From Uniform Log-Sobolev Inequality to Mass Gap for Lattice Yang–Mills at Weak Coupling: a Conditional and Windowed Assembly

(Version 2 of ai.viXra:2602.0054; v1 title without the subtitle claimed an unconditional gap)

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Abstract

This paper assembles the route from the uniform log-Sobolev inequality (LSI) on periodic tori to a transfer-matrix spectral gap for SU(N_c) lattice Yang–Mills in $d \geq 3$ at weak coupling: periodic LSI + boundary-uniform RG outputs $\Rightarrow$ DLR-LSI $\Rightarrow$ Stroock–Zegarlinski mixing $\Rightarrow$ exponential clustering $\Rightarrow$ (reflection positivity) $\Rightarrow$ $\Delta_{\mathrm{phys}} > 0$. **Version 2 corrects the logical status of this assembly after a quantitative audit** (companion numerical suite; Appendix A). (i) v1 claimed the route "bypasses any explicit Dobrushin contraction estimate". This is withdrawn: the DLR-LSI input (Theorem 5.1) invokes the multiscale fiber assembly of [2], whose inter-block step *is* a Dobrushin-type condition — made explicit as (H-DOB-blk) in ai.viXra:2602.0053(v2), the detailed companion treatment which v1 did not cite. v1's supporting claim in the proof of Theorem 5.1, that the fiber oscillation is "$O(1)$ regardless of β", is also withdrawn: the conditional fast potential obeys osc $= 2\beta n_{\mathrm{plaq}} + C_{\mathrm{poly}}$, *linear* in β (2602.0053(v2), Lemma 3.2; reproduced numerically here). (ii) The quantitative absorption in Proposition 4.7 inherits hypothesis (H-P0) of ai.viXra:2602.0052(v2), and the corrected asymptotic-freedom flow restricts all statements to the volume window $L \leq e^{C/g^2+O(1)}$: Theorem 1.1 is restated as windowed and conditional on (H-DOB-blk)+(H-P0). (iii) The erratum for [2] in §10 is corrected: v1's items (a) and (c) ("Assumption 6.3 is removed", "Theorem 1.1(ii) of [2] is now unconditional") are withdrawn — the assumption is *reduced*, not removed; item (b) (withdrawal of Lemma 6.4 of [2] due to the volume factor $(MR^{n_{\max}})^d$) was correct and stands. (iv) A new technical finding (Remark 2.8): the row-normalized transfer operator $\widehat{T}$ of Definition 2.2 satisfies the correlation identities (12)/(29) *exactly only when its normalizer* $D(\sigma) = \int K(\sigma, \sigma')d\sigma'$ *is constant* (true in the $d = 2$ toy, false for $d \geq 3$ where the spatial factor $e^{(\beta/2)S(\sigma)}$ survives); the correct identities hold in kernel form (with K, or the symmetrized $D^{-1/2}KD^{-1/2}$). Since $D^{-1}K$ and $D^{-1/2}KD^{-1/2}$ are similar, the spectrum — hence Δ_{phys} — is unaffected; adjudicated numerically (spectra equal to 10^{-16}; the $\widehat{T}$-form of (29) deviates from the exact path integral by 0.30 in a $d = 3$ toy). What survives and is validated end-to-end in exact toys: the slab splitting (Definition 2.1), self-adjointness and detailed balance (Lemma 2.3), the spectral clustering-to-gap step (Proposition 2.4), Osterwalder–Seiler reflection positivity including the Peter–Weyl positive-definiteness of $\mathrm{Re}\,\mathrm{tr}(UV^{-1})$ (Theorem 2.6), and the gauge-invariance lemmas of §8. The contribution is retagged: a correct and verifiable *transfer-matrix back end* for the program, whose front end (DLR-LSI) is conditional and windowed.

(H-DOB-blk) Block Dobrushin contraction (imported from ai.viXra:2602.0053(v2) [28]). *At every scale of the multiscale fiber assembly and for every frozen boundary configuration, the block interdependence matrix satisfies $\delta < 1$ uniformly.* Per [28], no worst-case influence bound can verify this at weak coupling (the printed bounds give $\delta < 1$ only for $\beta \lesssim 10^{-2}$, and the genuine worst-case influence tends to 1); an averaged or transport-based criterion would be required.

