cdot \rangle_a := \langle \cdot \rangle_{\beta(a),a}$ at fixed physical volume.

Remark 3.6 (Summability is genuinely stronger than the naive AF estimate)*.* Along a dyadic sequence $a_k = a_0 2^{-k}$, a naive asymptotic-freedom estimate typically gives one-step changes of size $|\langle \mathcal{O} \rangle_{a_{k+1}} - \langle \mathcal{O} \rangle_{a_k}| \lesssim g_k^2$, with $g_k^2 \sim c/k$ as $k \to \infty$. This is not summable in k, hence it does not suffice for a telescoping/Cauchy argument. Assumption 3.5 postulates an improved, summable control of one-step changes for bounded local observables.

Remark 3.7 (Examples of summable rates; status)*.* A sufficient condition for Assumption 3.5 is, for instance, $\varepsilon(a) \leq C(a/\ell)^\eta$ with some $\eta > 0$, or the log-improved rate $\varepsilon(a) \leq C/[\ln(a_0/a)]^{1+\delta}$ with $\delta > 0$, both of which are summable along $a_k = a_0 2^{-k}$. We do not derive such a rate in this paper; Assumption 3.5 is the single hard analytic input of Layer 1. Appendix B discusses why naive RG estimates lead to non-summable $O(g_k^2)$ terms and what additional structure would be needed to upgrade them to a summable bound.

3.3 Uniqueness of local Schwinger functions

Proposition 3.8 (Dyadic convergence of local expectations)**.** *Under Assumption 3.5, for each $\mathcal{O} \in \mathfrak{A}_\ell$ the dyadic limit*

$$\lim_{k\to\infty} \langle \mathcal{O} \rangle_{a_k,\beta(a_k)}, \qquad a_k := a_0/2^k, \tag{9}$$

exists.

Proof. Let $a_k := a_0/2^k$. For any $m > n \geq 0$, telescoping and Assumption 3.5 give

$$|\langle \mathcal{O} \rangle_{a_m} - \langle \mathcal{O} \rangle_{a_n}| \leq \sum_{j=n}^{m-1} \varepsilon(a_j). \tag{10}$$

Since $\sum_{j=0}^{\infty} \varepsilon(a_j) < \infty$ by (8), the sequence $\{\langle \mathcal{O} \rangle_{a_k}\}_{k\geq 0}$ is Cauchy in $\mathbb{R}$ and therefore convergent.

For a general sequence $a' \downarrow 0$, the convergence of the full-sequence limit requires controlling $|\langle \mathcal{O} \rangle_{a'} - \langle \mathcal{O} \rangle_{a_k}|$ for $a' \in [a_{k+1}, a_k]$, which is not provided by Assumption 3.5 (stated only for halving steps). We therefore state the main convergence result (Theorem 3.9) along the dyadic sequence. Extension to arbitrary sequences is deferred to future work. □

Theorem 3.9 (Existence and uniqueness of continuum local expectations (dyadic))**.** *Assume Assumption 3.5. Then for every $\ell > 0$ and every $\mathcal{O} \in \mathfrak{A}_\ell$, the limit along dyadic scales*

$$\langle \mathcal{O} \rangle := \lim_{k\to\infty} \langle \mathcal{O} \rangle_{a_k}, \qquad a_k := a_0/2^k, \tag{11}$$

exists and is unique (independent of subsequences of $\{a_k\}$). The limiting functional $\mathcal{O} \mapsto \langle \mathcal{O} \rangle$ extends uniquely by linearity to a positive normalized linear functional on $\mathcal{V}_{\mathrm{loc}} := \bigcup_{\ell>0} \mathrm{span}(\mathfrak{A}_\ell)$.

Proof. Fix $\mathcal{O} \in \mathfrak{A}_\ell$ and $a = a_0/2^n$. By telescoping,

$$|\langle \mathcal{O} \rangle_{a_0/2^n} - \langle \mathcal{O} \rangle_{a_0/2^m}| \leq \sum_{k=\min(m,n)}^{\max(m,n)-1} \varepsilon(a_0/2^k). \tag{12}$$

The summability condition in Assumption 3.5 ensures this is a Cauchy sequence. Since $|\langle \mathcal{O} \rangle_a| \leq 1$ (by $\|\mathcal{O}\|_\infty \leq 1$), the limit exists. Positivity and normalization are inherited from the lattice measures. □

Remark 3.10 (Extension to smeared curvature monomials)*.* The smeared curvature monomials $\mathcal{O}_a[\varphi_1, \ldots, \varphi_n]$ are *not* in $\mathfrak{A}_\ell$ (their L^∞ norm diverges as a^{-2n}). Extending Theorem 3.9 to these observables requires proving uniform moment bounds after vacuum subtraction (renormalization), which is beyond the scope of this paper.

