Large-field suppression for lattice gauge theories: from Balaban's renormalization group to conditional concentration — a conditional and windowed verification*

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Abstract

We verify, at the level of form, the large-field hypothesis (Hypothesis 4.2) of the companion paper on integrated cross-scale derivative bounds for Wilson lattice gauge theory (Paper III). The proof rests on three ingredients: (i) a dictionary lemma translating the Hilbert–Schmidt large-field condition on plaquette holonomies into Balaban's Lie-algebra formulation; (ii) an interface lemma connecting conditional measures with Balaban's T-operation and its uniform small-factor bound on admissible background fields (Eq. (1.89) of [2]); (iii) the uniformity estimate (Eq. (1.75) of [2]) ensuring that slow-field dependence contributes only an $O(1)$ multiplicative constant. For $d = 2$, we give an independent proof via character-positive convolutions that avoids the Balaban machinery entirely. Version 2 corrects the status of these results after the quantitative audit of the series (ai.viXra:2602.0051–0055, all v2). (a) v1's claim that the bound is "more than sufficient" for the absorption condition of Paper III is withdrawn: the printed small factor is $\exp(-c\,p_0(g_k))$ with $c = 2/(1 + \beta_0)$, and with the polylog floor on p_0 the suppression trivializes ($e^{-c\,p_0(\gamma_0)} \approx 0.95$) and the absorption inequality fails at every scale (excess $\geq 10^{4.6}$); effectiveness requires the power-law hypothesis (H-P0) of 2602.0052 v2. (b) v1's premise "$p_0(g_k) \to \infty$ as $g_k \to 0$ along the flow" and §7's appeal to a stability theorem rely on the inverted-sign running-coupling flow of the series erratum; with the correct asymptotic-freedom flow the small-field condition $g_k \leq \gamma_0$ holds only for $k \leq k^*(\beta)$, and all statements are windowed: $L_{\mathrm{vol}} \leq e^{C/g^2+O(1)}$. (c) v1's Remarks 4.1–4.2 (slow-field identification and Balaban conditional representation) are unproved interface statements; they are made explicit here as hypothesis (H-SFI), cf. the interface lemmas of 2602.0052 v2. (d) In $d = 2$ the prefactor $K_\beta(1)/Z(\bar{U}_B)$ of Proposition 6.4 is not uniform in β, so the $d = 2$ route verifies a fixed-β variant only; this is now stated in the theorem. What survives unconditionally and is validated in the companion numerical suite: the HS/Lie-algebra dictionary (Lemma 2.1), the gauge-invariance identity (Remark 2.2), the block event inclusion (Lemma 3.2), and the character-positivity mechanism of Section 6 (Peter–Weyl positivity of the Wilson weight, convolution stability, maximum at the identity, and conditional tail domination in an exact $d = 2$ toy).

*Version 1 of this paper carried the title "Large-field suppression for lattice gauge theories: From Balaban's renormalization group to conditional concentration" and asserted that its main bound is "more than sufficient" for the absorption condition of Paper III. That sufficiency claim is withdrawn in this version; see the Note added in v2 (Section 8).

Contents

1 Introduction and main result

1.1 Setup and notation

We work on a finite periodic lattice $\Lambda \subset \mathbb{Z}^d$ with gauge group $G = SU(N_c)$ and Wilson action at coupling $\beta = 2N_c/g^2$ (we write $\beta^{\mathrm{red}} = g^{-2} = \beta/(2N_c)$ when the reduced normalization is needed; cf. Paper III, eq. (9)). We adopt the notation and conventions of [3] (henceforth "Paper III") throughout. We write β_0 for Balaban's fixed ultraviolet reference coupling (as in [2]), and $\gamma_0 > 0$ for the small-coupling threshold such that Balaban's RG construction and bounds apply uniformly whenever the effective couplings satisfy $g_k \le \gamma_0$ for all scales k. We further choose γ_0 small enough so that

$$\varepsilon_k^{\mathrm{Bal}} \;\le\; \varepsilon_* \qquad \text{whenever } g_k \le \gamma_0, \tag{1}$$

where ε_* is the chart radius of Lemma 2.1 and $\varepsilon_k^{\mathrm{Bal}}$ is the Balaban HS threshold of (3). This ensures that in the regime of Theorem 1.3 the cutoff in (4) is inactive and $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$.

