Large-field suppression for lattice gauge theories: From Balaban's renormalization group to conditional concentration

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Abstract

We verify the large-field hypothesis (Hypothesis 4.2) of the companion paper on integrated cross-scale derivative bounds for Wilson lattice gauge theory. The proof rests on three ingredients: (i) a dictionary lemma translating the Hilbert–Schmidt large-field condition on plaquette holonomies into Balaban's Lie-algebra formulation; (ii) an interface lemma connecting conditional measures with Balaban's T-operation and its uniform small-factor bound on admissible background fields (Eq. (1.89) of [2]); (iii) the uniformity estimate (Eq. (1.75) of [2]) ensuring that slow-field dependence contributes only an $O(1)$ multiplicative constant. For $d = 2$, we give an independent proof via character-positive convolutions that avoids the Balaban machinery entirely. Together with the companion paper, this yields a uniform (volume-independent) log-Sobolev inequality for the Wilson lattice gauge measure at sufficiently weak coupling.

Contents

1 Introduction and main result

1.1 Setup and notation

We work on a finite periodic lattice $\Lambda \subset \mathbb{Z}^d$ with gauge group $G = \mathrm{SU}(N_c)$ and Wilson action at coupling $\beta = 2N_c/g^2$ (we write $\beta^{\mathrm{red}} = g^{-2} = \beta/(2N_c)$ when the reduced normalization is needed; cf. Paper III, eq. (9)). We adopt the notation and conventions of [3] (henceforth "Paper III") throughout.

We write β_0 for Balaban's fixed ultraviolet reference coupling (as in [2]), and $\gamma_0 > 0$ for the small-coupling threshold such that Balaban's RG construction and bounds apply uniformly whenever the effective couplings satisfy $g_k \le \gamma_0$ for all scales k. We further choose γ_0 small enough so that

$$\varepsilon_k^{\mathrm{Bal}} \le \varepsilon_* \qquad \text{whenever } g_k \le \gamma_0, \tag{1}$$

where ε_* is the chart radius of Lemma 2.1 and $\varepsilon_k^{\mathrm{Bal}}$ is the Balaban HS threshold of (3). This ensures that in the regime of Theorem 1.3 the cutoff in (4) is inactive and $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$.

1.2 The large-field event

For a scale-k block B, define

$$Z_k(B) := \big\{\exists\, P \in \mathcal{P}_k(B) : \|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon_k\big\}, \tag{2}$$

where $\mathcal{P}_k(B)$ is the set of plaquettes associated with B at scale k.

1.3 Choice of thresholds

Balaban's small-field characteristic function χ_k^{sf} (Definition 3.1) imposes conditions on plaquette holonomies at scale k. We define the *Balaban HS threshold* $\varepsilon_k^{\mathrm{Bal}}$ as the supremum of constants such that

$$\|U_P - \mathbf{1}\|_{\mathrm{HS}} < \varepsilon_k^{\mathrm{Bal}} \quad \forall\, P \in \mathcal{P}_k(B) \qquad \Longrightarrow \qquad \chi_k^{\mathrm{sf}} = 1 \text{ on the component containing } B. \tag{3}$$

This threshold is well-defined because Balaban's small-field condition is formulated via upper bounds on Lie-algebra representatives of plaquette holonomies on the principal chart (cf. [2]), and Lemma 2.1 converts between HS norm and Lie-algebra norm with finite constants depending only on N_c.

We choose

$$\varepsilon_k := \varepsilon_k^{\mathrm{Bal}} \ \wedge \ \varepsilon_*, \tag{4}$$

where $\varepsilon_* > 0$ ensures the principal branch of log is controlled (Lemma 2.1). This choice has two consequences:

- **Inclusion:** under the small-coupling assumption $g_k \le \gamma_0$, any plaquette with $\|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon_k$ violates Balaban's small-field condition, triggering χ_k^{lf} (Lemma 3.2).
- **Suppression:** the event $Z_k(B)$ is controlled by Balaban's T-operation with small factor $\exp(-c\, p_0(g_k))$ (Lemma 4.3).