(H-P0) Power-law large-field penalty (imported from ai.viXra:2602.0052(v2) [27]). *Balaban's large-field penalty satisfies $p_0(g) \gtrsim g^{-2}$, rather than only the polylog floor; equivalently the absorption inequalities behind Proposition 4.7 hold at every scale of the window.*

1 Introduction

1.1 The mass gap problem on the lattice

The Yang–Mills mass gap problem asks whether pure $\mathrm{SU}(N_c)$ gauge theory in $\mathbb{R}^d$ (for $d = 4$ and $N_c \geq 2$) possesses a strictly positive mass gap — i.e., the Hamiltonian H satisfies $\mathrm{spec}(H) \subset \{0\} \cup [m, \infty)$ for some $m > 0$. This is one of the seven Millennium Prize Problems posed by the Clay Mathematics Institute [1].

On the lattice $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$, the Euclidean formulation replaces the Hamiltonian by a *transfer matrix* $\widehat{T}$ that propagates states along one lattice direction ("time"). The mass gap becomes the spectral gap of $\widehat{T}$:

$$\Delta_{\mathrm{phys}} := -\log \|\widehat{T}|_{\mathbf{1}^\perp}\| > 0, \tag{1}$$

where $\mathbf{1}^\perp$ denotes the orthogonal complement of the vacuum (constant) state. The central question is whether Δ_{phys} remains bounded away from zero as $L \to \infty$ (along even L).

1.2 Main result

Theorem 1.1 (Lattice mass gap; v2: windowed and conditional)**.** *Fix $d \geq 3$ and $N_c \geq 2$. Assume* (H-DOB-blk) *and* (H-P0). *There exists $\beta_0 = \beta_0(N_c, d) < \infty$ such that for all $\beta \geq \beta_0$ and all even L with $2 \leq L \leq e^{C/g^2 + O(1)}$ ($g^2 = 2N_c/\beta$, $C = 24\pi^2/(11N_c)$), the transfer matrix $\widehat{T}$ of $\mathrm{SU}(N_c)$ lattice Yang–Mills theory on $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ satisfies*

$$\Delta_{\mathrm{phys}}(\beta, L) \;\geq\; \frac{1}{\xi(\beta)} \;>\; 0, \tag{2}$$

where $\xi(\beta) < \infty$ is the correlation length, independent of L within the window. (v1 claimed this for all even $L \geq 2$ without hypotheses; see the Note added in v2.)

Remark 1.2 (Relation to [2]). The paper [2] established the uniform log-Sobolev inequality (LSI) for $\mu_{\beta,\Lambda}$ on periodic tori, and derived the mass gap conditionally on an Assumption 6.3 (a Dobrushin-type contraction of multiscale influence coefficients; (H-DOB) in [2](v3)). v1 of the present paper claimed to remove this conditionality "by a different route". **v2 corrects this**: the route through DLR-LSI does not remove the Dobrushin-type input — it relocates it inside the fiber assembly of the DLR-LSI proof, where it appears as the block-level condition (H-DOB-blk) made explicit in the companion ai.viXra:2602.0053(v2) [28]. See Remark 10.1 for the corrected erratum to [2].

Remark 1.3 (v2: status summary). The chain below is architecturally correct and its *back end* (§§2, 6–8: transfer matrix, Stroock–Zegarlinski, clustering → gap, reflection positivity, gauge invariance) is unconditional and validated end-to-end in exact toy models (Appendix A). The *front end* (§§3–5: periodic LSI and its DLR extension) is conditional on (H-DOB-blk)+(H-P0) and windowed, per the current versions of the companions [2](v3), [27], [28].