Correction (v2). Under the correct asymptotic-freedom flow (2602.0051 v2; series erratum), the reduced coupling $\beta_k = g_k^{-2}$ *decreases* along the cascade, $2N_c\beta_k = 2N_c\beta_0' - c_{\mathrm{flow}}k + O(1/\beta)$ with $c_{\mathrm{flow}} > 0$, so the premise "$g_k \le \gamma_0$ for all k" is available only inside the window

$$k \;\le\; k^*(\beta) \;=\; \frac{2N_c}{c_{\mathrm{flow}}}\Big(\frac{1}{g^2} - \frac{1}{\gamma_0^2}\Big) + O(1), \qquad \text{equivalently} \qquad L_{\mathrm{vol}} \;\le\; e^{C/g^2+O(1)}.$$

All results of this paper are stated within this window. v1 asserted them for all scales; that assertion relied on the inverted-sign flow (flag F1 of the series erratum) and is withdrawn.

1.2 The large-field event

For a scale-k block B, define

$$Z_k(B) \;:=\; \{\exists\, P \in \mathcal{P}_k(B) :\; \|U_P - \mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon_k\}, \tag{2}$$

where $\mathcal{P}_k(B)$ is the set of plaquettes associated with B at scale k.

1.3 Choice of thresholds

Balaban's small-field characteristic function χ_k^{sf} (Definition 3.1) imposes conditions on plaquette holonomies at scale k. We define the Balaban HS threshold $\varepsilon_k^{\mathrm{Bal}}$ as the supremum of constants such that

$$\|U_P - \mathbf{1}\|_{\mathrm{HS}} < \varepsilon_k^{\mathrm{Bal}} \;\;\forall\, P \in \mathcal{P}_k(B) \quad\Longrightarrow\quad \chi_k^{\mathrm{sf}} = 1 \text{ on the component containing } B. \tag{3}$$

This threshold is well-defined because Balaban's small-field condition is formulated via upper bounds on Lie-algebra representatives of plaquette holonomies on the principal chart (cf. [2]), and Lemma 2.1 converts between HS norm and Lie-algebra norm with finite constants depending only on N_c. We choose

$$\varepsilon_k \;:=\; \varepsilon_k^{\mathrm{Bal}} \wedge \varepsilon_*, \tag{4}$$

where $\varepsilon_* > 0$ ensures the principal branch of log is controlled (Lemma 2.1). This choice has two consequences:

- **Inclusion:** under the small-coupling assumption $g_k \leq \gamma_0$, any plaquette with $\|U_P - \mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon_k$ violates Balaban's small-field condition, triggering χ_k^{lf} (Lemma 3.2).
- **Suppression:** the event $Z_k(B)$ is controlled by Balaban's T-operation with small factor $\exp(-c\, p_0(g_k))$ (Lemma 4.3), *conditionally on (H-SFI).*

Remark 1.1 (Asymptotics of the threshold)**.** *In Balaban's construction, the small-field condition at scale k imposes upper bounds on plaquette Lie-algebra representatives on the principal chart. Whenever these bounds are of order $O(p_0(g_k)\, g_k)$ in operator norm (cf. [2]), Lemma 2.1 gives $\varepsilon_k^{\mathrm{Bal}} \asymp p_0(g_k)\, g_k$.* (v2) *With the correct flow, g_k does* not *tend to 0 along the cascade; the asymptotics "$\varepsilon_k \to 0$ as $g_k \to 0$" of v1 concern the* bare*-coupling limit $g \to 0$ at fixed k, not the infrared limit $k \to \infty$ at fixed g.*

Remark 1.2 (Interpretation)**.** *Heuristically, $\varepsilon_k^{\mathrm{Bal}}$ is the largest HS-radius such that all plaquettes remain inside Balaban's small-field domain on the principal chart. The choice $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$ is therefore the sharp threshold at which the large-field characteristic χ_k^{lf} must trigger.*

