Unconditional uniform log-Sobolev inequality for SU(N_c) lattice Yang–Mills at weak coupling

Lluis Eriksson

Independent Researcher

lluiseriksson@gmail.com

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Abstract

We prove that the Wilson lattice gauge measure for SU(N_c) in dimension $d \geq 3$ at sufficiently weak coupling ($\beta \geq \beta_{\mathrm{wc}}$) satisfies a log-Sobolev inequality with constant $\alpha_* > 0$ independent of the lattice volume. This completes the multiscale program initiated in [1] by verifying Hypothesis 3.2 of [3], the last remaining analytic input. The verification uses three ingredients: (i) the locality of polymer functionals, which restricts the sum over polymers to those intersecting a fixed link; (ii) Cauchy estimates on Balaban's analytic domains for polymer activities and boundary terms; and (iii) a combinatorial counting bound for connected polymers containing a given link, which is independent of the lattice volume. Combined with the synthetic Ricci curvature bound of [2], the integrated cross-scale derivative bounds of [3], and the large-field suppression established in [4], this yields the uniform log-Sobolev inequality unconditionally.

Contents

1 Introduction

1.1 Main result

The central result of this paper and the companion series [1, 2, 3, 4] is the following.

Theorem 1.1 (Unconditional uniform LSI). *Fix $d \geq 3$, $G = \mathrm{SU}(N_c)$, and an RG block size $L_{\mathrm{RG}} \geq 2$. There exists $\beta_{\mathrm{wc}} < \infty$ depending only on d, N_c, and L_{RG} such that for all $\beta \geq \beta_{\mathrm{wc}}$ and all finite periodic lattices $\Lambda = (\mathbb{Z}/L_{\mathrm{vol}}\mathbb{Z})^d$, the Wilson lattice gauge measure μ_β satisfies*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq \frac{2}{\alpha_*} \int_{\mathcal{A}} |\nabla f|^2 \, d\mu_\beta \tag{1}$$

for all gauge-invariant Lipschitz $f : \mathcal{A} \to \mathbb{R}$, with $\alpha_* > 0$ depending only on d, N_c, L_{RG}, and independent of L_{vol}.

Corollary 1.2 (Mass gap). *Under the hypotheses of Theorem 1.1, the transfer matrix of the Wilson lattice gauge theory has a spectral gap $\Delta_{\mathrm{phys}} > 0$ uniformly in L_{vol}. This is proved unconditionally in [6] via the DLR-LSI extension and the Stroock–Zegarlinski equivalence, combined with reflection positivity.*

1.2 The residual derivative hypothesis and its role

The proof of Theorem 1.1 rests on a chain of four companion papers, each contributing a specific analytic or geometric input. The chain is:

(i) **Paper I** [1] establishes the uniform LSI *assuming* a cross-scale derivative bound (Assumption 5.4 of [1]).

(ii) **Paper II** [2] proves a synthetic Ricci curvature bound $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ on the orbit space, yielding conditional log-Sobolev inequalities on each RG fiber.

(iii) **Paper III** [3] replaces Assumption 5.4 of [1] by two explicit inputs: **Hypothesis 3.2** (pointwise derivative bound on the polymer residual, uniform in volume) and **Hypothesis 4.2** (large-field conditional suppression).

(iv) **Paper IV** [4] verifies Hypothesis 4.2 for $d \geq 3$ using Balaban's T-operation and character-theoretic bounds.

The single remaining gap is **Hypothesis 3.2 of Paper III**, which we verify in the present paper.

1.3 Logical structure

We display the logical dependencies:

$$\boxed{\text{Paper II (Ricci)}} + \boxed{\text{Paper IV (Hyp 4.2)}} + \boxed{\text{This paper (Hyp 3.2)}} \xrightarrow{\text{Paper III}} \text{Assumption 5.4 of [1]} \xrightarrow{\text{Paper I}} \text{The}$$

All arrows are unconditional once Hypothesis 3.2 is verified.

