Integrated cross-scale derivative bounds for Wilson lattice gauge theory: closing the log-Sobolev gap — a conditional and windowed closure*

Lluis Eriksson
Independent Researcher
lluiseriksson@gmail.com

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Abstract

We prove integrated cross-scale derivative bounds that replace the unverified Assumption 5.4 of [6]. Combined with two explicit large-field inputs (Hypotheses 3.2 and 4.2) and the conditional inequalities of [7], this yields — under the audited hypotheses listed below and within the stated volume window — the corresponding log-Sobolev assembly for the Wilson lattice gauge measure at sufficiently weak coupling, with constant independent of L_{vol} inside the window. The key decomposition into small-field and large-field contributions survives verbatim from v1, as do the sweeping-out modification (an L^1 bound in place of an essential supremum, and a shifted essential supremum over $\mathcal{G}_{k+1}$), the Rothaus closure, the $SU(2)$, $d = 2$ toy-model analysis, and the correction $\lambda_1 \geq \alpha_*$ to [6], Proposition 6.1(2). Version 2 corrects the logical status of the assembly after the quantitative audit of the series (ai.viXra:2602.0051–0056, all v2). (a) The Absorption step in the proof of Theorem 1.1 relied on the premises "$p_0(g) \to \infty$ as $g \to 0$ along the flow" and "if β_k grows sufficiently with k": both use the inverted-sign flow of the series erratum and are withdrawn; with the correct asymptotic-freedom flow all statements hold in the window $k \leq k^*(\beta)$, i.e. $L_{\mathrm{vol}} \leq e^{C/g^2+O(1)}$. (b) A new finding (F-ABS): the absorption condition (16) consumes Hypothesis 4.2 in the strong exponent form $e^{-c\beta_k\varepsilon_k^2}$, but the companion verification (2602.0056 v2) delivers only $e^{-c_{\mathrm{sf}}p_0(g_k)}$ with $c_{\mathrm{sf}} = 2/(1+\beta_0)$ — bounded along the window — so (16) fails at every scale $k \geq 2$ with the Balaban-compatible thresholds (9), even under (H-P0). The strong form is therefore an additional explicit hypothesis (H-ABS), and its saturated variant $\varepsilon_k \equiv \varepsilon_*$ opens a second window $k \leq k_{\mathrm{abs}} \propto \beta\varepsilon_*^2$. (c) The inputs are retagged per their v2 verifications: Hypothesis 3.2 is windowed and conditional on Balaban's small-field inputs (2602.0055 v2); Hypothesis 4.2 is form-level and conditional on (H-SFI)+(H-P0) (2602.0056 v2); the fiber LSI of [7] consumed by Corollary 1.2 inherits (H-YGZ) (2602.0053 v2). (d) Reference hygiene: v1's reference [8] was a self-citation of the present paper and is removed; the Wilson duplicate is removed; the audited chain 2602.0051–0056 (v2) is cited. What survives and is validated in the companion suite: the per-direction Wilson bound (Lemma 3.1), the energy–distance

*Version 1 of this paper carried the title "Integrated cross-scale derivative bounds for Wilson lattice gauge theory: closing the log-Sobolev gap" and presented Theorem 1.1 and Corollary 1.2 as closing the gap unconditionally, citing the companions [9], [10] as verifications of Hypotheses 4.2 and 3.2. Those citations referred to the v1 companions, whose unconditional claims have since been withdrawn (ai.viXra:2602.0055/0056, v2); the present version restates all results with their true hypothesis load and volume window. See the Note added in v2 (Section 8).

identity (13), the single-plaquette tail (Proposition 7.1, Table 1 reproduced digit-by-digit), the honest impossibility Remark 7.2, the factorization step (21), the Rothaus closure, and Appendix A's $\lambda_1 \geq \alpha_*$.

Contents

1 Introduction

1.1 Context and motivation

The construction of a rigorous non-perturbative framework for Yang–Mills theories on the lattice has been a central goal of mathematical physics since the foundational work of Wilson [12] and the analytical program of Balaban [1,2,3,4]. A key milestone is the establishment of uniform functional inequalities — in particular, a log-Sobolev inequality (LSI) with a constant independent of the lattice volume — for the Wilson lattice gauge measure. In the companion paper [6] we developed a multiscale framework reducing a uniform LSI to two inputs: (i) conditional functional inequalities on each RG fiber, from Ricci curvature lower bounds ([7]); (ii) cross-scale derivative bounds (Assumption 5.4 of [6]). **(v2)** Per the series audit, the *quantitative use* of (i) inside the multiscale assembly inherits (H-YGZ) (2602.0053 v2): the Ricci bound itself is unconditional, but the fiber LSI constant degrades by a Holley–Stroock factor $e^{-2\beta n_{\mathrm{plaq}}}$ once the conditional Wilson interaction on the fiber is accounted for. v1's sentence "the conditional inequalities (i) are fully established in [7]" is accordingly weakened.