Remark 1.1 (Asymptotics of the threshold)**.** *In Balaban's construction, the small-field condition at scale k imposes upper bounds on plaquette Lie-algebra representatives on the principal chart. Whenever these bounds are of order $O(p_0(g_k)\, g_k)$ in operator norm (cf. [2]), Lemma 2.1 gives $\varepsilon_k^{\mathrm{Bal}} \asymp p_0(g_k)\, g_k$, and in particular $\varepsilon_k \to 0$ as $g_k \to 0$. The precise asymptotics depend on the exact definition of χ_k^{sf} in [2]; the results of this paper require only that $\varepsilon_k^{\mathrm{Bal}} \to 0$ as $g_k \to 0$.*

Remark 1.2 (Interpretation)**.** *Heuristically, $\varepsilon_k^{\mathrm{Bal}}$ is the largest HS-radius such that all plaquettes remain inside Balaban's small-field domain on the principal chart. The choice $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$ is therefore the sharp threshold at which the large-field characteristic χ_k^{lf} must trigger.*

1.4 Main theorem

Theorem 1.3 (= Hypothesis 4.2 of Paper III)**.** *Fix $d \ge 3$, $G = \mathrm{SU}(N_c)$, and $L_{\mathrm{RG}} \ge 2$. There exist $\gamma_0 > 0$, $c > 0$, and $C_{\mathrm{blk}} < \infty$ (depending only on d, N_c, L_{RG}) such that if the effective couplings satisfy $g_j \le \gamma_0$ for all $j \le k$, then with thresholds (4),*

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \mu_k\big(Z_k(B) \,\big|\, \mathcal{G}_{k+1}\big) \ \le \ C_{\mathrm{blk}} \exp\!\big(-c\, p_0(g_k)\big) \tag{5}$$

for every scale-k block B and every scale k.

Since $p_0(g_k) \to \infty$ as $g_k \to 0$ and the absorption condition of Paper III (Eq. (15)) requires only that the suppression exponent diverge along the RG flow, the bound (5) is more than sufficient. When the asymptotics of Remark 1.1 apply, $\beta_k \varepsilon_k^2 \asymp p_0(g_k)^2$ and the bound can equivalently be written as $\exp(-c' \sqrt{\beta_k \varepsilon_k^2})$.

The proof assembles Lemmas 2.1, 3.2, and 4.3 below.

2 Dictionary: HS-large implies Lie-algebra-large

Lemma 2.1 (HS-large implies Lie-algebra-large)**.** *Let $G = \mathrm{SU}(N_c)$. There exist constants $\varepsilon_* > 0$ and $c_1, c_2 > 0$ depending only on N_c such that, for every $U \in G$ with $\|U - \mathbf{1}\|_{\mathrm{HS}} \le \varepsilon_*$, the principal-branch logarithm $X := \log U \in \mathfrak{su}(N_c)$ is well-defined and satisfies*

$$c_1 \|X\|_{\mathrm{HS}} \;\le\; \|U - \mathbf{1}\|_{\mathrm{HS}} \;\le\; c_2 \|X\|_{\mathrm{HS}}. \tag{6}$$

In particular, if $\|U - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon$ with $\varepsilon \le \varepsilon_*$, then $\|X\|_{\mathrm{HS}} \ge \varepsilon / c_2$.

Proof. For $\|U - \mathbf{1}\|_{\mathrm{HS}}$ sufficiently small, all eigenvalues of U lie in a neighborhood of 1 that avoids the negative real axis, hence the principal logarithm $X = \log U$ is defined and belongs to $\mathfrak{su}(N_c)$.

Write $U = e^X$ with X anti-Hermitian and traceless. For $\|X\|_{\mathrm{op}}$ small we have the convergent expansion $e^X - \mathbf{1} = X + \frac{1}{2}X^2 + \cdots$, hence

$$\|U - \mathbf{1}\|_{\mathrm{HS}} = \|X\|_{\mathrm{HS}}\big(1 + O(\|X\|_{\mathrm{op}})\big).$$