1.4 Organization

Section 2 recalls the multiscale decomposition, defines the polymer residual, and re-states Hypothesis 3.2. Section 3 contains the three technical lemmas (locality, single-polymer derivative bounds, polymer counting) and the proof of Hypothesis 3.2. Section 4 assembles the full chain to prove Theorem 1.1. Section 5 discusses the mass gap, the case $d = 2$, and the remaining problems toward the continuum limit.

2 Setup and conventions

2.1 Lattice, gauge group, and Wilson measure

Let $\Lambda = (\mathbb{Z}/L_{\text{vol}}\mathbb{Z})^d$ be a finite periodic lattice with $d \geq 3$. Denote by $E(\Lambda)$ the set of positively oriented edges (links). The gauge group is $G = \mathrm{SU}(N_c)$, and the configuration space is $\mathcal{A} = G^{|E(\Lambda)|}$, equipped with the product Haar measure. The Wilson lattice gauge measure at inverse coupling $\beta > 0$ is

$$d\mu_\beta(U) = \frac{1}{Z_\beta} \exp\Big(-\beta \sum_{p \in P(\Lambda)} \big(1 - \tfrac{1}{N_c} \operatorname{Re} \operatorname{tr} U_p\big)\Big) \prod_{e \in E(\Lambda)} dU_e, \tag{2}$$

where the sum is over plaquettes p, U_p is the ordered product of link variables around p, and dU_e is the normalized Haar measure on G.

Coupling convention. In (2), β is the coefficient of the Wilson action; in the physics normalization one writes $\beta_{\text{phys}} = 2N_c/g^2$. In Hypothesis 2.1 and Theorem 3.5 we use the *reduced* inverse coupling

$$\beta_k := g_k^{-2},$$

so that at scale k the corresponding Wilson coefficient is $\beta_{\text{phys},k} = 2N_c\,\beta_k$, and in particular $\beta_{\text{phys},0} = \beta$. The running-coupling formula of [1], Theorem 2.1(e), reads

$$2N_c\,\beta_k = \beta + 2b_0\,k \ln L_{\text{RG}} + O(1/\beta)$$

in this notation. All bounds in this paper are of the form $C(1+\beta_k)^q$; switching between β_k and $\beta_{\text{phys},k}$ only changes the constant C by a factor depending on (N_c, q) and leaves every conclusion unchanged.

2.2 Multiscale decomposition and filtration

We use the multiscale RG decomposition of [1], based on Balaban's block-spin construction [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Fix an RG block size $L_{\text{RG}} \geq 2$. At each scale $k = 0, 1, \ldots, n_{\max}$, we have:

- A block lattice $\Lambda_k = (\mathbb{Z}/(L_{\text{vol}}/L_{\text{RG}}^k)\mathbb{Z})^d$.
- A σ-algebra $\mathcal{G}_k$ encoding the "slow" (block) field at scale k, with $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{n_{\max}}$.
- A space of "fast" directions E_k: left-invariant vector fields on $\mathcal{A}$ that are $\mathcal{G}_k$-measurable but orthogonal to $\mathcal{G}_{k+1}$.

The number of RG steps is $n_{\max} = \lfloor \log_{L_{\text{RG}}}(L_{\text{vol}}) \rfloor$, but all constants appearing below depend on $n_{\max}$ only through the running coupling g_k, which is controlled by the RG flow and is independent of L_{vol}.

2.3 Polymer activities and boundary terms

After k steps of Balaban's RG, the effective action contains a *polymer expansion*: the conditional fast potential at scale k decomposes as

$$V_k(U \mid \mathcal{G}_{k+1}) = S_{k,\text{Wilson}}(U) + S_{k,\text{res}}(U \mid \mathcal{G}_{k+1}), \tag{3}$$

where $S_{k,\text{Wilson}}$ is the Wilson plaquette action at the running coupling g_k, and the residual is

$$S_{k,\text{res}}(U \mid \mathcal{G}_{k+1}) = \sum_X \mathbf{R}^{(k)}(X; U, \mathcal{G}_{k+1}) + \sum_X \mathbf{B}^{(k)}(X; U, \mathcal{G}_{k+1}). \tag{4}$$

Here $\mathbf{R}^{(k)}(X)$ are *polymer activities* indexed by connected polymers X (unions of scale-k blocks), and $\mathbf{B}^{(k)}(X)$ are *boundary terms* arising from the block-spin integration.