1.4 The audit hypotheses (new in v2)

Hypothesis ((H-SFI) slow-field interface)**.** *The content of Remarks 4.1 and 4.2: for gauge fields in the small-field domain, (i) the block-averaged holonomies generating $\mathcal{G}_{k+1}$ of Paper III encode the same coarse information as Balaban's background field $U_0^{(k)}$, modulo blockwise gauge transformations, and (ii) the conditional fast density $d\mu_k(\cdot \,|\, \mathcal{G}_{k+1})$ admits Balaban's representation via the scale-k small/large-field decomposition and the T-operation. v1 stated (i)–(ii) as remarks; no proof is given there or here. The corresponding interface lemmas are analyzed in 2602.0052 v2, where they are shown to require quantitative inputs not available at the printed level of generality.*

Hypothesis ((H-P0) power-law penalty exponent (2602.0052 v2))**.** *There exist $c_0 > 0$ and $\theta > 0$ such that Balaban's large-field penalty exponent satisfies $p_0(g) \geq c_0\, g^{-\theta}$ for $g \leq \gamma_0$. The printed polylog floor $p_0(g) \geq c_0|\log g|^{1+\epsilon}$ is insufficient: with $c = 2/(1+\beta_0)$ the suppression factor is $e^{-c\,p_0(\gamma_0)} \approx 0.95$ at $\gamma_0 = 0.1$, $\beta_0 = 100$, and the absorption inequality of Paper III fails at every scale (companion suite, check 4).*

1.5 Main theorem

Theorem 1.3 (= Hypothesis 4.2 of Paper III; v2: windowed, conditional)**.** *Fix $d \geq 3$, $G = SU(N_c)$, and $L_{\mathrm{RG}} \geq 2$. Assume* (H-SFI)*. There exist $\gamma_0 > 0$, $c > 0$, and $C_{\mathrm{blk}} < \infty$ (depending only on d, N_c, L_{RG}) such that if the effective couplings satisfy $g_j \leq \gamma_0$ for all $j \leq k$ — which, under the correct asymptotic-freedom flow, holds precisely for $k \leq k^*(\beta)$, i.e. $L_{\mathrm{vol}} \leq e^{C/g^2+O(1)}$ — then with thresholds* (4)*,*

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \; \mu_k\big(Z_k(B)\,\big|\,\mathcal{G}_{k+1}\big) \;\leq\; C_{\mathrm{blk}}\,\exp\big(-c\,p_0(g_k)\big) \tag{5}$$

for every scale-k block B and every scale $k \leq k^*(\beta)$, with $c = 2/(1+\beta_0)$.

Moreover — correcting v1 — the bound (5) *satisfies the absorption condition of Paper III (Eq. (15) there)* if and only if *the exponent clears the geometric rate: $c\,p_0(g_k) \geq (d-1)\ln L_{\mathrm{RG}} + C_{\mathrm{abs}}$. With the polylog floor on p_0 this fails at every scale; it holds under* (H-P0)*. v1's assertion that* (5) *is "more than sufficient" is withdrawn.*

The proof assembles Lemmas 2.1, 3.2, and 4.3 below.

2 Dictionary: HS-large implies Lie-algebra-large

Lemma 2.1 (HS-large implies Lie-algebra-large)**.** *Let $G = SU(N_c)$. There exist constants $\varepsilon_* > 0$ and $c_1, c_2 > 0$ depending only on N_c such that, for every $U \in G$ with $\|U - \mathbf{1}\|_{\mathrm{HS}} \leq \varepsilon_*$, the principal-branch logarithm $X := \log U \in \mathfrak{su}(N_c)$ is well-defined and satisfies*

$$c_1\|X\|_{\mathrm{HS}} \;\leq\; \|U - \mathbf{1}\|_{\mathrm{HS}} \;\leq\; c_2\|X\|_{\mathrm{HS}}. \tag{6}$$

In particular, if $\|U - \mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon$ with $\varepsilon \leq \varepsilon_*$, then $\|X\|_{\mathrm{HS}} \geq \varepsilon/c_2$.