1.3 Structure of the proof

The proof of Theorem 1.1 follows the chain

$$\boxed{\text{LSI (periodic)}}+\boxed{\text{Boundary-uniform RG (§4)}} \xrightarrow{\text{§5}} \boxed{\text{DLR-LSI}} \xrightarrow{\text{SZ}} \boxed{\text{DS mixing}} \xrightarrow{\text{DS}} \boxed{\text{Clustering}} \xrightarrow{\text{RP}} \boxed{\Delta \tag{3}$$

Step 1 Uniform LSI (periodic) and DLR-LSI. By [2], for $d \geq 3$ and $\beta \geq \beta_0$ the Gibbs measure $\mu_{\beta,\Lambda}$ on $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ (with L even) and periodic boundary condition satisfies LSI(α_*) with $\alpha_* > 0$ independent of L — (v2) per [2](v3), under its stated hypotheses and on the volume window. By Theorem 4.8 (Section 4), this extends in Section 5 to the DLR-LSI property, (v2) under (H-DOB-blk)+(H-P0) and on the window.

Step 2 Complete analyticity / mixing (Section 6). By the Stroock–Zegarlinski theorem [11, 14] in its standard DLR formulation, DLR-LSI for the specification implies the Dobrushin–Shlosman mixing condition (complete analyticity), hence exponential clustering with correlation length $\xi(\beta) < \infty$ uniformly in even L (within the window).

Step 3 RP → transfer matrix gap (Sections 7 and 8). Osterwalder–Seiler reflection positivity yields a positive self-adjoint transfer operator on $L^2(\nu)$. Exponential decay of temporal correlations implies $\|\widehat{T}|_{\mathbf{1}^\perp}\| \leq e^{-1/\xi}$ and therefore $\Delta_{\text{phys}} \geq 1/\xi$. (This step is unconditional.)

1.4 Notation and conventions

Throughout, $G = \mathrm{SU}(N_c)$ with its bi-invariant Riemannian metric normalized so that $\mathrm{Ric} \geq \frac{N_c}{4} g$ (the $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$ convention of the series; verified numerically in the companion suites). dU or $d\mu_{\text{Haar}}(U)$ is normalized Haar measure on G.

The lattice is $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ with edge set $E(\Lambda)$. A configuration is a map $U : E(\Lambda) \to G$; the configuration space is $\mathcal{A} = G^{|E(\Lambda)|}$. The Wilson action is

$$S_W(U) = \beta \sum_{p\in\mathcal{P}(\Lambda)} \mathrm{Re}\,\mathrm{tr}(U_p), \tag{4}$$

where the sum is over oriented plaquettes and $U_p = U_{e_1}U_{e_2}U_{e_3}^{-1}U_{e_4}^{-1}$. The lattice Yang–Mills measure is

$$d\mu_{\beta,\Lambda}(U) = \frac{1}{Z(\beta,\Lambda)} \exp\bigl(S_W(U)\bigr) \prod_{e\in E(\Lambda)} dU_e. \tag{5}$$

For a sub-volume $\Lambda' \subset \mathbb{Z}^d$ and boundary condition $\omega = \{U_e\}_{e\notin E(\Lambda')}$, the conditional measure is

$$d\mu^{\omega}_{\Lambda'}(U) = \frac{1}{Z_{\Lambda'}(\omega)} \exp\Bigl(\beta \sum_{p:\, p\cap E(\Lambda')\neq\emptyset} \mathrm{Re}\,\mathrm{tr}(U_p)\Bigr) \prod_{e\in E(\Lambda')} dU_e, \tag{6}$$

with links outside $E(\Lambda')$ frozen to ω. A gauge transformation is $g : V(\Lambda) \to G$ acting by $U_e \mapsto g_{s(e)} U_e g_{t(e)}^{-1}$; an observable is gauge-invariant if invariant under all such maps. (v2) The corrected coupling flow of the series erratum is $1/g_k^2 = 1/g_0^2 - 2b_0 k \ln L_{\mathrm{RG}}$, $b_0 = 11N_c/(48\pi^2)$: g_k increases toward the infrared, and the small-field regime $g_k \le \gamma_0$ holds for $k \le k^*(\beta)$ only; this is the source of the volume window in Theorem 1.1.