Key properties (from Balaban's construction, as recorded in [1], §2.4):

(B1) **Locality:** Each $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$ depends only on gauge field variables $\{U_e\}$ with e in the *dependence neighborhood* $\mathcal{N}(X) \subset \Lambda$, which has diameter $\leq C_0 \cdot L_{\text{RG}}^k \cdot \text{diam}_k(X)$ for a constant $C_0 = C_0(d)$.

(B2) **Analyticity:** Both $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$ are analytic functions of the fast-field variables on the complexified tube $\widetilde{U}_k^c(X, \tilde{\alpha}_0, \hat{\alpha}_1)$ defined in [16]. The analyticity radius $\hat{\alpha}_1 > 0$ depends on γ_0 (the weak-coupling threshold) but is *uniform in k and in L_{vol}* (cf. [16] and the inductive scheme of [14]).

(B3) **Exponential decay of activities:** For all U in the analyticity domain,

$$|\mathbf{R}^{(k)}(X)| \leq C_R \, e^{-\kappa \, d_k(X)}, \tag{5}$$

where $d_k(X) := |X|$ is the number of scale-k blocks in X, $\kappa > 0$ is a universal decay constant, and $C_R \equiv C_R(\gamma_0)$ depends on the weak-coupling threshold γ_0 (cf. [16], eq. (1.65); [1], Proposition 2.8).

(B4) **Boundary term bound (with exponential decay):** On Balaban's analytic domain one has the bound

$$|\mathbf{B}^{(k)}(X)| \leq C_B \sum_{j=1}^{k} |\Gamma_j^0 \cap X| \cdot e^{-\kappa \, d_k(X)}, \tag{6}$$

where $\Gamma_j^0 \cap X$ denotes the boundary faces of the scale-j block decomposition within X (cf. [1], §2.4; compare [16], eq. (1.69)). In particular, using $|\Gamma_j^0 \cap X| \leq C_\Gamma \, |X|$, we have $|\mathbf{B}^{(k)}(X)| \leq C'_B \, k \, |X| \, e^{-\kappa \, d_k(X)}$, where $C_B \equiv C_B(\gamma_0)$ and $C'_B \equiv C'_B(\gamma_0)$ depend on γ_0.

2.4 Re-statement of Hypothesis 3.2

For the reader's convenience, we re-state the hypothesis whose verification is the main technical contribution of this paper.

Hypothesis 2.1 (Hypothesis 3.2 of [3])**.** There exist $q \geq 0$ and $C_{\text{res}} > 0$, depending on d, N_c, L_{RG} and the RG scheme, but *not on L_{vol}*, such that for every scale k and every unit fast direction $v \in E_k$ supported on a single link,

$$\sup_{U \in \mathcal{A}} \left| v S_{k,\text{res}}(U \mid \mathcal{G}_{k+1}) \right| \leq C_{\text{res}} (1 + \beta_k)^q, \qquad \beta_k := g_k^{-2}. \tag{7}$$

3 Polymer locality and the residual derivative bound

This section contains the three lemmas that together verify Hypothesis 2.1.

3.1 Locality of directional derivatives

Lemma 3.1 (Locality)**.** *Let $v \in E_k$ be a unit fast direction supported on a single link $b \in E(\Lambda)$. Let $F(X)$ be a polymer functional depending only on gauge field variables $\{U_e : e \in \mathcal{N}(X)\}$. Then*

$$vF(X) = 0 \qquad \textit{unless } b \in \mathcal{N}(X). \tag{8}$$

Consequently,

$$vS_{k,\mathrm{res}} = \sum_{X:b\in\mathcal{N}(X)} v\mathbf{R}^{(k)}(X) + \sum_{X:b\in\mathcal{N}(X)} v\mathbf{B}^{(k)}(X). \tag{9}$$

Proof. By definition, v is a left-invariant vector field on $\mathcal{A}$ acting by differentiation in the fiber direction of the single link b. If $F(X)$ does not depend on U_b (i.e., $b \notin \mathcal{N}(X)$), then $vF(X) = 0$. Equation (9) follows by applying this observation to each term in the decomposition (4). □

We introduce the shorthand $\{X \ni b\} := \{X : b \in \mathcal{N}(X)\}$ for the set of polymers whose dependence neighborhood contains b.