4 Layer 2: Step-scaling confinement

4.1 Wilson loops and Creutz ratios at fixed physical scale

Definition 4.1 (Wilson loops in lattice and physical units)**.** For integers $I, J \in \mathbb{N}$ define the $I \times J$ rectangular Wilson loop (in lattice units) by

$$W_{I,J}^{(a)} := \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr} \prod_{e \in \mathcal{C}_{I,J}} U_e. \tag{13}$$

For physical dimensions $R, T > 0$, define

$$W_{R,T}^{(a)} := W_{\lfloor R/a \rfloor,\, \lfloor T/a \rfloor}^{(a)}. \tag{14}$$

Remark 4.2 (Resolution of the fixed-I, J error)*.* A previous draft incorrectly held I, J fixed as $a \to 0$, which would cause the physical area $R \cdot T = (Ia)(Ja) \to 0$, measuring only ultraviolet fluctuations. The correct procedure is to scale $I(a) \sim R/a$, $J(a) \sim T/a$ so that R and T remain constant.

Definition 4.3 (Creutz ratio (finite volume and infinite volume))**.** Let $a > 0$ and let $L > 0$ be a physical side length with $L/a \in \mathbb{N}$. Write $\langle \cdot \rangle_{a,L}$ for expectation with respect to the $\mathrm{SU}(N_c)$ lattice Yang–Mills measure on the four-torus of side length L at lattice spacing a (i.e. $N_a = L/a$ in each direction). For physical dimensions $R, T > 0$, set $\hat{R} = \lfloor R/a \rfloor$, $\hat{T} = \lfloor T/a \rfloor$ and define the finite-volume lattice Creutz ratio by

$$\chi_L^{(a)}(R,T) := -\ln \frac{\langle W_{\hat{R},\hat{T}}^{(a)} \rangle_{a,L} \, \langle W_{\hat{R}-1,\hat{T}-1}^{(a)} \rangle_{a,L}}{\langle W_{\hat{R}-1,\hat{T}}^{(a)} \rangle_{a,L} \, \langle W_{\hat{R},\hat{T}-1}^{(a)} \rangle_{a,L}}. \tag{15}$$

When it exists, the infinite-volume Creutz ratio is $\chi_\infty^{(a)}(R,T) := \lim_{L\to\infty} \chi_L^{(a)}(R,T)$.

4.2 Physical string tension

Assumption 4.4 (Thermodynamic limit for Creutz ratios (conditional))**.** For each lattice spacing a along the chosen trajectory and each fixed physical rectangle $R \times T$, the infinite-volume limit of the lattice Creutz ratio exists:

$$\chi_\infty^{(a)}(R,T) := \lim_{L\to\infty} \chi_L^{(a)}(R,T), \tag{16}$$

where $\chi_L^{(a)}(R,T)$ denotes the Creutz ratio computed on a four-torus of physical side length L (with $L/a \in \mathbb{N}$). Moreover, for each fixed $R, T > 0$, the dyadic continuum limit $\chi_\infty(R,T) := \lim_{k\to\infty} \chi_\infty^{(a_k)}(R,T)$ exists.

Definition 4.5 (Physical string tension (infinite volume; conditional))**.** Assume Assumption 4.4. For each fixed $R, T > 0$ with $R, T \gg \ell_0$, define the infinite-volume continuum Creutz ratio by

$$\chi_\infty(R,T) := \lim_{k\to\infty} \chi_\infty^{(a_k)}(R,T), \qquad a_k := a_0/2^k. \tag{17}$$

Then define the physical string tension by

$$\sigma_{\mathrm{phys}} := \liminf_{\substack{R,T\to\infty \\ R/T \text{ fixed}}} \chi_\infty(R,T). \tag{18}$$

Proposition 4.6 (Existence of the infinite-volume dyadic Creutz-ratio limit)**.** *Assume Assumption 4.4 and Assumption 4.7. Then for each fixed $R, T > 0$, the dyadic limit $\chi_\infty(R,T) = \lim_{k\to\infty} \chi_\infty^{(a_k)}(R,T)$ exists, where $a_k := a_0/2^k$.*

Proof. This is exactly the second part of Assumption 4.4, together with Assumption 4.7 ensuring that the logarithm in the definition of $\chi_\infty^{(a_k)}(R,T)$ is well-defined for all sufficiently large k. □

Assumption 4.7 (Nonvanishing Creutz denominators (thermodynamic; conditional))**.** For the rectangles used in (15) along the dyadic trajectory, and for all sufficiently large k, the thermodynamic limits

$$\lim_{L\to\infty}\langle W^{(a_k)}_{\hat{R}-1,\hat{T}}\rangle_{a_k,L} \quad \text{and} \quad \lim_{L\to\infty}\langle W^{(a_k)}_{\hat{R},\hat{T}-1}\rangle_{a_k,L} \tag{19}$$

exist and are nonzero.

Remark 4.8 (Mass gap does not imply confinement)*.* Exponential clustering (mass gap) implies that correlators of gauge-invariant *local* operators decay exponentially. It does *not* by itself imply an area law for Wilson loops: the mass gap controls local observables, while the string tension is an infrared property of extended (non-local) observables. In Abelian theories (compact QED in $d=3$), a mass gap can coexist with a perimeter law. We therefore formulate confinement as a conditional result relying on an explicit infrared positivity input stated in terms of Creutz ratios.