2 Lattice Yang–Mills: Transfer Matrix and Reflection Positivity

2.1 Temporal decomposition

We distinguish one of the d periodic directions as "time." Write $\Lambda = \Lambda_{\mathrm{s}} \times \Lambda_{\mathrm{t}}$ where $\Lambda_{\mathrm{s}} = (\mathbb{Z}/L\mathbb{Z})^{d-1}$ and $\Lambda_{\mathrm{t}} = \mathbb{Z}/L\mathbb{Z}$ ($T := L$, even so that the reflection plane at $t = 0$ bisects the lattice). The edges decompose into *spatial links* $E_s(t)$ (within the slice $\Lambda_{\mathrm{s}} \times \{t\}$) and *temporal links* $E_t(t)$ (connecting slices t and $t+1$). A time-slice configuration is $\sigma_t = \{U_e : e \in E_s(t)\} \in G^{|E_s|}$; $d\sigma$ is product Haar measure.

2.2 The transfer operator

Definition 2.1 (Slab kernel)**.** *Fix t and let* $\mathrm{slab}(t, t+1)$ *denote the plaquettes with time coordinates in $\{t, t+1\}$. Define the slab weight*

$$W(\sigma, \tau, \sigma') := \exp\Big(\frac{\beta}{2} \sum_{p \subset \mathrm{slice}(t)} \mathrm{Re}\,\mathrm{tr}(U_p(\sigma)) + \beta \sum_{\substack{p \text{ temporal} \\ \text{across } (t,t+1)}} \mathrm{Re}\,\mathrm{tr}(U_p(\sigma, \tau, \sigma')) + \frac{\beta}{2} \sum_{p \subset \mathrm{slice}(t+1)} \mathrm{Re}\,\mathrm{tr}(U_p(\sigma'))\Big), \tag{7}$$

where $\sigma, \sigma' \in G^{|E_s|}$ are the spatial configurations at t and $t+1$ and $\tau \in G^{|E_t|}$ the temporal links. The $\beta/2$ factors ensure that composing adjacent slabs reproduces the full Euclidean weight without double counting (validated exactly in Appendix A, check 1). The slab kernel is

$$K(\sigma, \sigma') := \int_{G^{|E_t|}} W(\sigma, \tau, \sigma')\, d\tau. \tag{8}$$

Definition 2.2 (Slice measure and normalized transfer operator)**.** *Define the one-slice marginal ν on $G^{|E_s|}$ by*

$$d\nu(\sigma) := \frac{1}{Z_\nu} \Big(\int K(\sigma, \sigma')\, d\sigma' \Big) d\sigma, \tag{9}$$

and the normalized transfer operator $\widehat{T} : L^2(\nu) \to L^2(\nu)$ by

$$(\widehat{T} f)(\sigma) := \frac{\displaystyle\int K(\sigma, \sigma') f(\sigma')\, d\sigma'}{\displaystyle\int K(\sigma, \eta)\, d\eta}. \tag{10}$$

$\widehat{T}$ is a Markov operator with invariant measure ν and $\widehat{T}\mathbf{1} = \mathbf{1}$, so its spectrum lies in $[-1, 1]$. (v2: see Remark 2.8 — the correlation identities (12) and (29) below hold exactly in kernel form*; the row-normalized $\widehat{T}$ reproduces them exactly only when $D(\sigma) = \int K(\sigma, \eta)\, d\eta$ is constant.)*

Lemma 2.3 (Self-adjointness)**.** *The operator $\widehat{T}$ is self-adjoint on $L^2(\nu)$.*

Proof. The Wilson action is invariant under time reversal $t \mapsto -t$ (combined with $U_e \mapsto U_e^{-1}$ on temporal links), which implies $K(\sigma, \sigma') = K(\sigma', \sigma)$. The detailed balance identity $\nu(d\sigma)\widehat{T}(\sigma, d\sigma') = \nu(d\sigma')\widehat{T}(\sigma', d\sigma)$ follows, giving self-adjointness on $L^2(\nu)$; see [8, 9]. (Validated exactly in the toy models of Appendix A, check 2.) □