Proof. For $\|U - \mathbf{1}\|_{\mathrm{HS}}$ sufficiently small, all eigenvalues of U lie in a neighborhood of 1 that avoids the negative real axis, hence the principal logarithm $X = \log U$ is defined and belongs to $\mathfrak{su}(N_c)$. Write $U = e^X$ with X anti-Hermitian and traceless. For $\|X\|_{\mathrm{op}}$ small we have the convergent expansion $e^X - \mathbf{1} = X + \frac{1}{2}X^2 + \cdots$, hence $\|U - \mathbf{1}\|_{\mathrm{HS}} = \|X\|_{\mathrm{HS}}\big(1 + O(\|X\|_{\mathrm{op}})\big)$. Choose $\varepsilon_* > 0$ such that $\|X\|_{\mathrm{op}}$ is small whenever $\|U - \mathbf{1}\|_{\mathrm{HS}} \leq \varepsilon_*$, and absorb the error term into constants c_1, c_2 depending only on N_c. Finally, $\|X\|_{\mathrm{op}} \leq \|X\|_{\mathrm{HS}} \leq \sqrt{N_c}\,\|X\|_{\mathrm{op}}$ gives the required uniform control. □

Remark 2.2 (Gauge invariance)**.** *The quantity $\|U_P - \mathbf{1}\|_{\mathrm{HS}}^2 = 2N_c - 2\,\mathrm{Re}\,\mathrm{tr}(U_P)$ is a class function (gauge-invariant). The eigenvalues of $X = \log U_P$ are likewise gauge-invariant (up to permutation), so Lemma 2.1 is independent of gauge-fixing conventions.*

3 Block event inclusion

Definition 3.1 (Balaban large-field trigger at scale k)**.** *Fix a scale k. Let $\chi_k^{\rm sf}$ denote Balaban's small-field characteristic function (as defined in [2]; cf. Definition 2.1 and Eq. (2.3) therein), and set $\chi_k^{\rm lf} := 1-\chi_k^{\rm sf}$. For a block B, let $L_k(B)$ be the event that $\chi_k^{\rm lf}$ is triggered in the connected component of the scale-k decomposition that contains B. The Balaban HS threshold $\varepsilon_k^{\rm Bal}$ is as in (3).*

Lemma 3.2 (Block event inclusion)**.** *Let B be a scale-k block and assume $g_k \le \gamma_0$. With the choice $\varepsilon_k = \varepsilon_k^{\rm Bal} \wedge \varepsilon_*$ as in (4),*

$$Z_k(B) \;\subset\; L_k(B). \tag{7}$$

Proof. Under the small-coupling assumption $g_k \le \gamma_0$, we have $\varepsilon_k^{\rm Bal} \le \varepsilon_*$ by (1), so the cutoff in (4) is inactive and $\varepsilon_k = \varepsilon_k^{\rm Bal}$. On $Z_k(B)$, there exists a plaquette $P \in \mathcal{P}_k(B)$ with $\|U_P - \mathbf{1}\|_{\rm HS} \ge \varepsilon_k = \varepsilon_k^{\rm Bal}$. By the contrapositive of (3), $\chi_k^{\rm sf} \neq 1$ on the component containing B, so $\chi_k^{\rm lf}$ is triggered and $Z_k(B) \subset L_k(B)$. $\square$

4 Interface with Balaban's construction

This is the technical heart of the paper: we connect the conditional measure $\mu_k(\cdot \,|\, \mathcal{G}_{k+1})$ of Paper III with the T-operation of Balaban. **(v2)** The two identifications on which this section rests were stated in v1 as Remarks 4.1 and 4.2 without proof; they jointly constitute hypothesis (H-SFI) of §1.4 and everything downstream of them is conditional on it.

4.1 Identification of slow-field structures

Paper III defines $\mathcal{G}_{k+1}$ as the σ-algebra generated by the block-averaged holonomies at scale $k+1$. In Balaban's construction, the slow-field variables at step k are the minimizers $U_0^{(k)}$ of the gauge-fixed effective action, determined by the averaging operations (cf. [1], Section 1).

Remark 4.1 (Slow-field identification (interface statement; part of (H-SFI)))**.** *For gauge fields in the small-field domain, the block-averaged holonomies that generate $\mathcal{G}_{k+1}$ encode the same coarse information as Balaban's background field $U_0^{(k)}$, modulo blockwise gauge transformations. In particular, Balaban's bounds, which are uniform over admissible backgrounds, may be applied uniformly in the conditioning on $\mathcal{G}_{k+1}$.* (v2) *This is an unproved identification; see 2602.0052 v2 for the obstructions to making it quantitative.*

Remark 4.2 (Balaban conditional representation (interface statement; part of (H-SFI)))**.** *Under Balaban's RG hypotheses at scale k (effective coupling $g_k \le \gamma_0$), the conditional fast density $d\mu_k(\cdot \,|\, \mathcal{G}_{k+1})$ admits Balaban's representation in terms of the scale-k small/large-field decomposition and the associated T-operation (cf. the representation around Eq. (1.72) in [2]).* (v2) *Unproved; part of* (H-SFI).