1.2 The obstruction and its resolution

Assumption 5.4 of [6] posits that

$$\operatorname*{ess\,sup}_{\mathcal{G}_k} E\big[\,|\nabla_{\mathcal{E}_k} V_{<k}|^2 \,\big|\, \mathcal{G}_k\big] \;\le\; D_k, \qquad \sum_k D_k < \infty, \tag{1}$$

uniformly in the lattice volume. The difficulty is that the essential supremum over $\mathcal{G}_k$ requires pointwise control for every slow configuration, including the large-field region where Balaban's polymer expansion gives no analytic control. Our resolution (unchanged from v1): (1) the sweeping-out argument needs only the L^1 bound for the defective LSI and a *shifted* essential supremum over $\mathcal{G}_{k+1}$ for the Poincaré inequality; (2) both are proved by decomposing into small-field (polymer expansion) and large-field (pointwise bound $\times$ measure suppression) contributions.

1.3 Main results

Theorem 1.1 (Integrated cross-scale derivative bounds; v2: windowed, conditional)**.** *Let μ_β denote the Wilson lattice gauge measure on a finite lattice Λ with gauge group $G = SU(N_c)$, and let $\{\mathcal{E}_k, \mathcal{G}_k\}$ be the multiscale decomposition of [6]. Assume Hypothesis 3.2 (windowed; verified conditionally on Balaban's small-field inputs (B1)–(B4) in [10]), Hypothesis 4.2 (form-level; verified conditionally on (H-SFI)+(H-P0) in [9]), and the absorption hypothesis* (H-ABS) *of §1.5. Then for β sufficiently large and all scales $k \le \min\{k^*(\beta), k_{\mathrm{abs}}(\beta)\}$ (see §1.5), and any unit vector $v \in \mathcal{E}_k$ supported in a single block:*

(a) (L^1 bound)

$$E_{\mu_\beta}\big[|vV_{<k}|^2\big] \;\le\; D_k \;:=\; 2C_{\mathrm{SF}}^2\, L_{\mathrm{RG}}^{-(d-1)k}; \tag{2}$$

(b) (Shifted essential supremum)

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} E\big[\,|vV_{<k}|^2 \,\big|\, \mathcal{G}_{k+1}\big] \;\le\; D_k. \tag{3}$$

In both cases $\sum_k D_k < \infty$ independently of L_{vol} (the sum running over the windowed scales). (v1 asserted (a)–(b) for all k; that assertion relied on the inverted flow and on the strong exponent form of Hypothesis 4.2, and is replaced by the present windowed, conditional statement.)

Corollary 1.2 (Uniform log-Sobolev inequality; v2: windowed, conditional)**.** *Under the hypotheses of Theorem 1.1* and additionally (H-YGZ) *(for the quantitative fiber LSI constant $\hat{\alpha}_0 > 0$ consumed from [7]), the Wilson measure μ_β satisfies*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \;\leq\; \frac{2}{\alpha_*}\,\mathcal{E}(f,f) \qquad \textit{for all Lipschitz } f,$$

with $\alpha_* > 0$ independent of L_{vol} within the window $L_{\text{vol}} \leq e^{C/g^2+O(1)}$.

1.4 Organization

Section 2 recalls the framework. Section 3 gives the per-direction bound and Hypothesis 3.2. Section 4 gives the large-field suppression and Hypothesis 4.2. Section 5 proves Theorem 1.1, with the corrected Absorption step. Section 6 closes the LSI. Section 7 contains the $SU(2)$, $d = 2$ validation. Appendix A records the correction to [6]; Appendix B the reproducibility material.

1.5 The audit hypotheses (new in v2)

Hypothesis ((H-ABS) strong-form absorption input)**.** *Hypothesis 4.2 holds with the exponent as printed in (14): $\operatorname{ess\,sup}\mu_k(Z_k(B)|\mathcal{G}_{k+1}) \leq C_{\text{blk}} e^{-c\beta_k\varepsilon_k^2}$ with $c > 0$ depending only on* (d, N_c, L_{RG}), for the saturated thresholds $\varepsilon_k \equiv \varepsilon_*$. *This is strictly stronger than what the companion verification [9] delivers: there the small factor is $e^{-c_{\text{sf}}p_0(g_k)}$ with $c_{\text{sf}} = 2/(1+\beta_0)$, and [9]'s own asymptotics give $\beta_k\varepsilon_k^2 \asymp p_0(g_k)^2$, i.e. exponent $\asymp \sqrt{\beta_k\varepsilon_k^2}$, not $\beta_k\varepsilon_k^2$. Under (H-ABS) the absorption condition (16) holds for*

$$k \;\leq\; k_{\text{abs}}(\beta) \;:=\; \frac{c\,\beta_k\varepsilon_*^2 - \ln(M_k^2 C_{\text{blk}}/C_{\text{SF}}^2)}{(d-1)\ln L_{\text{RG}}},$$

a second window, linear in β like $k^*(\beta)$. Without (H-ABS), (16) fails at every scale $k \geq 2$ with the thresholds (9) (companion suite, check 5).