The mass gap is

$$\Delta_{\text{phys}} := -\log \|\widehat{T}|_{\mathbf{1}^\perp}\|, \tag{11}$$

where $\mathbf{1}^\perp$ is the orthogonal complement of the constants in $L^2(\nu)$. Eigenvalues e^{-E_i} of $\widehat{T}$ correspond to energy levels E_i; $\Delta_{\text{phys}} = E_1 - E_0$ with $E_0 = 0$. Temporal correlations are controlled by the transfer structure:

$$\langle f, \widehat{T}^n g\rangle_\nu = \int f(\sigma_0)\, g(\sigma_n)\, d\mu_{\beta,\Lambda} \tag{12}$$

for $f, g \in L^2(\nu)$ and $0 \le n \le L$ — (v2) *exactly in kernel form and, for the row-normalized* $\widehat{T}$, *exactly iff* D *is constant*; see Remark 2.8. In general the right-hand side equals the kernel-form expression (29) below, and (12) with $\widehat{T}$ holds up to the D-conjugation, which leaves the spectrum (hence (11)) unchanged.

2.3 Spectral representation of temporal correlations

Proposition 2.4 (Spectral representation)**.** *For any gauge-invariant observable* $\mathcal{O} \in L^2(\nu)$ *with* $\langle \mathcal{O}\rangle_\nu = 0$, *there exists a positive Borel measure* $\rho_{\mathcal{O}}$ *on* $[-\|\widehat{T}|_{\mathbf{1}^\perp}\|, \|\widehat{T}|_{\mathbf{1}^\perp}\|]$ *such that*

$$\langle \mathcal{O}, \widehat{T}^n \mathcal{O}\rangle_\nu = \int \lambda^n\, d\rho_{\mathcal{O}}(\lambda). \tag{13}$$

In particular, if $\langle \mathcal{O}, \widehat{T}^n \mathcal{O}\rangle_\nu \le Ce^{-n/\xi}$ *for all* $n \ge 0$, *then* $\|\widehat{T}|_{\mathbf{1}^\perp}\| \le e^{-1/\xi}$ *and hence* $\Delta_{\text{phys}} \ge 1/\xi$.

Proof. The spectral theorem applied to the self-adjoint $\widehat{T}$ on $\mathbf{1}^\perp$ gives (13) with $d\rho_{\mathcal{O}} \ge 0$. If the left side decays as $Ce^{-n/\xi}$, then every λ in the support of $\rho_{\mathcal{O}}$ satisfies $|\lambda|^n \le Ce^{-n/\xi}$ for all n, giving $|\lambda| \le e^{-1/\xi}$. (Validated in Appendix A, check 4: the measured covariance decay rate equals $-\log(\lambda_1/\lambda_0)$ to 10^{-6}.) □

2.4 Reflection positivity

Definition 2.5 (Time reflection)**.** *The time reflection* θ *acts by* $t \mapsto -t$: $(\theta U)_{(x,t),(x,t+\hat{0})} = U^{-1}_{(x,-t-\hat{0}),(x,-t)}$ *(temporal links),* $(\theta U)_{(x,t),(x+\hat{\mu},t)} = U_{(x,-t),(x+\hat{\mu},-t)}$ *(spatial links).*

Theorem 2.6 (Osterwalder–Seiler [8])**.** *For the Wilson action, the Euclidean measure* $\mu_{\beta,\Lambda}$ *is reflection positive with respect to* θ*: for any function* F *depending only on link variables with* $t \ge 0$,

$$\int F \cdot (\theta F)^*\, d\mu_{\beta,\Lambda} \;\ge\; 0. \tag{14}$$

Proof. The Wilson action decomposes as $S_W = S_+ + S_- + S_{\text{cross}}$, where $S_\pm$ involves only plaquettes in the upper/lower half-space and S_{cross} the crossing plaquettes. The key identity is that $\operatorname{Re}\operatorname{tr}(UV^{-1})$ is a positive-definite kernel on G (Peter–Weyl), making each crossing factor $\exp(\beta \operatorname{Re}\operatorname{tr}(U_p))$ a sum of positive terms. The standard argument yields (14); see [8, 9]. (Validated in Appendix A, check 5: the OS Gram matrix is PSD by exact enumeration, and the Peter–Weyl kernels $\operatorname{Re}\operatorname{tr}(UV^{-1})$ and $e^{\beta \operatorname{Re}\operatorname{tr}(UV^{-1})}$ have PSD Gram matrices on Haar samples of SU(2).) □

Remark 2.7. Reflection positivity implies the existence of a positive self-adjoint transfer operator implementing one-step time translation in the Osterwalder–Schrader reconstruction. In the present paper we work directly with the normalized operator $\widehat{T}$ of Definition 2.2, whose self-adjointness follows independently from the symmetry of the slab kernel (Lemma 2.3).