Assumption 4.9 (Uniform positive string tension input via Creutz ratios (infinite volume))**.** Assume Assumption 4.4. There exist $\sigma_0>0$ and $R_*,T_*>0$ such that for all sufficiently small a along the chosen trajectory and all physical rectangles $R\times T$ with $R\geq R_*$ and $T\geq T_*$, the infinite-volume lattice Creutz ratio satisfies

$$\chi_\infty^{(a)}(R,T)\geq\sigma_0. \tag{20}$$

Remark 4.10 (Status of Assumption 4.9)*.* Assumption 4.9 is a uniform infrared positivity input stated directly in terms of Creutz ratios at fixed physical scale. In strong coupling, area-law behavior and positivity of Creutz ratios are classical [15]. In weak coupling, the existence of a nonzero continuum string tension is supported by extensive Monte Carlo evidence (notably for SU(2) and SU(3)) and by large-N_c considerations, but a rigorous proof of a uniform lower bound of the form (20) along an asymptotic-freedom trajectory remains open.

Theorem 4.11 (Confinement (conditional))**.** *Assume Assumption 4.4 (thermodynamic/dyadic limits for Creutz ratios), Assumption 4.9 (uniform Creutz-ratio positivity input), and Assumption 4.7 (nonvanishing Creutz denominators). Then*

$$\sigma_{\text{phys}}>0. \tag{21}$$

Proof. The argument proceeds in two steps.

Step 1 (Existence of the infinite-volume continuum Creutz ratio). By Proposition 4.6, for each fixed $R,T>0$ the dyadic limit $\chi_\infty(R,T)=\lim_{k\to\infty}\chi_\infty^{(a_k)}(R,T)$ exists. This is conditional on the thermodynamic-limit input (Assumption 4.4) and on nonvanishing denominators (Assumption 4.7).

Step 2 (Positivity from the lattice input). By Assumption 4.9, for all sufficiently small a and all $R\geq R_*$, $T\geq T_*$ we have $\chi_\infty^{(a)}(R,T)\geq\sigma_0$. Passing to the dyadic limit $a=a_k\to 0$ yields $\chi_\infty(R,T)\geq\sigma_0$ for all such R,T. Taking the large-rectangle lim inf in Definition 4.5 gives

$$\sigma_{\text{phys}}=\liminf_{\substack{R,T\to\infty\\ R/T\ \text{fixed}}}\chi_\infty(R,T)\geq\sigma_0>0. \tag{22}$$

□

5 Reflection positivity and reconstruction (bounded observables)

The limiting state on bounded gauge-invariant observables obtained in Theorem 3.9 must satisfy reflection positivity to permit a Hilbert-space reconstruction. In this section we work at the level of the bounded observable algebra $\mathfrak{A}_\ell$, not with distribution-valued curvature fields. Accordingly, we verify reflection positivity and the associated reconstruction for bounded observables; OS axioms for curvature-field Schwinger functions $\{G_n\}$ are deferred to future work.

5.1 OS-Positivity

Lemma 5.1 (Osterwalder–Schrader positivity)**.** *For each $a > 0$, the lattice Yang–Mills measure satisfies reflection positivity with respect to any lattice hyperplane. If the expectations converge along dyadic scales $a_k \to 0$ (Theorem 3.9), then the limiting state inherits reflection positivity.*

Proof. **Step 1 (Lattice RP).** The Wilson action satisfies reflection positivity on the lattice [14]: for every lattice functional $\mathcal{A}$ supported on links in the half-lattice $\{x^0 > 0\}$,

$$\langle (\Theta\mathcal{A})^* \, \mathcal{A}\rangle_{\mu_{\beta(a),a}} \geq 0. \tag{23}$$

Step 2 (Bounded observables). For bounded observables $\mathcal{O} \in \mathfrak{A}_\ell$ supported in $\{x^0 > \delta\}$ for some $\delta > 0$, the product $(\Theta\mathcal{O})^* \cdot \mathcal{O}$ is itself a bounded gauge-invariant observable (with $\|(\Theta\mathcal{O})^* \cdot \mathcal{O}\|_\infty \leq 1$) supported in a region of diameter $\leq 2\ell$. The lattice RP inequality (23) gives $\langle (\Theta\mathcal{O})^* \, \mathcal{O}\rangle_a \geq 0$ for every a.

Step 3 (Inheritance for bounded observables). By Theorem 3.9, the expectations of bounded observables converge along dyadic scales: $\langle (\Theta\mathcal{O})^* \, \mathcal{O}\rangle_{a_k} \to \omega((\Theta\mathcal{O})^* \, \mathcal{O})$ as $k \to \infty$. Since $\langle (\Theta\mathcal{O})^* \, \mathcal{O}\rangle_{a_k} \geq 0$ for every k, and the limit of nonnegative real numbers is nonnegative: $\omega((\Theta\mathcal{O})^* \, \mathcal{O}) \geq 0$.

Step 4 (Extension to general bounded observables). For any finite collection of bounded gauge-invariant observables $\mathcal{O}_1, \ldots, \mathcal{O}_N \in \mathfrak{A}_\ell$ supported in $\{x^0 > 0\}$, the relevant bilinear form involves products that remain in $\mathfrak{A}_{2\ell}$. By Theorem 3.9, each lattice expectation converges, and the lattice inequality ≥ 0 (Step 1) passes to the limit.

□

Remark 5.2 (Subtlety for unbounded observables)*.* For smeared curvature monomials (which are not in $\mathfrak{A}_\ell$), OS-positivity of the continuum limit would require first establishing convergence of the relevant Schwinger functions after renormalization. Since we do not treat unbounded observables in this paper (Remark 2.5), the OS-positivity result above applies only to bounded observables in $\mathfrak{A}_\ell$.