Hypothesis (Imported hypotheses)**.** (H-P0) *(power-law large-field penalty exponent) and* (H-SFI) *(slow-field interface) are inherited from Hypothesis 4.2 via 2602.0056 v2 / 2602.0052 v2.* (H-YGZ) *(fiber LSI constant not degraded by Holley–Stroock) is inherited by Corollary 1.2 via 2602.0053 v2. The volume window $k \leq k^*(\beta)$, $L_{\text{vol}} \leq e^{C/g^2+O(1)}$, is inherited from the corrected asymptotic-freedom flow (2602.0051 v2): at the series' representative arithmetic ($g = 0.05$, $\gamma_0 = 0.1$, $N_c = 3$, $L_{\text{RG}} = 2$), $k^* = 18639$ and $\log L_{\text{vol}} \leq 32.3/g^2$.*

2 Setup and notation

We work within the framework of [6]. Let $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ be a periodic lattice of side $L = L_{\text{RG}}^{n_{\max}} L_0$, gauge group $G = SU(N_c)$, configuration space $\mathcal{A} = G^{|\text{edges}|}$. The normalized Wilson plaquette functional is

$$S_W(U) \;:=\; \sum_{P\in\mathcal{P}} \Big(1 - \tfrac{1}{N_c}\operatorname{Re}\operatorname{tr} U_P\Big), \tag{4}$$

the Wilson action is $S_\beta = \beta S_W$, and $d\mu_\beta = Z(\beta)^{-1} e^{-\beta S_W} \prod_e dU_e$ with dU_e normalized Haar.

2.1 Multiscale decomposition

Following [6], $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{n_{\max}}$ encode the slow variables, and $\mathcal{E}_k \subset T\mathcal{A}$ the scale-k fast fluctuations.

Definition 2.1 (Conditional fast potential)**.** *Fix k. With $\mu_k(\cdot\,|\,\mathcal{G}_{k+1})$ the conditional measure of the scale-k fast variables and ν_k the Haar product measure on the fiber,*

$$V_{<k}(U\,|\,\mathcal{G}_{k+1}) \;:=\; -\log \frac{d\mu_k(\cdot\,|\,\mathcal{G}_{k+1})}{d\nu_k}(U) + \mathrm{const}(\mathcal{G}_{k+1}). \tag{5}$$

In Balaban's framework the effective action at scale k decomposes as

$$V_{<k} \;=\; \beta_k S_W + S_{k,\mathrm{res}} + \mathrm{const}(\mathcal{G}_{k+1}), \tag{6}$$

with β_k the effective inverse coupling and $S_{k,\mathrm{res}}$ the polymer/boundary residual. For the toy model ($n_{\max}=1$), $\beta_0=\beta$ and $S_{0,\mathrm{res}}\equiv 0$. For a unit fast vector $v\in\mathcal{E}_k$,

$$vV_{<k} \;=\; v(\beta_k S_W) + vS_{k,\mathrm{res}}. \tag{7}$$

2.2 Small-field and large-field regions

For each scale k and block B, the local large-field event is

$$Z_k(B) \;:=\; \big\{U : \exists\, P\in\mathcal{P}_k(B) \text{ with } \|U_P-\mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon_k\big\}. \tag{8}$$

We fix the Balaban-compatible thresholds

$$\varepsilon_k^2 \;:=\; c_*\, g_k^2\, p_0(g_k)\wedge \varepsilon_*^2, \qquad \beta_k := g_k^{-2}, \tag{9}$$

with p_0 Balaban's large-field function and ε_* the chart radius. **(v2)** The parenthetical of v1, "$p_0(g)\to\infty$ as $g\to 0$", concerns the bare-coupling limit at fixed k; along the cascade, with the correct flow, g_k *increases* and $p_0(g_k)$ *decreases* (from 3.34 to 2.50 over the window at the representative arithmetic).

Proposition 2.2 (Small-field analyticity (local form), [6, App. A])**.** *Fix k and a scale-k block B. On $\Omega^{\mathrm{sf}}_{k,B}$, for any unit $v\in\mathcal{E}_k$ supported in B: $|vV_{<k}|\le C_{\mathrm{SF}}\,L_{\mathrm{RG}}^{-(d-1)k/2}$.*

2.3 What was Assumption 5.4

Assumption 2.3 ([6, Assumption 5.4], now superseded)**.** *For each k and each unit $v\in\mathcal{E}_k$,* $\operatorname{ess\,sup}_{\mathcal{G}_k} E[|vV_{<k}|^2|\mathcal{G}_k]\le D_k$*, $\sum_k D_k<\infty$.*

The present paper replaces this assumption by Theorem 1.1 — in v2, at the price of the explicit hypothesis load and the window stated there.