Remark 2.8 (v2: normalization defect of Definition 2.2 — kernel form vs. row-normalized form). Let $D(\sigma) := \int K(\sigma, \eta)\, d\eta$, so $\widehat{T} = D^{-1}K$ and $d\nu \propto D\, d\sigma$. The exact consequence of the slab algebra (Definition 2.1) on the periodic-time lattice is the *kernel form*

$$\mathbb{E}_{\mu_{\beta,\Lambda}}\big[\mathcal{O}(0)\, \mathcal{O}(n)\big] = \frac{\operatorname{tr}\big(K^{L-n}\, \mathcal{O}\, K^n\, \mathcal{O}\big)}{\operatorname{tr}(K^L)},$$

i.e. (29) below with K in place of $\widehat{T}$. Because $K^n \neq D^{1/2}(D^{-1/2}KD^{-1/2})^n D^{1/2} \cdot (\text{const})$ unless D is constant, the row-normalized $\widehat{T}$ does *not* satisfy (12)/(29) exactly in general. In $d \geq 3$ the normalizer is genuinely non-constant: the purely spatial factor $e^{(\beta/2)S(\sigma)}$ in (7) survives the τ- and σ'-integrations (in a $d = 3$ toy, $\operatorname{osc}(\log D) = 2.09$ and the $\widehat{T}$-form of (29) deviates from the exact path integral by 3.0×10^{-1}; Appendix A, check 3). In $d = 2$-type reductions D is constant by link-flip homogeneity and both forms coincide exactly (check 2). Two facts limit the damage: (i) $D^{-1}K = D^{-1/2}\big(D^{-1/2}KD^{-1/2}\big)D^{1/2}$ is a similarity, so $\widehat{T}$ and the symmetrized kernel have *identical spectra* (verified to 4×10^{-16}) and Δ_{phys} in (11) is unaffected; (ii) $\widehat{T}$ is genuinely reversible with invariant measure $\nu \propto D$ (Lemma 2.3 stands). What must be corrected is the *interpretation*: ν of (9) is the invariant measure of the auxiliary Markov chain, not the Gibbs time-slice marginal, and all correlation identities in §§2.3, 7, 8.3 are to be read in kernel form. All conclusions of the paper survive with this reading.

3 Uniform Log-Sobolev Inequality on Periodic Tori

The key analytic input is the uniform LSI established in [2] using a multiscale RG analysis based on Balaban's block-spin transformations [20, 21, 22], with the fiber LSI from [3], the entropy telescoping from [5], large-field suppression from [6], and Cauchy-type residual bounds from [7].

Proposition 3.1 (Uniform LSI on tori [2]; v2: as per [2](v3))**.** *Fix $d \geq 3$ and $N_c \geq 2$. There exist $\beta_0 = \beta_0(N_c, d) < \infty$ and $\alpha_* > 0$ such that for all $\beta \geq \beta_0$ and all even L in the volume window $L \leq e^{C/g^2 + O(1)}$, the Gibbs measure $\mu_{\beta,\Lambda}$ on $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ with periodic boundary condition satisfies*

$$\operatorname{Ent}_{\mu_{\beta,\Lambda}}(f^2) \leq \frac{2}{\alpha_*} \sum_{e \in E(\Lambda)} \int |\nabla_e f|^2\, d\mu_{\beta,\Lambda} \tag{15}$$

for all smooth f, with α_* independent of L within the window, under the hypotheses stated in [2](v3) *(in particular the site-level (H-DOB) there, discharged only conditionally).* (v1 stated this for all even $L \geq 2$ unconditionally.)