4.2 The interface lemma

Lemma 4.3 (Interface: conditional large-field mass; conditional on (H-SFI))**.** *Let $\mu_k(\cdot \,|\, \mathcal{G}_{k+1})$ denote the scale-k fast conditional measure of Paper III, and let X be a connected component of Balaban's large-field region at scale k. Assume* (H-SFI) *and that the effective couplings satisfy*

$g_j \le \gamma_0$ *for all* $j \le k$. *Then there exists* $C_{\rm uni} < \infty$, *depending only on* $(d, N_c, L_{\rm RG})$, *such that for* $\mathcal{G}_{k+1}$*-a.e. slow-field configuration,*

$$\mu_k\big(\textit{large-field component} = X \,\big|\, \mathcal{G}_{k+1}\big) \;\le\; C_{\rm uni}\, \exp\Big(-\frac{2}{1+\beta_0}\, p_0(g_k)\Big). \tag{8}$$

The constant $C_{\rm uni}$ *absorbs the uniformity factor* $e^{3\sup|\sigma|}$ *from Balaban's estimate (cf. Eq. (1.75) in [2]), where* $\sup|\sigma| = O(1)$ *uniformly in the slow field.*

Proof. By Remark 4.2 (i.e. under (H-SFI)), the conditional fast density $d\mu_k(\cdot\,|\,\mathcal{G}_{k+1})$ admits Balaban's representation at scale k. In particular, the conditional contribution of configurations with large-field component X is controlled by the T-operation $T_k(X)$ (cf. Eq. (1.72) in [2]).

Step 1: The small-factor bound. By Eq. (1.89) of [2]: $\|T_k(X)\,1\| \le \exp\big(-\frac{2}{1+\beta_0} p_0(g_k)\big)$.

Step 2: Uniformity in the slow field. The slow-field dependence enters through the analytically extended T-operation $T'_k(X, (U, J))$. By Eq. (1.75) of [2]: $\|(T'_k(X, (U, J))\,1)^{-1}\, T'_k(X, (U, J))\, F\| \le e^{3\sup|\sigma|} \sup|F|$. The analyticity bounds of Balaban's construction ensure $\sup|\sigma| \le C_\sigma$ with C_σ depending only on the geometric parameters $(d, N_c, L_{\rm RG})$.

Step 3: Assembly. Setting $C_{\rm uni} = e^{3C_\sigma}$ gives the claimed bound uniformly in $\mathcal{G}_{k+1}$. □

Remark 4.4 (The small factor trivializes without (H-P0); new in v2)**.** *The exponent in* (8) *is* $c\,p_0(g_k)$ *with* $c = 2/(1+\beta_0)$. *Since* β_0 *is a* fixed large *reference coupling* ($\beta_0 = 100$ *in the series' representative arithmetic),* $c \approx 0.02$, *and with the polylog floor* $p_0(\gamma_0) \approx 2.5$ *at* $\gamma_0 = 0.1$ *the factor is* $e^{-c\,p_0} \approx 0.95$*: no suppression. This is the same trivialization adjudicated in 2602.0052 v2 (F2) and reproduced in the suites of 2602.0053 v2/0055 v2; check 4 of the companion suite reproduces it for this paper. All effectiveness claims are therefore conditional on* (H-P0)*.*

5 Proof of the main theorem

Proof of Theorem 1.3. 1. Inclusion. By Lemma 3.2, $Z_k(B) \subset L_k(B)$.

2. Suppression. By Lemma 4.3 (under (H-SFI)), $\mu_k(L_k(B)\,|\,\mathcal{G}_{k+1}) \le C_{\rm uni} \exp\big(-\frac{2}{1+\beta_0} p_0(g_k)\big)$.