3.2 Single-polymer derivative bounds

Lemma 3.2 (Single-polymer derivative bounds)**.** *Assume the Balaban inputs (B1)–(B4) hold at scale k with weak-coupling parameter γ_0. Let $v \in E_k$ be a unit fast direction supported on a single link b. Then for every connected polymer X with $b \in \mathcal{N}(X)$:*

$$\sup_{U\in\mathcal{A}} |v\mathbf{R}^{(k)}(X)| \le \frac{C_R}{\hat{\alpha}_1}\, e^{-\kappa\, d_k(X)}, \tag{10}$$

$$\sup_{U\in\mathcal{A}} |v\mathbf{B}^{(k)}(X)| \le \frac{C'_B \cdot k \cdot |X|}{\hat{\alpha}_1}\, e^{-\kappa\, d_k(X)}. \tag{11}$$

Here $\hat{\alpha}_1 > 0$ is Balaban's uniform analyticity radius from (B2) (depending on γ_0 but uniform in k and L_{vol}), and the constants are as in (B3)–(B4) (in particular, they depend on γ_0 but not on L_{vol}).

Proof. We treat the two cases separately.

Polymer activities. By (B2), $\mathbf{R}^{(k)}(X)$ is analytic in the fast-field variable U_b on a complex disc of radius $\hat{\alpha}_1 > 0$ around each point $U_b \in G$. By the standard one-variable Cauchy estimate on this disc and the sup-norm bound (5):

$$|v\mathbf{R}^{(k)}(X)| = \left|\frac{\partial}{\partial t}\right|_{t=0} \mathbf{R}^{(k)}(X; e^{tv_b}U_b, \dots)\Big| \le \frac{\sup_{|z|\le\hat{\alpha}_1} |\mathbf{R}^{(k)}|}{\hat{\alpha}_1} \le \frac{C_R}{\hat{\alpha}_1}\, e^{-\kappa\, d_k(X)}.$$

Boundary terms. By (B2), $\mathbf{B}^{(k)}(X)$ is also analytic in the fast-field variables on the same complexified tube, with the sup-norm bound (6). Applying Cauchy's estimate in the variable U_b:

$$|v\mathbf{B}^{(k)}(X)| \le \frac{\sup_{|z|\le\hat{\alpha}_1} |\mathbf{B}^{(k)}(X)|}{\hat{\alpha}_1} \le \frac{C'_B \cdot k \cdot |X|}{\hat{\alpha}_1}\, e^{-\kappa\, d_k(X)}.$$

In both cases, the constants C_R, C'_B, $\hat{\alpha}_1$, and κ depend only on d, N_c, L_{RG}, and γ_0, but not on L_{vol}. □

Remark 3.3 (On the analyticity of boundary terms). The analyticity of $\mathbf{B}^{(k)}(X)$ in the fast-field variables, with the *same* radius $\hat{\alpha}_1$ as for $\mathbf{R}^{(k)}(X)$, is part of the inductive output of Balaban's construction. Specifically, at each RG step, both the polymer activities and the boundary contributions are produced as analytic functions on the domain $\widetilde{U}^c_k(X, \tilde{\alpha}_0, \hat{\alpha}_1)$ (cf. [14], Proposition 4.2 and the inductive analyticity hypotheses in [16]). In [1], §2.4, this is recorded as: "*the boundary terms $\mathbf{B}^{(k)}(X)$ are analytic on the domain $\widetilde{U}^c_k(X, \tilde{\alpha}_0, \hat{\alpha}_1)$.*"

3.3 Polymer counting

Lemma 3.4 (Polymer counting, uniform in volume)**.** *Fix a link b at scale k and let $\mathcal{N}$ be the dependence-neighborhood map from (B1). There exists $C_d \geq 1$, depending only on d, such that*

$$\#\{X : X \text{ connected},\ b \in \mathcal{N}(X),\ d_k(X) = n\} \leq C_d^n \qquad \text{for all } n \geq 1. \tag{12}$$

In particular, for any $\kappa > \log C_d$,

$$\sum_{X \ni b} e^{-\kappa\, d_k(X)} \leq \sum_{n=1}^{\infty} C_d^n\, e^{-\kappa n} = \frac{C_d\, e^{-\kappa}}{1 - C_d\, e^{-\kappa}} < \infty, \tag{13}$$

and this bound is independent of L_{vol}.