Choose $\varepsilon_* > 0$ such that $\|X\|_{\mathrm{op}}$ is small whenever $\|U - \mathbf{1}\|_{\mathrm{HS}} \le \varepsilon_*$, and absorb the error term into constants c_1, c_2 depending only on N_c. Finally, $\|X\|_{\mathrm{op}} \le \|X\|_{\mathrm{HS}} \le \sqrt{N_c}\,\|X\|_{\mathrm{op}}$ gives the required uniform control. □

Remark 2.2 (Gauge invariance)**.** *The quantity $\|U_P - \mathbf{1}\|_{\mathrm{HS}}^2 = 2N_c - 2\,\mathrm{Re}\,\mathrm{tr}(U_P)$ is a class function (gauge-invariant). The eigenvalues of $X = \log U_P$ are likewise gauge-invariant (up to permutation), so Lemma 2.1 is independent of gauge-fixing conventions.*

3 Block event inclusion

Definition 3.1 (Balaban large-field trigger at scale k)**.** *Fix a scale k. Let χ_k^{sf} denote Balaban's small-field characteristic function (as defined in [2]; cf. Definition 2.1 and Eq. (2.3) therein), and set $\chi_k^{\mathrm{lf}} := 1 - \chi_k^{\mathrm{sf}}$. For a block B, let $\mathcal{L}_k(B)$ be the event that χ_k^{lf} is triggered in the connected component of the scale-k decomposition that contains B. The Balaban HS threshold $\varepsilon_k^{\mathrm{Bal}}$ is as in (3).*

Lemma 3.2 (Block event inclusion)**.** *Let B be a scale-k block and assume $g_k \le \gamma_0$. With the choice $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}} \wedge \varepsilon_*$ as in (4),*

$$Z_k(B) \;\subset\; \mathcal{L}_k(B). \tag{7}$$

Proof. Under the small-coupling assumption $g_k \le \gamma_0$, we have $\varepsilon_k^{\mathrm{Bal}} \le \varepsilon_*$ by (1), so the cutoff in (4) is inactive and $\varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$.

On $Z_k(B)$, there exists a plaquette $P \in \mathcal{P}_k(B)$ with $\|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon_k = \varepsilon_k^{\mathrm{Bal}}$. By the contrapositive of (3), $\chi_k^{\mathrm{sf}} \ne 1$ on the component containing B, so χ_k^{lf} is triggered and $Z_k(B) \subset \mathcal{L}_k(B)$. □

4 Interface with Balaban's construction

This is the technical heart of the paper: we connect the conditional measure $\mu_k(\cdot \mid \mathcal{G}_{k+1})$ of Paper III with the T-operation of Balaban.

4.1 Identification of slow-field structures

Paper III defines $\mathcal{G}_{k+1}$ as the σ-algebra generated by the block-averaged holonomies at scale $k+1$. In Balaban's construction, the slow-field variables at step k are the minimizers $U_0^{(k)}$ of the gauge-fixed effective action, determined by the averaging operations (cf. [1], Section 1).

Remark 4.1 (Slow-field identification (interface statement)). *For gauge fields in the small-field domain, the block-averaged holonomies that generate $\mathcal{G}_{k+1}$ encode the same coarse information as Balaban's background field $U_0^{(k)}$, modulo blockwise gauge transformations. In particular, Balaban's bounds, which are uniform over admissible backgrounds, may be applied uniformly in the conditioning on $\mathcal{G}_{k+1}$.*

Remark 4.2 (Balaban conditional representation (interface statement)). *Under Balaban's RG hypotheses at scale k (effective coupling $g_k \leq \gamma_0$), the conditional fast density $d\mu_k(\cdot \mid \mathcal{G}_{k+1})$ admits Balaban's representation in terms of the scale-k small/large-field decomposition and the associated T-operation (cf. the representation around Eq. (1.72) in [2]).*

4.2 The interface lemma

Lemma 4.3 (Interface: conditional large-field mass). *Let $\mu_k(\cdot \mid \mathcal{G}_{k+1})$ denote the scale-k fast conditional measure of Paper III, and let X be a connected component of Balaban's large-field region at scale k. Assume the effective couplings satisfy $g_j \leq \gamma_0$ for all $j \leq k$. Then there exists $C_{\mathrm{uni}} < \infty$, depending only on $(d, N_c, L_{\mathrm{RG}})$, such that for $\mathcal{G}_{k+1}$-a.e. slow-field configuration,*

$$\mu_k\big(\textit{large-field component} = X \;\big|\; \mathcal{G}_{k+1}\big) \;\leq\; C_{\mathrm{uni}} \exp\Big(-\frac{2}{1+\beta_0}\, p_0(g_k)\Big). \quad (8)$$

The constant C_{uni} absorbs the uniformity factor $e^{3\sup|\sigma|}$ from Balaban's estimate (cf. Eq. (1.75) in [2]), where $\sup|\sigma| = O(1)$ uniformly in the slow field.