3. Absorption — corrected in v2. v1 argued: "since $p_0(g_k) \to \infty$ as $g_k \to 0$, for β sufficiently large we have $c\,p_0(g_k) \ge (d-1)\ln L_{\rm RG} + C_{\rm abs}$." Both steps fail: under the correct flow g_k increases along the cascade (F1), and at any fixed scale the polylog floor gives $c\,p_0 \approx 0.05 \ll (d-1)\ln L_{\rm RG} \ge \ln 2$ (suite check 4: the absorption inequality fails at every sampled scale, excess $10^{4.6}$–$10^{23.2}$). The absorption condition of Paper III therefore holds only under (H-P0), and then only for $k \le k^*(\beta)$.

4. Essential supremum. Constants are uniform in $\mathcal{G}_{k+1}$ by Lemma 4.3 (under (H-SFI)). Set $c = 2(1+\beta_0)^{-1}$ and $C_{\rm blk} = C_{\rm uni}$. □

6 The two-dimensional case via character positivity

In $d = 2$, Hypothesis 4.2 can be verified independently of Balaban's RG machinery, using harmonic analysis on compact groups. **(v2)** The verification below is *pointwise in* β: the prefactor of Proposition 6.4 is finite for each fixed β but is not claimed uniform in β; see Remark 6.6.

Definition 6.1 (Character-positive class functions)**.** *A continuous class function* $f : G \to \mathbb{R}$ *is* character-positive *if* $f(g) = \sum_{\lambda\in\widehat{G}} a_\lambda \chi_\lambda(g)$ *with* $a_\lambda \ge 0$*.*

Lemma 6.2 (Maximum at the identity)**.** *If f is character-positive, then $f(g) \le f(\mathbf{1})$ for all $g \in G$.*

Proof. For each irrep λ, $|\chi_\lambda(g)| \le d_\lambda = \chi_\lambda(\mathbf{1})$. With $a_\lambda \ge 0$, $f(g) = \sum_\lambda a_\lambda \chi_\lambda(g) \le \sum_\lambda a_\lambda |\chi_\lambda(g)| \le \sum_\lambda a_\lambda d_\lambda = f(\mathbf{1})$. □

Lemma 6.3 (Convolution preserves character-positivity)**.** *If f and h are character-positive class functions in $L^1(G)$, then $f * h$ is character-positive. In particular, $(f*h)(g) \le (f*h)(\mathbf{1})$ for all $g \in G$.*

Proof. For central L^1 functions, Fourier coefficients satisfy $\widehat{f*h}(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda)/d_\lambda$. If $f = \sum a_\lambda \chi_\lambda$ and $h = \sum b_\lambda \chi_\lambda$ with $a_\lambda, b_\lambda \ge 0$, then $f*h = \sum (a_\lambda b_\lambda / d_\lambda)\chi_\lambda$ has nonnegative coefficients. The maximum statement follows from Lemma 6.2. □

Proposition 6.4 (Prop. 7.2 of Paper III, corrected; $d = 2$, fixed β)**.** *In $d = 2$, with axial gauge in a 2×2 block, the conditional density of the plaquette variable U_{P_0} given the boundary holonomy $\bar{U}_B$ is*

$$\rho(U_{P_0} \,|\, \bar{U}_B) \ \propto \ w_\beta(U_{P_0})\, K_\beta(U_{P_0}^{-1}\bar{U}_B),$$

where $K_\beta = w_\beta^{*3}$ is character-positive. By Lemma 6.2, $K_\beta(g) \le K_\beta(\mathbf{1})$ for all $g \in G$. Hence $\rho(U_{P_0} \,|\, \bar{U}_B) \le K_\beta(\mathbf{1})\, w_\beta(U_{P_0})$ (up to the normalization $Z(\bar{U}_B)$), and the tail probability is dominated by the single-plaquette measure:

$$\mu_\beta\big(\|U_{P_0} - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon \,\big|\, \bar{U}_B\big) \ \le \ \frac{K_\beta(\mathbf{1})}{Z(\bar{U}_B)}\, \nu_\beta\big(\|U_{P_0} - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon\big).$$

For each fixed β, the ratio $K_\beta(\mathbf{1})/Z(\bar{U}_B)$ is bounded above by a constant $C(\beta) < \infty$, since $Z(\bar{U}_B) > 0$ by strict positivity of w_β and continuity of K_β on the compact group G. This pointwise bound eliminates the spurious $e^{\pm\beta}$ prefactor that appeared in the earlier version of Prop. 7.2; we do not claim uniformity in β or in $\bar{U}_B$ for this ratio.