5.2 Euclidean covariance: $SO(4)$ rotational invariance

Proposition 5.3 ($SO(4)$ invariance (conditional))**.** *The lattice action is hypercubic-invariant, and $SO(4)$-breaking operators in the effective action have engineering dimension ≥ 6, hence are suppressed by $O(a^2)$. Under the standard operator-classification input (Remark 5.4), no relevant $SO(4)$-breaking operators are generated, and full $SO(4)$ covariance is expected to hold in the infinite-volume continuum limit for any renormalized Schwinger functions $\{G_n\}$ once constructed. At the level of the bounded observable algebra $\mathfrak{A}_\ell$, hypercubic symmetry is exact on the lattice and inherited by the dyadic limit.*

Proof. **Step 1 (Symmetry on the lattice).** The Wilson action is invariant under the hypercubic group $\mathfrak{S}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^4 \subset SO(4)$, but not under the full $SO(4)$. The breaking is by irrelevant operators of engineering dimension ≥ 6; the leading $SO(4)$-breaking term in the Symanzik effective action is of order $O(a^2)$.

Step 2 (Status and scope). At the level of $\mathfrak{A}_\ell$, only hypercubic symmetry is asserted; it is exact at each a and inherited by the dyadic limit. Promoting this to full $SO(4)$ covariance for renormalized curvature Schwinger functions requires (i) the operator-classification input ruling out relevant $SO(4)$-breaking terms (Remark 5.4) and (ii) a construction of renormalized G_n, which is deferred to future work.

Step 3 (Conditional conclusion). Full $SO(4)$ invariance of renormalized curvature Schwinger functions $\{G_n\}$ would follow from a quantitative $O(a^2)$ bound on the $SO(4)$-breaking contribution (as outlined in Step 1) combined with the construction of G_n; both are deferred to future work. For the bounded observable algebra $\mathfrak{A}_\ell$, hypercubic invariance is exact at each a and is inherited by the dyadic limit. □

Remark 5.4 (Precise citation needed). The uniform bound in Step 2 requires that the RG does not generate relevant ($d_{\text{op}} \leq 4$) $SO(4)$-breaking operators through mixing. In pure Yang–Mills with the Wilson action, this is excluded by gauge invariance and the discrete symmetries of the hypercubic lattice, which forbid all dimension-4 operators not already present in the continuum action. The detailed argument uses the classification of gauge-invariant operators under the hypercubic group; see [17] and Papers 75–76 for the multiscale control.

5.3 Translation invariance and cluster property

In finite volume (a four-torus of side L), translation invariance holds with respect to the torus translations and is inherited by the dyadic continuum limit. Statements involving $|R| \to \infty$ (and hence the cluster property in $\mathbb{R}^4$) require an infinite-volume limit. We do not prove the thermodynamic limit in this paper; accordingly, the cluster property

$$\omega(\mathcal{O}_1\, \tau_R \mathcal{O}_2) \xrightarrow{|R|\to\infty} \omega(\mathcal{O}_1) \cdot \omega(\mathcal{O}_2) \tag{24}$$

is treated as a conditional property of the infinite-volume continuum theory, following from exponential clustering (Lemma 6.1) under Assumption A.2.

5.4 Reconstruction

Assumption 5.5 (Strong continuity of Euclidean time translations)**.** In the OS reconstruction associated to the limiting state ω on bounded observables, the Euclidean time-translation semigroup $(T(t))_{t\geq 0}$ is strongly continuous on the reconstructed Hilbert space $\mathcal{H}$.

Corollary 5.6 (Hilbert-space reconstruction from reflection positivity)**.** *Assume the limiting functional $\omega(\mathcal{O}) := \langle \mathcal{O} \rangle$ on bounded gauge-invariant observables is translation invariant and reflection positive (Lemma 5.1). Let $\mathfrak{A}_+$ be the subalgebra of observables supported in $\{x^0 > 0\}$. Define*

$$\langle A, B \rangle_{\text{RP}} := \omega\big((\Theta A)^* B\big), \qquad A, B \in \mathfrak{A}_+. \tag{25}$$

Then:

1. *$\langle \cdot, \cdot \rangle_{\text{RP}}$ is positive semidefinite; quotienting null vectors and completing yields a separable Hilbert space $\mathcal{H}$ with cyclic vacuum $\Omega = [\mathbf{1}]$.*
2. *Euclidean time translations induce a contraction semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$.*
3. *Under Assumption 5.5, there exists a self-adjoint $H \geq 0$ with $T(t) = e^{-tH}$ and $H\Omega = 0$.*

This follows from the general reflection-positivity reconstruction [12, 13, 8] applied to the bounded observable algebra.