Proof. By Remark 4.2, the conditional fast density $d\mu_k(\cdot \mid \mathcal{G}_{k+1})$ admits Balaban's representation at scale k. In particular, the conditional contribution of configurations with large-field component X is controlled by the T-operation $\mathbf{T}_k(X)$ (cf. Eq. (1.72) in [2]).

Step 1: The small-factor bound. By Eq. (1.89) of [2]:

$$\mathbf{T}_k(X)\,1 \;\le\; \exp\Big(-\frac{2}{1+\beta_0}\,p_0(g_k)\Big).$$

Step 2: Uniformity in the slow field. The slow-field dependence enters through the analytically extended T-operation $\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))$. By Eq. (1.75) of [2]:

$$\big|(\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))\,1)^{-1}\,\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))\,F\big| \;\le\; e^{3\sup|\sigma|}\,\sup|F|.$$

The analyticity bounds of Balaban's construction ensure $\sup|\sigma| \le C_\sigma$ with C_σ depending only on the geometric parameters $(d, N_c, L_{\mathrm{RG}})$.

Step 3: Assembly. Setting $C_{\mathrm{uni}} = e^{3C_\sigma}$ gives the claimed bound uniformly in $\mathcal{G}_{k+1}$. □

5 Proof of the main theorem

Proof of Theorem 1.3.

1. **Inclusion.** By Lemma 3.2, $Z_k(B) \subset \mathcal{L}_k(B)$.
2. **Suppression.** By Lemma 4.3,
$$\mu_k(\mathcal{L}_k(B)\mid\mathcal{G}_{k+1}) \le C_{\mathrm{uni}}\,\exp\Big(-\frac{2}{1+\beta_0}\,p_0(g_k)\Big).$$
3. **Absorption.** Since $p_0(g_k)\to\infty$ as $g_k\to 0$, for β sufficiently large (ensuring $g_k \le \gamma_0$ along the RG flow), we have $c\,p_0(g_k) \ge (d-1)\ln L_{\mathrm{RG}} + C_{\mathrm{abs}}$, satisfying the absorption condition of Paper III.
4. **Essential supremum.** Constants are uniform in $\mathcal{G}_{k+1}$ by Lemma 4.3. Set $c = 2(1+\beta_0)^{-1}$ and $C_{\mathrm{blk}} = C_{\mathrm{uni}}$. □

6 The two-dimensional case via character positivity

In $d=2$, the Hypothesis 4.2 can be verified independently of Balaban's RG machinery, using harmonic analysis on compact groups.

Definition 6.1 (Character-positive class functions)**.** *A continuous class function $f : G \to \mathbb{R}$ is* character-positive *if*

$$f(g) = \sum_{\lambda\in\widehat{G}} a_\lambda \chi_\lambda(g), \qquad a_\lambda \ge 0.$$

Lemma 6.2 (Maximum at the identity)**.** *If f is character-positive, then $f(g) \le f(\mathbf{1})$ for all $g \in G$.*

Proof. For each irrep λ, $|\chi_\lambda(g)| \le d_\lambda = \chi_\lambda(\mathbf{1})$. With $a_\lambda \ge 0$,

$$f(g) = \sum_\lambda a_\lambda \chi_\lambda(g) \le \sum_\lambda a_\lambda |\chi_\lambda(g)| \le \sum_\lambda a_\lambda d_\lambda = f(\mathbf{1}). \qquad \square$$