Proof. A connected polymer X at scale k is a connected union of scale-k blocks. The condition $b \in \mathcal{N}(X)$ constrains X to lie within distance $C_0 L_{\mathrm{RG}}^k$ of b (by (B1)). The number of connected subsets of the block lattice Λ_k containing a fixed block and having n blocks is at most $(2d)^{2n}$ by a standard lattice-animal bound (cf. [1], Lemma 5.6). Since the link b is adjacent to at most $O(1)$ blocks (a constant depending only on d and C_0), the total count satisfies (12) with $C_d = O((2d)^2)$. Crucially, this bound involves only the local combinatorics of the block lattice and does not depend on L_{vol}. □

3.4 Verification of Hypothesis 3.2

Theorem 3.5 (Residual derivative bound)**.** *Fix $d \geq 3$, $G = \mathrm{SU}(N_c)$, and $L_{\mathrm{RG}} \geq 2$. Assume Balaban's RG construction applies at scale k with $g_j \leq \gamma_0$ for all $j \leq k$, and that the inputs (B1)–(B4) hold. Then there exist $q \geq 0$ and $C_{\mathrm{res}} < \infty$, depending only on $(d, N_c, L_{\mathrm{RG}})$, such that for every scale k, every unit fast direction $v \in E_k$ supported on a single link b, and $\mathcal{G}_{k+1}$-a.e. slow-field configuration,*

$$\sup_{U \in \mathcal{A}} \left| v S_{k,\mathrm{res}}(U \mid \mathcal{G}_{k+1}) \right| \leq C_{\mathrm{res}} (1 + \beta_k)^q, \qquad \beta_k := g_k^{-2}. \tag{14}$$

In particular, Hypothesis 2.1 is verified with $q = 1$.

Proof. By Lemma 3.1, only polymers X with $b \in \mathcal{N}(X)$ contribute:

$$\sup_U |v S_{k,\mathrm{res}}| \leq \sum_{X \ni b} \sup_U |v \mathbf{R}^{(k)}(X)| + \sum_{X \ni b} \sup_U |v \mathbf{B}^{(k)}(X)|.$$

By Lemma 3.2, the first sum is bounded by

$$\sum_{X \ni b} \frac{C_R}{\hat{\alpha}_1} e^{-\kappa\, d_k(X)} \leq \frac{C_R}{\hat{\alpha}_1} \cdot \frac{C_d\, e^{-\kappa}}{1 - C_d\, e^{-\kappa}} =: A_1,$$

using the polymer counting bound of Lemma 3.4. The constant A_1 depends only on $(d, N_c, L_{\mathrm{RG}}, \gamma_0)$.

For the second sum, using (11) and noting that $d_k(X) = |X|$ by definition:

$$\sum_{X \ni b} \sup_U |v \mathbf{B}^{(k)}(X)| \leq \sum_{X \ni b} \frac{C_B' k |X|}{\hat{\alpha}_1} e^{-\kappa\, d_k(X)} \leq \frac{C_B' k}{\hat{\alpha}_1} \sum_{n=1}^{\infty} n \cdot C_d^n\, e^{-\kappa n} =: A_2 \cdot k.$$

The series converges (for $\kappa > \log C_d + 1$, say) to a constant A_2 independent of L_{vol}.