6 Mass gap

6.1 Uniform exponential clustering

Lemma 6.1 (Uniform exponential clustering)**.** *There exists* $m_0 > 0$ *such that for all* $a \in (0, a_0]$ *and all gauge-invariant local observables* $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{A}_\ell$ *supported in regions separated by physical distance* d*:*

$$\left|\langle \mathcal{O}_1\, \mathcal{O}_2\rangle_a - \langle \mathcal{O}_1\rangle_a \langle \mathcal{O}_2\rangle_a\right| \le C_{\mathcal{O}_1,\mathcal{O}_2}\, e^{-m_0\, d}. \tag{26}$$

Proof sketch. This follows from Assumption A.2 (the uniform physical transfer-matrix gap). For each fixed spatial size L and all sufficiently small a, the assumption gives $m_{\text{phys}}(a, L) \ge m_0 > 0$. By the spectral expansion of time-separated connected correlators in eigenvalues of $\widehat{T}_a$, this implies $|\langle \mathcal{O}_1\mathcal{O}_2\rangle_a - \langle \mathcal{O}_1\rangle_a\langle \mathcal{O}_2\rangle_a| \le C\, e^{-m_0 d}$ for observables separated by Euclidean time distance d. Extension to general Euclidean separation uses a standard chessboard / finite-propagation estimate [8]. The detailed argument is in Appendix A. □

6.2 From clustering to mass gap

Theorem 6.2 (Mass gap (conditional))**.** *Assume Assumption 3.5 (RG–Cauchy), Assumption A.2 (uniform transfer-matrix gap), and Assumption 5.5 (strong continuity). Then the Hamiltonian* H *of the reconstructed quantum theory satisfies*

$$\inf\big(\operatorname{spec}(H) \setminus \{0\}\big) \ge m_0 > 0. \tag{27}$$

Proof. **Step 1 (Continuum clustering for bounded observables).** By Lemma 6.1, for any $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{A}_\ell$ supported at physical separation d: $|\langle \mathcal{O}_1\mathcal{O}_2\rangle_a - \langle \mathcal{O}_1\rangle_a\langle \mathcal{O}_2\rangle_a| \le C\, e^{-m_0 d}$ uniformly in a. By Theorem 3.9, the expectations of bounded observables converge along dyadic scales, and the exponential bound is preserved under pointwise limits:

$$\left|\omega(\mathcal{O}_1\mathcal{O}_2) - \omega(\mathcal{O}_1)\,\omega(\mathcal{O}_2)\right| \le C\, e^{-m_0 d}. \tag{28}$$

Step 2 (Spectral gap from exponential decay). By Corollary 5.6 and Assumption 5.5, the limiting state reconstructs a Hilbert space $\mathcal{H}$ with vacuum Ω and a self-adjoint $H \ge 0$ with $T(t) = e^{-tH}$. Let $A \in \mathfrak{A}_+$ with $\omega(A) = 0$. The exponential clustering (28) applied to time-translated copies gives

$$|\langle [A],\, T(t)[A]\rangle| \le C_A\, e^{-m_0 t}, \qquad t \ge 0. \tag{29}$$

By the spectral theorem, $t \mapsto \langle [A], e^{-tH}[A]\rangle$ is the Laplace transform of a positive spectral measure on $\operatorname{spec}(H)$. The decay (29) forces $\operatorname{supp}(\rho) \subset [m_0, \infty)$, hence $\operatorname{spec}(H) \cap (0, m_0) = \emptyset$. □

7 Nontriviality

Theorem 7.1 (Nontriviality (conditional))**.** *Under the hypotheses of Theorem 4.11 and the area-law implication stated in Step 1 of the proof below, the continuum theory is not a generalised free field.*

Proof. **Step 1 (Area-law input).** By Theorem 4.11, the physical string tension satisfies $\sigma_{\text{phys}} > 0$ (defined via Creutz ratios). We assume that $\sigma_{\text{phys}} > 0$ implies the quantitative area law: for large planar Wilson loops of area $\mathcal{A}$,

$$-\ln\langle W(\mathcal{C})\rangle \;\ge\; \sigma_{\text{phys}} \cdot \mathcal{A} \;-\; C \cdot \text{Perimeter}(\mathcal{C}). \tag{30}$$

This implication (from positive Creutz-ratio limit to a lower bound with perimeter correction) is standard in the lattice literature but is not proved in this paper; we treat it as a conditional input.

Step 2 (Free field obeys perimeter law). In any generalised free field theory, the connected n-point functions are determined by the 2-point function. For a non-abelian gauge theory, a "free" theory means the path integral is Gaussian in the Lie-algebra-valued field A_μ. In such a theory:

- The Wilson loop $W(\mathcal{C}) = \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\,\mathcal{P}\exp(ig\oint_{\mathcal{C}} A)$ has expectation $\langle W(\mathcal{C})\rangle = \exp(-\frac{g^2}{2}\oint_{\mathcal{C}}\oint_{\mathcal{C}} G_{\mu\nu}(x,y)\,dx^\mu\,dy^\nu$ $\cdots)$ where $G_{\mu\nu}$ is the free propagator.
- For a massless or massive free propagator in $d = 4$, the double line integral grows at most as the perimeter: $|\oint\oint G| \le C' \cdot \mathrm{Perimeter}(\mathcal{C})$.
- Hence $-\ln\langle W(\mathcal{C})\rangle \le C'' \cdot \mathrm{Perimeter}(\mathcal{C})$.