3 Per-direction pointwise bound

Lemma 3.1 (Pointwise bound: Wilson contribution)**.** *Let $G=SU(N_c)$ with bi-invariant metric $\langle X,Y\rangle = -2\,\mathrm{tr}(XY)$ on $\mathfrak{su}(N_c)$. For any unit tangent vector $v\in\mathcal{E}_k$ (supported on a single link b in left-trivialization) and any $U\in\mathcal{A}$:*

$$\big|v(\beta_k S_W)(U)\big| \;\le\; c_W(k) \;:=\; \frac{2(d-1)\beta_k}{\sqrt{2N_c}}. \tag{10}$$

Proof. Write $v(U_b) = XU_b$ with $X \in \mathfrak{su}(N_c)$ and $|X| = 1$ (so $\|X\|_{\mathrm{HS}} = 1/\sqrt{2}$). Only the $2(d-1)$ plaquettes containing b contribute, and for each, $|v\,\frac{1}{N_c}\operatorname{Re}\operatorname{tr}(U_P)| \le \|X\|_{\mathrm{HS}}\|U_{P'}\|_{\mathrm{HS}}/N_c = 1/\sqrt{2N_c}$, using $\|U_{P'}\|_{\mathrm{HS}} = \sqrt{N_c}$. Summing and multiplying by β_k gives the result. □

Theorem 3.2 (Pointwise bound for residual derivatives; = Hypothesis 3.2, windowed per [10])**.** *There exist $q \ge 0$ and $C_{\mathrm{res}} > 0$ such that for every k and every unit $v \in \mathcal{E}_k$ supported on a single link,*

$$\sup_{U\in\mathcal{A}}\left|vS_{k,\mathrm{res}}(U \mid \mathcal{G}_{k+1})\right| \;\le\; C_{\mathrm{res}}\,(1+\beta_k)^q. \tag{11}$$

(v2) *Per the verification [10] (2602.0055 v2): the bound holds* windowed *($k \le k^*(\beta)$, with the* bare *β on the right-hand side) and* conditionally on Balaban's small-field inputs (B1)–(B4)*; the constants depend additionally on γ_0. v1 cited [10]'s v1 as an unconditional verification; that citation is corrected.*

Remark 3.3 (Status of Hypothesis 3.2)**.** *In the small-field region the polymer activities are analytic and the bound follows from Balaban's construction and the Cauchy estimate. The hypothesis extends this to a global bound including the large-field region. In the toy model ($n_{\max} = 1$), $S_{0,\mathrm{res}} \equiv 0$ and the hypothesis holds trivially.*

Remark 3.4 (Explicit values for Wilson part)**.** *For $d = 2$, $N_c = 2$: $c_W(0) = \beta$. For $d = 4$, $N_c = 2$: $c_W(0) = 3\beta$.*

Definition 3.5 (Combined pointwise bound)**.** *For each scale k, define*

$$M_k^2 \;:=\; 2\,c_W(k)^2 + 2\,C_{\mathrm{res}}^2(1+\beta_k)^{2q}. \tag{12}$$

By $(a+b)^2 \le 2a^2 + 2b^2$, Lemma 3.1 and Hypothesis 3.2 give $|vV_{<k}|^2 \le M_k^2$ for all $U \in \mathcal{A}$ (within the window and under the conditions of Theorem 3.2). For the toy model, $M_0^2 = 2c_W(0)^2 = 2\beta^2$.

4 Large-field measure suppression

4.1 Energy–distance identity

For any $U \in SU(N_c)$:

$$1 - \frac{1}{N_c}\operatorname{Re}\operatorname{tr}(U) \;=\; \frac{1}{2N_c}\|U - \mathbf{1}\|_{\mathrm{HS}}^2. \tag{13}$$

Each plaquette with $\|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon$ contributes at least $\frac{\beta}{2N_c}\varepsilon^2$ to the Wilson action.

4.2 Toy model: $SU(2)$, $d = 2$

Lemma 4.1 (Toy block large-field suppression)**.** *For all $\beta \ge 1$ and $\varepsilon \in (0, 2\sqrt{2}\,]$: $\mu_\beta(Z_0(B)) \le 4C\beta^{3/2}e^{-\beta\varepsilon^2/4}$, where $C > 0$ is a universal constant and $Z_0(B)$ is the event that at least one of the 4 plaquettes in the block satisfies $\|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon$.*

4.3 General case: Balaban hypothesis

Theorem 4.2 (Balaban-type conditional large-field suppression; = Hypothesis 4.2, conditional per [9])**.** *In $d \geq 2$, for each block B there exist $c > 0$ and $C_{\mathrm{blk}} < \infty$, depending on $(d, N_c, L_{\mathrm{RG}})$ but independent of L_{vol} and k, such that*

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \mu_k\big(Z_k(B) \,\big|\, \mathcal{G}_{k+1}\big) \;\leq\; C_{\mathrm{blk}} \exp\!\big(-c\, \beta_k \varepsilon_k^2\big). \tag{14}$$

(v2) *Per the verification [9] (2602.0056 v2): the verified form is windowed, conditional on* (H-SFI)+(H-P0)*, and carries the exponent $c_{\mathrm{sf}}\, p_0(g_k)$ with $c_{\mathrm{sf}} = 2/(1+\beta_0)$ — equivalently $\asymp \sqrt{\beta_k \varepsilon_k^2}$ under the asymptotics of [9],* not *the printed $\beta_k \varepsilon_k^2$. The strong form* (14) *as printed is hypothesis* (H-ABS) *when used in the Absorption step. In $d = 2$ the route of [9] is additionally fixed-β.*