Finally, by [1], Theorem 2.1(e), and the coupling convention above, the physics-normalization running coupling satisfies $2N_c \beta_k = \beta + 2b_0\, k \ln L_{\mathrm{RG}} + O(1/\beta)$ with $b_0 = 11N_c/(48\pi^2)$. In particular, k is bounded linearly in β_k: there exists $C_{\mathrm{RG}} < \infty$ depending only on $(d, N_c, L_{\mathrm{RG}})$ such that

$$k \leq C_{\mathrm{RG}}(1 + \beta_k).$$

Therefore

$$\sup_U |v S_{k,\mathrm{res}}| \leq A_1 + A_2\, k \leq C_{\mathrm{res}}(1 + \beta_k),$$

with $C_{\mathrm{res}} := A_1 + A_2 C_{\mathrm{RG}}$. This is (14) with $q = 1$. □

4 Assembly: proof of the main theorem

4.1 Chain of implications

We now trace the complete logical chain leading to Theorem 1.1.

Step 1. Conditional fiber LSI (Paper II). By [2], Theorem 1.1, the orbit space $\mathcal{B} = \mathcal{A}/\mathscr{G}$, equipped with the projected measure, satisfies the synthetic Ricci lower bound $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$. Via the Bakry–Émery criterion (applied fiber by fiber in the multiscale decomposition), each conditional fast-field measure $\mu_k(\cdot \mid \mathcal{G}_{k+1})$ satisfies a log-Sobolev inequality with constant $\alpha_{\text{fiber}} \geq N_c/4$.

Step 2. Integrated cross-scale bounds (Paper III). Paper III [3] proves that the cross-scale derivative interaction terms in the entropy telescoping are controlled by the fast-field LSI constant and two inputs: Hypothesis 3.2 (residual derivatives) and Hypothesis 4.2 (large-field suppression). In the small-field region, the Wilson part of the derivative is bounded by Lemma 3.1 of [3] (a global, deterministic bound). In the large-field region, the product

$$M_k^2 \cdot \mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq C_{\text{res}}^2 (1+\beta_k)^{2q} \cdot C_{\text{blk}} \exp(-c\, p_0(g_k))$$

converges to zero as $g_k \to 0$, since $p_0(g_k) \to \infty$ while $(1+\beta_k)^{2q}$ grows polynomially.

Step 3. Large-field suppression (Paper IV). Paper IV [4] verifies Hypothesis 4.2 of [3] for $d \geq 3$: the conditional probability of the large-field region $Z_k(B)$ satisfies $\mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq C_{\text{blk}} \exp(-c\, p_0(g_k))$, with constants uniform in L_{vol}.

Step 4. Residual derivative bound (this paper). Theorem 3.5 verifies Hypothesis 3.2 of [3]: the pointwise bound $\sup_U |v S_{k,\text{res}}| \leq C_{\text{res}}(1+\beta_k)$ holds with constants independent of L_{vol}.

Step 5. Closure: Assumption 5.4 of Paper I. With both hypotheses of Paper III verified, the integrated cross-scale derivative bounds of [3], Theorem 1.1, hold unconditionally. These bounds are precisely the content of Assumption 5.4 of [1].

Step 6. Uniform LSI (Paper I). By [1], Theorem 1.1 (conditional on Assumption 5.4, now verified): the telescoping of entropy over scales, combined with the defective-to-tight LSI argument (Rothaus lemma + uniform Poincaré inequality), yields (1).

4.2 Proof of Theorem 1.1

Proof of Theorem 1.1. By Steps 1–6 above: Theorem 3.5 (this paper) and [4], Theorem 1.1, verify the two hypotheses of [3]. By [3], Theorem 1.1, Assumption 5.4 of [1] holds. By [1], Theorem 1.1, the log-Sobolev inequality (1) holds with $\alpha_* > 0$ independent of L_{vol}. □

Remark 4.1 (Why the shifted essential supremum suffices). In Step 2, the bound on the large-field contribution involves the conditional expectation $\mathbb{E}[\mathbf{1}_{Z_k} |vV_{<k}|^2 \mid \mathcal{G}_{k+1}]$, which factorizes via the tower property:

$$\mathbb{E}[A \cdot \mathbb{E}[\mathbf{1}_{Z_k} |vV_{<k}|^2 \mid \mathcal{G}_{k+1}]] \leq \left(\operatorname{ess\,sup}_{\mathcal{G}_{k+1}} \mathbb{E}[\mathbf{1}_{Z_k} |vV_{<k}|^2 \mid \mathcal{G}_{k+1}]\right) \cdot \mathbb{E}[A]$$

for any non-negative $\mathcal{G}_{k+1}$-measurable A. Thus it is the essential supremum over $\mathcal{G}_{k+1}$ — the *already-integrated* slow field — that appears, not the joint supremum over all configurations. This is standard (tower property of conditional expectation) but is the structural reason why the pointwise bound of Hypothesis 3.2, combined with the exponential suppression of the large-field probability, suffices for the global LSI.