Remark 6.5. *The argument of Proposition 6.4 does not extend to $d \ge 3$ because the marginal density in higher dimensions involves coupling tensors (Clebsch–Gordan coefficients for $r = 2(d-1) \ge 4$ factors) that can have indefinite signs. The Balaban machinery of Sections 2–4 is essential in $d \ge 3$.*

Remark 6.6 (β-non-uniformity of the $d = 2$ prefactor; new in v2)**.** *Theorem 1.3 promises constants depending only on $(d, N_c, L_{\mathrm{RG}})$. The $d = 2$ route delivers less: the measured growth of $C(\beta) = K_\beta(\mathbf{1})/\min_{\bar{U}} Z(\bar{U})$ in the exact $U(1)$ toy of the companion suite (check 7) is already super-polynomial in β, and no β-uniform bound is claimed. The $d = 2$ verification of Hypothesis 4.2 is therefore* fixed-β*: for each β there is a finite constant, which suffices for the qualitative statement but not for the β-uniform constants advertised in v1's Theorem 1.3. In $d \ge 3$ nothing changes: there the route is (H-SFI)+(H-P0)-conditional anyway.*

7 Compatibility with the absorption condition

The absorption condition (Eq. (15) of Paper III) requires $c\,p_0(g_k) \ge (d-1)\ln L_{\mathrm{RG}} + C_{\mathrm{abs}}$.

Corrected in v2. v1 argued that this holds for all k "since $p_0(g) \to \infty$ as $g \to 0$" and appealed to Balaban's stability theorem to keep $g_k \le \gamma_0$ for all k. Both appeals rely on the inverted-sign flow (F1): under the correct asymptotic-freedom flow the effective coupling *grows* along the cascade and remains in $]0, \gamma_0]$ only for $k \le k^*(\beta)$, i.e. $L_{\text{vol}} \le e^{C/g^2+O(1)}$ (2602.0051 v2; suite check 5: at $g = 0.05, \gamma_0 = 0.1, N_c = 3, L_{\text{RG}} = 2$, $k^* = 18639$ and $\log L_{\text{vol}} \le 32.3/g^2$). Within the window, the inequality itself still fails with the polylog floor on p_0 (Remark 4.4); it holds under (H-P0). Summary: *compatibility = window + (H-P0)*, not unconditional validity.

8 Note added in v2

Version 1 of this paper claimed the unconditional verification of Hypothesis 4.2 of Paper III. Following the quantitative audit of the series (ai.viXra:2602.0051/0052/0053/0054/0055, all v2), the claims are corrected as follows.

- v1's "more than sufficient" (after Theorem 1.3) and §7's "holds for all k" are withdrawn: with $c = 2/(1+\beta_0)$ and the polylog floor, the suppression factor is ≈ 0.95 and absorption fails at every scale (excess $10^{4.6}$–$10^{23.2}$). Effectiveness requires (H-P0) (2602.0052 v2). [Suite check 4]

- v1's premises "$p_0(g_k) \to \infty$ as $g_k \to 0$ along the flow" and "Balaban's stability theorem ensures $g_k \le \gamma_0$ for all k" used the inverted-sign flow (F1). Corrected: all statements are windowed, $k \le k^*(\beta)$, $L_{\text{vol}} \le e^{C/g^2+O(1)}$. [Suite check 5]

- Remarks 4.1–4.2 (slow-field identification; Balaban conditional representation) are unproved interface statements; they are now the explicit hypothesis (H-SFI), on which Lemma 4.3 and Theorem 1.3 are conditional. Cf. the interface lemmas of 2602.0052 v2.

- The $d = 2$ verification (Section 6) is fixed-β: the prefactor $K_\beta(\mathbf{1})/Z(\bar{U}_B)$ is not uniform in β (Remark 6.6). The character-positivity mechanism itself is exact and fully validated. [Suite checks 6–7]

- Reference hygiene: v1's [3] carried no identifier (its earlier treatment is viXra:2505.0139); the audited companions 2602.0051–0055 (v2) are now cited. What survives unconditionally: Lemma 2.1, Remark 2.2, Lemma 3.2, and the character-positivity results of Section 6. [Suite checks 1–3, 6–7]

All numerical adjudications are reproduced by the deterministic companion suite `verify_2602_0056.py` (7 checks) at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME, directory `verification/2602-0`

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