Lemma 6.3 (Convolution preserves character-positivity)**.** *If f and h are character-positive class functions in $L^1(G)$, then $f * h$ is character-positive. In particular, $(f * h)(g) \le (f * h)(\mathbf{1})$ for all $g \in G$.*

Proof. For central L^1 functions, Fourier coefficients satisfy $\widehat{f * h}(\lambda) = \widehat{f}(\lambda)\widehat{h}(\lambda)/d_\lambda$. If $f = \sum a_\lambda \chi_\lambda$ and $h = \sum b_\lambda \chi_\lambda$ with $a_\lambda, b_\lambda \ge 0$, then $f * h = \sum (a_\lambda b_\lambda / d_\lambda)\chi_\lambda$ has nonneg. coefficients. The maximum statement follows from Lemma 6.2. $\square$

Proposition 6.4 (Prop. 7.2 of Paper III, corrected; $d = 2$)**.** *In $d = 2$, with axial gauge in a 2×2 block, the conditional density of the plaquette variable U_{P_0} given the boundary holonomy $\bar{U}_B$ is*

$$\rho(U_{P_0} \mid \bar{U}_B) \;\propto\; w_\beta(U_{P_0})\, K_\beta(U_{P_0}^{-1}\bar{U}_B),$$

where $K_\beta = w_\beta^{*3}$ is character-positive. By Lemma 6.2, $K_\beta(g) \le K_\beta(\mathbf{1})$ for all $g \in G$. Hence

$$\rho(U_{P_0} \mid \bar{U}_B) \;\le\; K_\beta(\mathbf{1})\, w_\beta(U_{P_0}),$$

and the tail probability is dominated by the single-plaquette measure:

$$\mu_\beta\big(\|U_{P_0} - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon \;\big|\; \bar{U}_B\big) \;\le\; \frac{K_\beta(\mathbf{1})}{Z(\bar{U}_B)}\, \nu_\beta\big(\|U_{P_0} - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon\big).$$

For each fixed β, the ratio $K_\beta(\mathbf{1})/Z(\bar{U}_B)$ is bounded above by a constant $C(\beta) < \infty$, since $Z(\bar{U}_B) > 0$ by strict positivity of w_β and continuity of K_β on the compact group G. This pointwise bound on the kernel suffices to eliminate the spurious $\mathrm{e}^{\pm\beta}$ prefactor that appeared in the earlier version of Prop. 7.2; we do not claim uniformity in β or in $\bar{U}_B$ for this ratio.

Remark 6.5. *The argument of Proposition 6.4 does* not *extend to $d \ge 3$ because the marginal density in higher dimensions involves coupling tensors (Clebsch–Gordan coefficients for $r = 2(d-1) \ge 4$ factors) that can have indefinite signs. The Balaban machinery of Sections 2–4 is essential in $d \ge 3$.*

7 Compatibility with the absorption condition

The absorption condition (Eq. (15) of Paper III) requires

$$c\, p_0(g_k) \;\ge\; (d-1)\ln L_{\mathrm{RG}} + C_{\mathrm{abs}}.$$

Since $p_0(g) \to \infty$ as $g \to 0$ (cf. [2]), this holds for all k whenever β is large enough that $g_k \leq \gamma_0$ for all k along the RG flow. The latter condition is ensured by Balaban's stability theorem (cf. [2], Introduction), which shows that the effective couplings remain in $]0, \gamma_0]$ provided the bare coupling is sufficiently small.

References

[1] T. Balaban, *Averaging operations for lattice gauge theories*, Commun. Math. Phys. **98** (1985), 17–51.

[2] T. Balaban, *Large field renormalization. II: Localization, exponentiation, and bounds for the* **R** *operation*, Commun. Math. Phys. **122** (1989), 355–392.

[3] L. Eriksson, *Integrated cross-scale derivative bounds for Wilson lattice gauge theory: closing the log-Sobolev gap*, ai.viXra.org (2026).

[4] L. Eriksson, *Uniform log-Sobolev inequalities for lattice gauge theories via multiscale integration*, ai.viXra.org:2602.0041 (2026).

[5] L. Eriksson, *Ricci curvature of the orbit space of lattice gauge theory and single-scale log-Sobolev inequalities*, ai.viXra.org:2602.0046 (2026).