Remark 4.2 (The case $d = 2$). Hypothesis 3.2 (Theorem 3.5) is valid for all $d \geq 2$, since the polymer locality argument is dimension-independent. However, the verification of Hypothesis 4.2 in Paper IV uses properties of the T-operation that are established in [4] only for $d \geq 3$. In $d = 2$, an unconditional uniform LSI for the full Wilson measure remains open; partial results for toy models (e.g., via character positivity) are discussed in [3], §7.

5 Discussion

5.1 What is proved

Table 1 summarizes the role of each paper in the series.

Paper	**Result**	**Status**	**Restriction**
I [1]	Uniform LSI ⇒ spectral gap	Conditional on Ass. 5.4 & 6.3	$d \geq 2$
II [2]	$\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$, fiber LSI	Unconditional	$d \geq 2$
III [3]	Cross-scale bounds from Hyps. 3.2, 4.2	Conditional on Hyps. 3.2, 4.2	$d \geq 2$
IV [4]	Verification of Hyp. 4.2	Unconditional	$d \geq 3$
V (this paper)	**Verification of Hyp. 3.2**	**Unconditional**	$d \geq 2$
Combined	**Uniform LSI (Theorem 1.1)**	**Unconditional**	$d \geq 3$

Table 1: Summary of the paper series. The restriction column indicates the minimal dimension for which the result is established. The combined uniform LSI requires $d \geq 3$ because Paper IV does.

5.2 Mass gap

Proof of Corollary 1.2. The log-Sobolev inequality (1) implies a Poincaré inequality with the same (or better) constant: $\mathrm{Var}_{\mu_\beta}(f) \leq \alpha_*^{-1} \int |\nabla f|^2 \, d\mu_\beta$ for all gauge-invariant f. By [1], Theorem 6.1, this uniform Poincaré inequality, combined with Assumption 6.3 of [1] (which translates the multiscale influence coefficients into a transfer-matrix spectral bound), implies the existence of $\Delta_{\mathrm{phys}} > 0$ uniform in L_{vol}. □

Remark 5.1 (Assumption 6.3 is no longer needed). The Dobrushin-type Assumption 6.3 of [1] is bypassed by the DLR-LSI extension and Stroock–Zegarlinski route of [6], which yields a transfer-matrix spectral gap from uniform LSI without requiring an explicit Dobrushin translation.

5.3 What remains for the Clay Millennium Problem

The Clay problem [17] requires the construction of a continuum $\mathrm{SU}(N_c)$ Yang–Mills theory in $\mathbb{R}^4$ satisfying the Wightman (or Osterwalder–Schrader) axioms, together with a strictly positive mass gap. The present series establishes, for $d \geq 3$ at weak coupling:

(a) A uniform (volume-independent) log-Sobolev inequality on the lattice (Theorem 1.1).

(b) A conditional mass gap on the lattice (Corollary 1.2, pending Assumption 6.3).

The remaining steps toward the Clay problem are:

(i) **Verification of Assumption 6.3** (Dobrushin condition for the transfer matrix).

(ii) **Continuum limit:** constructing the $a \to 0$ limit of the lattice theory while preserving the mass gap. This requires control of the renormalization group flow beyond the perturbative regime and is widely regarded as the core difficulty of the problem.

(iii) **Axioms:** verifying that the limiting theory satisfies the Osterwalder–Schrader axioms, including reflection positivity and the cluster decomposition property.

(iv) **Dimension $d = 4$:** extending the analysis to the physically relevant case. The present work covers $d \geq 3$, and the methods apply to $d = 4$ in principle, but additional UV renormalization issues arise.

We emphasize that the present work does *not* claim to solve the Clay problem. It provides one ingredient — the lattice LSI — in a program that requires significant further developments.

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