Step 3 (Contradiction). For loops with $\mathcal{A}/\mathrm{Perimeter} \to \infty$, the area law (30) is incompatible with the perimeter law. Hence the theory is not free. □

8 Discussion

8.1 Honest assessment of the logical structure

We summarize the logical status of each component:

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Result	Mechanism	Status
Existence of continuum expectations	Telescoping + summable RG–Cauchy	**Conditional** on Assumption 3.5
Tightness in $\mathcal{S}'$	Equicontinuity + Banach–Alaoglu (deterministic); Mitoma (random field, bounded densities)	**Proved** for bounded observables ($\|\mathcal{O}\|_\infty \le 1$, Proposition 3.1, Corollary 3.3); for unbounded curvature monomials, requires renormalization (not treated here)
OS-Positivity	Lattice RP + pointwise convergence	**Proved** (Lemma 5.1)
$SO(4)$ invariance	Irrelevant hypercubic breaking $\Rightarrow O(a^2) \to 0$	**Conditional:** hypercubic symmetry proved for $\mathfrak{A}_\ell$; full $SO(4)$ requires renormalized G_n (deferred)
Mass gap	Transfer-matrix gap + OS reconstruction	**Conditional** on Assumptions A.2 and 5.5
$\sigma_{\text{phys}} > 0$ (confinement)	Creutz-ratio convergence + lattice positivity input	**Conditional** on Theorems 4.4, 4.7 and 4.9
Nontriviality	Area law vs. perimeter law	**Conditional** on confinement

*Assumption 3.5 is mappable to the multiscale outputs of Papers 75–76 (see Appendix B), but the mapping requires verifying that the polymer bounds control expectations of *all* local observables uniformly across infinitely many RG scales. This is the single hardest analytical step.

**The absence of relevant $SO(4)$-breaking operators is guaranteed by the symmetry classification of gauge-invariant operators under the hypercubic group, combined with the uniform bounds of the multiscale RG. The detailed argument uses standard results [17] but must be verified within the Balaban framework.

8.2 What this paper does not do

1. We do not prove the RG–Cauchy estimate (Assumption 3.5) from first principles. It is motivated by the Balaban multiscale program but not formally derived from it. As discussed in Remark 3.6 and Appendix B (Remark B.1), the naive asymptotic-freedom bound gives a non-summable $O(1/k)$ rate; summability requires an improved rate that we do not establish.
2. The transfer-matrix spectral gap (Assumption A.2) is postulated, not derived. The LSI results of Papers 68–76 control a Glauber/Langevin dynamics gap, which is a *different*

operator (Remark A.1). Connecting the two requires additional arguments not provided here.

3. The uniform infrared positivity input for Creutz ratios (Assumption 4.9) is postulated, not derived. We do not derive area-law behavior from the mass gap (Remark 4.8).
4. We do not provide quantitative lower bounds on m_0 or σ_{phys} beyond strict positivity.
5. We work exclusively with bounded local observables (Definition 2.4). The construction of continuum field-strength correlators $\langle F_{\mu\nu}(x)F_{\rho\sigma}(y)\rangle$ requires renormalization (vacuum subtraction, wave-function renormalization) that is deferred to future work.
6. We do not extend the construction to include matter fields.

8.3 Comparison with previous work

The continuum limit program for Yang–Mills has a long history. Balaban's multiscale analysis [1, 2, 3, 4, 5, 6] provides the essential control of the ultraviolet regime. The present paper builds on this by:

- using the uniform functional inequalities of [E26I]–[E26IX] to obtain volume-independent bounds;
- formulating the continuum limit as a tightness + uniqueness problem in $\mathcal{S}'$, avoiding reliance on specific non-local observables;
- using step-scaling Creutz ratios to capture confinement without the pitfall of fixed lattice loop dimensions.

A Transfer matrix and uniform clustering

This appendix provides the detailed argument for Lemma 6.1.

A.1 Setup

Consider the lattice $\Lambda_a = a(\mathbb{Z}/N_a\mathbb{Z})^3 \times a(\mathbb{Z}/M_a\mathbb{Z})$ with spatial volume $L^3 = (N_a\, a)^3$ and temporal extent $T_{\text{phys}} = M_a\, a$. The transfer matrix $\widehat{T}_a$ acts on $L^2(\mu_{\beta(a),\Sigma_a})$, where $\Sigma_a = a(\mathbb{Z}/N_a\mathbb{Z})^3$ is a spatial slice and $\mu_{\beta(a),\Sigma_a}$ is the spatial lattice measure.

A.2 Mass gap: transfer-matrix spectral gap

Remark A.1 (LSI vs. transfer-matrix gap). The log-Sobolev inequality (LSI) of [E26VII,E26IX] controls the spectral gap of a *Glauber-type Markov dynamics* associated to μ_a. This is a different operator from the *transfer matrix* $\widehat{T}_a$, whose spectral gap governs the exponential decay of Euclidean correlators in the time direction. In general, an LSI bound does not imply a transfer-matrix gap bound (or vice versa). We therefore formulate the transfer-matrix gap as a separate assumption.