Remark 4.3 (On the role of β_k)**.** *The use of β_k (rather than the bare β) is essential for the absorption step, since the scale-k effective coupling governs the energy penalty under the conditional fast measure.* (v2) *With the correct flow β_k* decreases *with k; v1's contrary suggestion is withdrawn (see §5).*

Remark 4.4 (Toy model and status in $d \geq 3$)**.** *For $n_{\max} = 1$ the unconditional version is proved in Section 7 (Lemma 4.1). The conditional version cannot hold uniformly over all coarse configurations in $d = 2$ (Remark 7.2); we therefore retain Hypothesis 4.2 as an input even in $d = 2$. For $d \geq 3$, Hypothesis 4.2 is verified* at form level *in [9] — windowed and conditional as recorded in Theorem 4.2.*

5 Proof of the integrated derivative bounds

Proof of Theorem 1.1. Fix a scale $k \leq \min\{k^*, k_{\mathrm{abs}}\}$ and a unit $v \in \mathcal{E}_k$ supported in a block B. Using the partition $\mathcal{A} = \Omega^{\mathrm{sf}}_{k,B} \sqcup Z_k(B)$:

$$E\big[|vV_{<k}|^2 \,\big|\, \mathcal{G}_{k+1}\big] = \underbrace{E\big[\mathbf{1}_{\Omega^{\mathrm{sf}}_{k,B}} |vV_{<k}|^2 \,\big|\, \mathcal{G}_{k+1}\big]}_{\text{(I): small-field}} + \underbrace{E\big[\mathbf{1}_{Z_k(B)} |vV_{<k}|^2 \,\big|\, \mathcal{G}_{k+1}\big]}_{\text{(II): large-field}}. \tag{15}$$

Term (I). By Proposition 2.2, (I) $\leq C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}$.

Term (II). By Definition 3.5, $|vV_{<k}|^2 \leq M_k^2$ everywhere, so (II) $\leq M_k^2\, \mu_k(Z_k(B)|\mathcal{G}_{k+1}) \leq M_k^2 C_{\mathrm{blk}} e^{-c\beta_k \varepsilon_k^2}$ by (14) — i.e. under (H-ABS).

Absorption — corrected in v2. The requirement (II) $\leq C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}$ amounts to

$$c\, \beta_k \varepsilon_k^2 \;\geq\; (d-1)\, k \ln L_{\mathrm{RG}} + C_0, \qquad C_0 = C_0(C_{\mathrm{SF}}, C_{\mathrm{blk}}, C_{\mathrm{res}}, q) > 0, \tag{16}$$

whose right-hand side grows linearly in k. v1 argued that (16) holds "since $p_0(g) \to \infty$ as $g \to 0$" and that "if β_k grows sufficiently with k along the RG flow, one may take $\varepsilon_k \equiv \varepsilon_*$". Both premises are withdrawn (F1: inverted flow). The corrected accounting, adjudicated in the companion suite (check 5), is:

- With the thresholds (9) in the unsaturated regime, $\beta_k \varepsilon_k^2 = c_* p_0(g_k)$ is *bounded* along the window (and decreasing), so (16) fails for every $k \geq 2$ at the representative arithmetic — regardless of (H-P0), which controls $p_0(\gamma_0)$ but cannot make a bounded sequence grow linearly.

- With the exponent actually delivered by [9] ($c_{\rm sf} p_0(g_k) \approx 0.05$), (16) fails already at $k = 1$ by four orders of magnitude (excess $10^{4.6}$, growing to 10^{23} at $k = 32$).
- Under (H-ABS) with saturated thresholds $\varepsilon_k \equiv \varepsilon_*$, (16) holds precisely for $k \le k_{\rm abs}(\beta)$, linear in β; combined with the flow window this gives the stated range $k \le \min\{k^*, k_{\rm abs}\}$.

Combining, for k in the stated range, $E[|vV_{<k}|^2|\mathcal{G}_{k+1}] \le 2C_{\rm SF}^2 L_{\rm RG}^{-(d-1)k} = D_k$ μ-a.s., proving (b); taking full expectation gives (a).

Summability. $\sum_k D_k \le 2C_{\rm SF}^2/(1 - L_{\rm RG}^{-(d-1)}) < \infty$, independently of $n_{\max}$ and $L_{\rm vol}$. □

6 Closing the LSI: modified sweeping-out and Rothaus closure

Throughout, $E_k[\cdot] := E[\cdot \,|\, \mathcal{G}_k]$. This section is unchanged from v1 up to the hypothesis load: every statement consumes Theorem 1.1 (hence its window and hypotheses) and, for the fiber constant $\hat\alpha_0$, hypothesis (H-YGZ).

6.1 Defective log-Sobolev inequality

Lemma 6.1 (Defective LSI; replaces [6, Lemma 5.7]). *For every bounded Lipschitz $g : \mathcal{A} \to \mathbb{R}$,*

$$\mathrm{Ent}_\mu(g^2) \;\le\; \frac{2}{\hat\alpha_0}\,\mathcal{E}(g,g) + D_*\|g\|_\infty^2, \tag{17}$$

where $D_* := \hat\alpha_0^{-1}\sum_k \sum_{v\in \mathrm{ONB}(\mathcal{E}_k)} E[|vV_{<k}|^2] < \infty$ by Theorem 1.1(a).

Proof. As in [6, Lemma 5.7]: entropy telescoping, conditional LSI on each fiber, and the score-function identity for $g_k := \sqrt{E_k[g^2]}$:

$$vE_k[g^2] \;=\; E_k[v(g^2)] - \mathrm{Cov}_k(g^2, vV_{<k}). \tag{18}$$

The covariance term satisfies $|\,\mathrm{Cov}_k(g^2, vV_{<k})|^2/(2E_k[g^2]) \le \frac{1}{2}\|g\|_\infty^2 E_k[|vV_{<k}|^2]$. *Key modification relative to [6]:* instead of bounding $E_k[|vV_{<k}|^2]$ by its ess-sup over $\mathcal{G}_k$, take full expectation — only the L^1 bound of Theorem 1.1(a) is needed. Summing over v and k yields the defect $D_*\|g\|_\infty^2$. □

6.2 Poincaré inequality (spectral gap)

Lemma 6.2 (Spectral gap; replaces [6, Lemma 5.10]). *For every Lipschitz g,*

$$\mathrm{Var}_\mu(g) \;\le\; \frac{1}{\lambda_1}\,\mathcal{E}(g,g), \qquad \lambda_1 \;\ge\; \frac{\hat\alpha_0}{2\big(1+\sum_k D_k\big)} \;>\; 0, \tag{19}$$

with λ_1 independent of $L_{\rm vol}$.

Proof. With $m_k := E[g|\mathcal{G}_k]$, the martingale variance decomposition and the conditional Poincaré bound each term; for the gradient of m_k,

$$|vm_k|^2 \;\le\; 2E_k[|vg|^2] + 2\,\mathrm{Var}_k(g)\,E_k[|vV_{<k}|^2]. \tag{20}$$

Key factorization step. Set $A := E[|\nabla_{\mathcal{E}_k} g|^2|\mathcal{G}_{k+1}]$ ($\mathcal{G}_{k+1}$-measurable) and $B := E_k[|vV_{<k}|^2]$. The tower property gives

$$E[A\cdot B] \;=\; E\big[A \cdot E[B\,|\,\mathcal{G}_{k+1}]\big] \;\le\; D_k\,E[A], \tag{21}$$

using Theorem 1.1(b). Summing over v and k:

$$\mathrm{Var}(g) \;\leq\; \frac{2}{\hat{\alpha}_0}\Big(1+\sum_k D_k\Big)\,\mathcal{E}(g,g). \tag{22}$$

$\square$

6.3 Tight LSI via Rothaus closure

Theorem 6.3 (Uniform tight log-Sobolev inequality; v2: windowed, conditional)**.** *There exists $\alpha_* > 0$, depending on $\hat{\alpha}_0$, D_*, λ_1, and the block geometry, but independent of L_{vol}, such that for all Lipschitz f:*

$$\mathrm{Ent}_\mu(f^2) \;\leq\; \frac{2}{\alpha_*}\,\mathcal{E}(f,f). \tag{23}$$

Proof. Lemma 6.1 (defective LSI) plus Lemma 6.2 (spectral gap) yield a tight LSI by the standard Rothaus argument (cf. [5, Proposition 5.1.3]); extension to $f \in H^1(\mu)$ by truncation and lower semicontinuity. The hypothesis load and window are those of Theorem 1.1 and Corollary 1.2. $\square$

7 Toy model validation: $SU(2)$ in $d = 2$, one RG step

This section survives v1 unchanged (it is the unconditional core of the paper) and is now machine-validated in the companion suite.

7.1 Parametrization and Haar measure

Every $U \in SU(2)$ is $U = \cos\theta\,\mathbf{1} + i\sin\theta\,\hat{n}\cdot\vec{\sigma}$ with $\theta \in [0,\pi]$, so $\mathrm{Re}\,\mathrm{tr}(U) = 2\cos\theta$ and, for class functions, $\int f\,dU = \frac{2}{\pi}\int_0^\pi f(\theta)\sin^2\theta\,d\theta$. The Hilbert–Schmidt distance satisfies

$$\|U-\mathbf{1}\|_{\mathrm{HS}}^2 \;=\; 4 - 2\,\mathrm{Re}\,\mathrm{tr}(U) \;=\; 8\sin^2(\theta/2). \tag{24}$$

7.2 Single-plaquette Wilson measure

$$d\nu_\beta(\theta) \;=\; \frac{\sin^2\theta\, e^{\beta\cos\theta}}{Z_1(\beta)}\,d\theta, \qquad Z_1(\beta) \;=\; \frac{\pi}{\beta}\,I_1(\beta), \tag{25}$$

with I_1 the modified Bessel function.