Assumption A.2 (Uniform physical transfer-matrix gap along the trajectory)**.** There exists $m_0 > 0$ such that for every fixed physical spatial size $L > 0$ and all sufficiently small a (with $L/a \in \mathbb{N}$), the transfer matrix $\widehat{T}_a$ acting on gauge-invariant states has two largest eigenvalues $\lambda_0(a, L) > \lambda_1(a, L) \geq 0$ satisfying

$$m_{\text{phys}}(a, L) := -\frac{1}{a}\ln\Big(\frac{\lambda_1(a, L)}{\lambda_0(a, L)}\Big) \geq m_0. \tag{31}$$

Lemma A.3 (Uniform exponential clustering—from transfer-matrix gap)**.** *Assume Assumption A.2. Then for all $a \in (0, a_0]$ and all $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{A}_\ell$ with supports separated by physical Euclidean-time distance d,*

$$\big|\langle \mathcal{O}_1\, \mathcal{O}_2\rangle_a - \langle \mathcal{O}_1\rangle_a \langle \mathcal{O}_2\rangle_a\big| \leq C\, e^{-m_0\, d}. \tag{32}$$

Proof sketch. By reflection positivity, connected time-separated correlators admit a spectral expansion in eigenvalues of $\widehat{T}_a$. The gap (31) bounds the leading sub-dominant contribution by $e^{-m_0 d}$. Extension from pure time-separation to general spatial separation uses a standard chessboard / finite-propagation estimate [8]. □

Relation to transfer-matrix gap. The exponential decay rate $\Delta_a(L)$ in (32) is precisely the spectral gap of the transfer matrix $\widehat{T}_a = e^{-aH_a}$ (i.e. $\Delta_a(L) = -\ln(\lambda_1/\lambda_0)$ where $\lambda_0 > \lambda_1$ are the two largest eigenvalues). This identification is standard; see [8], Chapter 6.

A.3 Conversion to physical units

In physical distance $d = \tau \cdot a$, the decay rate is

$$e^{-\Delta_a(L)\,\tau} = e^{-(\Delta_a(L)/a)\,d} = e^{-m_{\text{phys}}(a,L)\,d} \tag{33}$$

with

$$m_{\text{phys}}(a, L) = \frac{\Delta_a(L)}{a}. \tag{34}$$

A.4 Uniform lower bound on m_{phys}

The Stroock–Zegarlinski theory and DLR-LSI control mixing properties for a Glauber-type dynamics associated to μ_a. This yields concentration and a Markov-generator spectral gap, but it does not directly identify the transfer-matrix gap (Remark A.1), nor does it provide a physical mass-gap bound uniform in a along an LCP. For this reason, in the present paper the transfer-matrix mass gap is taken as an explicit input (Assumption A.2).

Remark A.4 (Finite-volume spectral gap and the role of Assumption A.2)*.* For each fixed $a > 0$ and finite spatial size L, the transfer matrix $\widehat{T}_a$ can be realized as a compact positivity-improving integral operator on a compact configuration space, hence it has an isolated top eigenvalue and a strictly positive finite-volume gap; see, e.g., [14, 8]. The content of Assumption A.2 is the *uniform* lower bound $m_{\text{phys}}(a, L) \geq m_0 > 0$ along the continuum trajectory, which is the input used to deduce uniform exponential clustering and, via OS reconstruction, the mass gap in Theorem 6.2.

Remark A.5 (No circularity)*.* This argument does *not* invoke confinement (Theorem 4.11). The mass gap and confinement are logically independent conclusions, each conditional on its own assumption (Assumption A.2 and Assumption 4.9, respectively).

Remark A.6 (LSI concentration vs. spatial clustering vs. transfer-matrix gap)*.* The DLR-LSI constant α_{DLR} of [E26VII] controls the spectral gap of a Glauber-type Markov dynamics associated to μ_a, uniformly in boundary conditions and volumes. Via the Herbst argument [7], this implies sub-Gaussian concentration for suitable (Lipschitz) observables around their mean. This is not, by itself, a statement about exponential decay of spatial connected correlators between two local observables at separation d, and it does not identify the transfer-matrix gap governing Euclidean-time decay. Accordingly, Assumption A.2 is treated as an independent input in this paper.

Remark A.7 (Heuristic: m_{phys} vs. σ_{phys} (not used))*.* On the lattice, a bound of the form $m_{\text{phys}} \geq c\sqrt{\sigma_{\text{phys}}}$ is known [15]. This provides physical motivation for expecting both a mass gap and a positive string tension, but we do *not* use this bound in any proof. In particular, we do not claim that it transfers to the continuum under the Layer 1 framework of bounded observables; such a transfer would require the construction of renormalized field-strength correlators.

B Bridge lemma: mapping Assumption 3.5 to Papers 75–76

This appendix outlines how Assumption 3.5 (the RG–Cauchy estimate for local observables) maps to the multiscale outputs of the Balaban program as used in Papers 75–76 ([E26VIII]–[E26IX]).

B.1 The one-step difference

Consider a local observable $\mathcal{O} \in \mathfrak{A}_\ell$ at physical scale ℓ. The expectation $\langle\mathcal{O}\rangle_a$ at lattice spacing a differs from $\langle\mathcal{O}\rangle_{a/2}$ because the passage $a \to a/2$ integrates out one shell of fluctuations (one RG step in the multiscale decomposition).

In Balaban's framework, the effective action after k RG steps has the form

$$S_k^{\text{eff}} = \beta_k A(U_k) + S_k^{\text{loc,irr}} + \sum_X \mathbf{R}^{(k)}(X) + \sum_X \mathbf{B}^{(k)}(X). \tag{35}$$

The one-step change in a local expectation arises from:

(i) the shift $\beta_k \to \beta_{k+1}$ (running coupling);

(ii) the change in polymer activities $\mathbf{R}^{(k)}(X) \to \mathbf{R}^{(k+1)}(X)$;

(iii) the change in boundary terms $\mathbf{B}^{(k)}(X) \to \mathbf{B}^{(k+1)}(X)$.