7.3 Analytic tail bound

Proposition 7.1 (Single-plaquette tail)**.** *There exists a universal constant $C > 0$ such that for all $\beta \geq 1$ and $\varepsilon \in (0, 2\sqrt{2}\,]$,*

$$\nu_\beta\big(\|U-\mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon\big) \;\leq\; C\,\beta^{3/2}\exp\Big(-\frac{\beta\varepsilon^2}{4}\Big).$$

Proof. The event equals $\{\theta \geq \theta_0\}$ with $\theta_0 = 2\arcsin(\varepsilon/\sqrt{8})$, $\cos\theta_0 = 1-\varepsilon^2/4$. Numerator: $\int_{\theta_0}^{\pi}\sin^2\theta\, e^{\beta\cos\theta}d\theta \leq \pi e^{\beta(1-\varepsilon^2/4)}$. Denominator: $Z_1(\beta) \geq c\,e^\beta\beta^{-3/2}$ by Gaussian comparison. Divide. $\square$

7.4 Block large-field suppression (unconditional)

For a 2×2 block B with 4 plaquettes, the union bound gives $\mu_\beta(Z_0(B)) \le 4C\beta^{3/2}e^{-\beta\varepsilon^2/4}$ (Lemma 4.1).

7.5 Remark on the conditional version

Remark 7.2 (Conditional version in $d = 2$). *The shifted bound of Theorem 1.1(b) would require a uniform bound on $\mu_\beta(Z_0(B)|\mathcal{G}_1)$ over all coarse configurations $\bar{U}_B$. In $d = 2$ with axial gauge the fine plaquettes are i.i.d. and $\bar{U}_B$ is their product; for $\bar{U}_B = -\mathbf{1}$ the constraint $U_1U_2U_3U_4 = \bar{U}_B$ forces some $\|U_i - \mathbf{1}\|_{\mathrm{HS}} \ge c > 0$ (in the $U(1)$ reduction, $c = 2\sin(\pi/8) \approx 0.765$), so $\mu_\beta(Z_0(B)|\bar{U}_B) = 1$ for ε small: no bound $C\beta^{3/2}e^{-c\beta\varepsilon^2}$ uniform in $\bar{U}_B$ can hold. Accordingly, the toy closes Theorem 1.1(a) via Lemma 4.1, and Hypothesis 4.2 is retained even in $d = 2$.* (v2) *This remark is consistent with, and sharpened by, the β-non-uniformity of the conditional prefactor measured in [9] (2602.0056 v2, Remark 6.6): $C(\beta) = 1.3/33/4.7 \times 10^7$ at $\beta = 1/4/16$ in the exact $U(1)$ toy.*

Remark 7.3 (Higher dimensions). *For $d \ge 3$, plaquettes within a block share edges and are not independent; the conditional suppression requires Balaban-type density bounds, as formulated in Hypothesis 4.2.*

7.6 Verification of the full chain

Combining with $|vV_{<0}|^2 \le M_0^2 = 2\beta^2$ ($d = 2$, $N_c = 2$): $E[|vV_{<0}|^2] \le C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-1} + C_{\mathrm{toy}}\beta^{7/2}e^{-\beta\varepsilon^2/4}$. For fixed $\varepsilon > 0$ the large-field term vanishes as $\beta \to \infty$, confirming the absorption into the geometric factor *in the toy* (where $n_{\max} = 1$ and no cascade is involved — the k-growth obstruction of §5 does not arise).

7.7 Numerical validation

Table 1 reports the exact tail probability $P(\beta, \varepsilon)$ (30-digit quadrature in v1; reproduced in the v2 companion suite with scipy quadrature to 10^{-12}) and the normalized ratio $R := P/(\beta^{3/2}e^{-\beta\varepsilon^2/4})$, which remains bounded by ≈ 0.37 and decreases monotonically with β, confirming Proposition 7.1. (v1's Table 1 listed the full 8×7 grid and a two-panel figure; v2 prints a representative subset — the full grid is regenerated deterministically by the suite, check 3, including digit-level agreement with the v1 values.)

Remark 7.4 (Character expansion). *$Z_1(\beta)$ can also be obtained from the $SU(2)$ character expansion via $I_0(\beta) - I_2(\beta) = \frac{2}{\beta}I_1(\beta)$, recovering (25). For multi-plaquette objects the character representation provides an exact alternative to quadrature (cf. [9, Section 6]).*

8 Note added in v2

Version 1 presented Theorem 1.1 and Corollary 1.2 as an unconditional replacement of Assumption 5.4 of [6], citing [9] and [10] (v1) as verifications of Hypotheses 4.2 and 3.2. Following the quantitative audit of the series (ai.viXra:2602.0051–0056, all v2), the status is corrected:

- **F-ABS (new in this paper).** The Absorption step consumes Hypothesis 4.2 with exponent $c\beta_k\varepsilon_k^2$; the companion verification delivers $c_{\mathrm{sf}}p_0(g_k)$, $c_{\mathrm{sf}} = 2/(1 + \beta_0)$, with $\beta_k\varepsilon_k^2 \asymp p_0^2$

β	ε	$P(\beta,\varepsilon)$	$R(\beta,\varepsilon)$
2	0.3	9.91e-01	0.2924
2	2.0	1.53e-01	0.3197
6	1.0	3.69e-01	0.0897
10	1.3	3.35e-02	0.0577
20	1.0	1.74e-02	0.0231
30	1.5	1.90e-07	0.0196
50	1.3	3.16e-09	0.0106

Table 1: Exact tail probability and normalized ratio $R = P/(C\beta^{3/2}e^{-\beta\varepsilon^2/4})$ with $C = \pi/\sqrt{2\pi} \approx 1.2533$ (representative subset of v1's Table 1; full grid in the companion suite). (v2) v1's caption printed the ratio without C, but its tabulated values include it; the convention is fixed here (suite check 3).

(exponent $\asymp \sqrt{\beta_k\varepsilon_k^2}$). With thresholds (9), $\beta_k\varepsilon_k^2 = c_* p_0(g_k)$ is bounded along the window while the requirement (16) grows linearly in k: absorption fails for every $k \geq 2$ even under (H-P0). The strong form is now the explicit hypothesis (H-ABS), with its saturated window $k_{\text{abs}} \propto \beta\varepsilon_*^2$. [Suite check 5]

- **F1.** "$p_0(g) \to \infty$ as $g \to 0$ along the flow" and "if β_k grows sufficiently with k" used the inverted-sign flow; withdrawn. All statements are windowed: $k \leq k^*(\beta)$, $L_{\text{vol}} \leq e^{C/g^2+O(1)}$. [Suite check 6]
- **Inherited retags.** Hypothesis 3.2: windowed, conditional on (B1)–(B4) (2602.0055 v2). Hypothesis 4.2: form-level, conditional on (H-SFI)+(H-P0), fixed-β in $d = 2$ (2602.0056 v2). Fiber LSI constant $\hat{\alpha}_0$: inherits (H-YGZ) (2602.0053 v2). Corollary 1.2 carries the full load.
- **Reference hygiene.** v1's [8] was a self-citation of the present paper (removed; placeholder kept). The duplicated Wilson entry is removed. [9], [10], [11] now cite the v2 replacements by number (2602.0056, 2602.0055, 2602.0054), and the audit chain 2602.0051/0052/0053 (v2) is added as [13]–[15].
- **What survives.** Lemma 3.1 (with its constant verified by numerical optimization to saturation), the identities (13)/(24), Proposition 7.1 and Table 1 (v1 values reproduced digit-by-digit), Remark 7.2 (verified exactly in the $U(1)$ toy), the factorization step (21), the Rothaus closure (validated on exact finite toys), and Appendix A's correction $\lambda_1 \geq \alpha_*$ (validated numerically; it also propagates to [6] v3). [Suite checks 1–4, 7–8]

All numerical adjudications are reproduced by the deterministic companion suite `verify_2602_0057.py` (8 checks) at `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`, directory `verification/2602-`

A Correction to Proposition 6.1(2) of [6]

In the original statement, the spectral gap bound reads $\lambda_1 \geq \alpha_*/2$, where α_* is the LSI constant. The correct bound, from the standard relation between the log-Sobolev constant and the spectral gap (cf. [5, Corollary 5.1.4]), is

$$\lambda_1 \;\geq\; \alpha_*.$$

The factor of 2 arose from a notational inconsistency in the definition of the Dirichlet form. With the convention used throughout this paper and [7] ($\mathrm{Ent}(g^2) \le \frac{2}{\alpha}\mathcal{E}(g,g)$), the spectral gap satisfies $\lambda_1 \ge \alpha$. This does not affect any qualitative conclusion of [6]; it improves the spectral gap by a factor of 2. (v2) Validated numerically on exact finite toys (suite check 8: $\lambda_1/\alpha \in [1.00, 2.9]$ across test chains, never below 1); the correction should be incorporated in [6]'s next revision (v3 carries the erratum hooks).

B Reproducibility

v1's Appendix B listed the script `toy_validation.py` (mpmath, 30-digit precision), generating Table 1 and Figure 1. In v2 the validation is subsumed by the deterministic companion suite `verify_2602_0057.py` (NumPy/SciPy, 8 checks, $\sim$5 s), which reproduces the full v1 grid of Table 1 (check 3, including the printed values to displayed precision), the constant of Lemma 3.1 (check 1), the conditional impossibility of Remark 7.2 (check 4), the F-ABS adjudication (check 5), the window (check 6), the factorization identity (check 7), and the Rothaus/Appendix A relations (check 8). Repo: `THE-ERIKSSON-PROGRAMME`, `verification/2602-0057/`.

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[8] *[Removed in v2: v1's reference [8] was a self-citation of the present paper.]*

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