B.2 Bounding each contribution

Running coupling. By [E26IX], Theorem 4.3, $|g_{k+1}^{-2} - g_k^{-2} - 2b_0 \ln 2| \le C_{\text{run}} g_k^2$. The shift in β_k contributes to local expectations a change of order g_k^2 times the observable, which for a fixed observable at physical scale ℓ is bounded by $C_\mathcal{O} \cdot g_k^2$.

Polymer activities. By condition (IC4): $\sum_{X \ni \text{support of } \mathcal{O}} |\mathbf{R}^{(k)}(X)| \le C_R\, e^{-p_0(g_k)}$. The change from step k to $k+1$ of the polymer contribution to $\langle\mathcal{O}\rangle$ is controlled by the polymer norm, giving a bound $C_\mathcal{O}\, e^{-c\, p_0(g_k)}$.

Boundary terms. By condition (IC5): $\sum_{X \ni \text{support}} |\mathbf{B}^{(k)}(X)| \le C_B(k+1)$, but the one-step *change* is bounded by the same polymer-norm mechanism, giving $C_\mathcal{O}\, e^{-c'\, p_0(g_k)}$.

B.3 Summability over scales

The total one-step change is therefore

$$|\langle\mathcal{O}\rangle_{a_k} - \langle\mathcal{O}\rangle_{a_{k+1}}| \le C_\mathcal{O}\big(g_k^2 + e^{-c\, p_0(g_k)}\big) \le C'_\mathcal{O}\, g_k^2. \tag{36}$$

Since $g_k^{-2} = g_0^{-2} + 2b_0 k \ln 2 + O(1)$, we have $g_k^2 \le C/(g_0^{-2} + 2b_0 k)$, and

$$\sum_{k=n}^{\infty} g_k^2 \le C \sum_{k=n}^{\infty} \frac{1}{g_0^{-2} + 2b_0 k} \sim \frac{C'}{2b_0} \ln\Big(\frac{g_0^{-2} + 2b_0\, n_{\max}}{g_0^{-2} + 2b_0\, n}\Big) \xrightarrow{n_{\max}\to\infty} \infty. \tag{37}$$

A more careful bound: since $g_k^2 \sim \frac{1}{2b_0 k \ln 2}$ for large k, the sum $\sum_{k\ge n} g_k^2 \sim \frac{1}{2b_0 \ln 2} \sum_{k \ge n} 1/k$ diverges logarithmically. This means the naive telescoping gives logarithmic rather than power-law convergence in n.

Remark B.1 (Logarithmic divergence of the naive bridge bound (bridge version)). In $d = 4$, along an asymptotically free trajectory one typically has $g_k^2 \sim \frac{c}{k}$ as $k \to \infty$. Consequently, the naive one-step bound $|\langle\mathcal{O}\rangle_{a_k} - \langle\mathcal{O}\rangle_{a_{k+1}}| \lesssim g_k^2$ yields, for partial telescoping up to a finite scale N,

$$\sum_{k=n}^{N} g_k^2 \sim c\,(\ln N - \ln n), \tag{38}$$

which diverges as $N \to \infty$. Hence this naive bridge estimate is *not summable* and cannot by itself imply an RG–Cauchy estimate of the form postulated in Assumption 3.5. Establishing summability requires additional structure beyond the $O(g_k^2)$ control (e.g. cancellations in the one-step difference, or a genuinely improved nonperturbative decay rate).

Remark B.2 (Strengthening to power law). To obtain the power-law bound $(a/\ell)^\eta$ of Assumption 3.5, one must exploit the *large-field penalty* $e^{-p_0(g_k)}$ rather than the perturbative g_k^2 bound. Since $p_0(g_k) \geq c_0|\log g_k|^{1+\epsilon_0} \geq c_0'(\ln k)^{1+\epsilon_0}$ for large k, the sum $\sum_{k\geq n} e^{-p_0(g_k)}$ converges superpolynomially fast, giving $\eta > 0$. The perturbative contribution g_k^2 gives only logarithmic convergence. Whether this term cancels in the Cauchy difference for a suitably chosen trajectory $\beta(a)$ is an open question that we do not resolve here.

A complete verification would require a detailed analysis of the cancellation structure in the one-step RG map, which we defer to future work.

C Notational concordance with [E26IX]

This paper	**[E26IX]**	**Meaning**
$\beta(a)$	β	bare inverse coupling
$g(a)$	g_0	bare coupling
γ_0	γ_0	master threshold
α_*	α_*	LSI constant
α_{DLR}	α_{DLR}	DLR-LSI constant
$\Delta_{\mathrm{TM}}(a, L) := -\ln(\lambda_1/\lambda_0)$	—	transfer-matrix gap (dimensionless, in lattice units)
$m_{\mathrm{phys}}(a, L) := \Delta_{\mathrm{TM}}(a, L)/a$	—	physical mass gap (energy units)
$m_{\mathrm{phys}}(a)$	Δ_{phys}/a	physical mass
$G_n^{(a)}$	—	lattice Schwinger functions
G_n	—	continuum Schwinger functions
σ_{phys}	—	physical string tension

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