Remark 3.2 (Comparison with naive estimate). The naive Holley–Stroock single-site estimate gives $\alpha_{\text{ss}} = \frac{N_c}{4} e^{-4(d-1)N_c\beta}$, exponentially small in β. The multiscale RG approach of [2] targets $\alpha_* = O(1)$ for $\beta \geq \beta_0$, because the RG extracts the relevant part of the interaction at each scale. (v2: the quantitative realization of this $O(1)$ claim is exactly what the audits of [27, 28] condition on (H-P0)/(H-YGZ)-type hypotheses.)

Remark 3.3 (Extension to arbitrary boundary conditions). The proof in [2] is carried out on periodic tori. The stronger DLR-LSI property is addressed in Sections 4 and 5 by verifying boundary-uniform RG outputs (Theorem 4.8). (v2: the detailed treatment is the companion [28]; the extension is *structural* — the quantitative closure requires (H-DOB-blk) and (H-P0).)

Remark 3.4 (v2: volume window). On the corrected flow, $g_k \le \gamma_0$ holds only for $k \le k^*(\beta) = (g_0^{-2} - \gamma_0^{-2})/(2b_0 \ln L_{\mathrm{RG}})$; since the multiscale argument uses $n_{\max} \sim \log_{L_{\mathrm{RG}}} L$ scales, every statement in §§3–5 is restricted to $L \le e^{C/g^2+O(1)}$. This is the common boundary of ai.viXra:2602.0032/0033/0051/0052 (v2); removing it requires the strong-coupling handoff (H-XOVER) of [26].

4 Verification of boundary-uniform RG outputs

In this section we state the boundary-uniform RG outputs needed to extend the periodic uniform LSI of [2] toward the DLR-LSI property. (v2: the proofs below are assertion-level summaries; the detailed statements, proofs and the numerical adjudication are in the companion [28], whose conditional structure — (H-P0) for the quantitative absorption — is inherited here.)

4.1 Finite domains with frozen boundary and a boundary-adapted large-field event

Let $\Lambda' \Subset \mathbb{Z}^d$ be finite and $\omega = \{U_e\}_{e \notin E(\Lambda')}$ a frozen boundary configuration; the conditional measure $\mu^{\omega}_{\Lambda'}$ is (6).

Definition 4.1 (Fully dynamical plaquettes). *Let $\mathcal{P}(\Lambda')$ denote the plaquettes whose four links belong to $E(\Lambda')$. For a scale-k block B, let $\mathcal{P}^{\mathrm{dyn}}_k(B)$ be the scale-k plaquettes in the support of B that belong to $\mathcal{P}(\Lambda')$.*

Definition 4.2 (Boundary-adapted large-field event). *For a scale-k block B, define*

$$Z_k(B;\omega) := \big\{\exists\, P \in \mathcal{P}^{\mathrm{dyn}}_k(B) : \|U_P - \mathbf{1}\|_{\mathrm{HS}} \ge \varepsilon_k\big\}. \tag{16}$$

Remark 4.3 (Why fully dynamical plaquettes). If one allowed plaquettes involving frozen links, an adversarial ω could make the corresponding holonomies deterministic and force a "large-field" trigger with probability 1. Restricting to fully dynamical plaquettes keeps $Z_k(B;\omega)$ a genuine event under $\mu^{\omega}_{\Lambda'}$ for every ω.

Lemma 4.4 (Compact-parameter supremum). *Let $G = \mathrm{SU}(N_c)$ and let ω range over a product G^m, $m < \infty$. If a bound holds uniformly over Balaban-admissible background fields (hence uniformly in ω as part of the background), then taking $\sup_\omega$ preserves the same constant.*

Proof. Immediate: G^m is compact and the background-uniform constants are deterministic. □

4.2 Uniform polymer bounds

Proposition 4.5 (Uniform polymer bounds with frozen boundary). *For $\beta \ge \beta_0$ in the weak-coupling regime where Balaban's construction applies (v2: $k \le k^*(\beta)$), the RG expansion for $\mu^{\omega}_{\Lambda'}$ produces polymer activities $R^{(k)}(X;\omega)$ satisfying*

$$\sup_{\Lambda',\omega} |R^{(k)}(X;\omega)| \le C_R e^{-\kappa d_k(X)}, \tag{17}$$

with $C_R, \kappa > 0$ depending only on $(d, N_c, L_{\mathrm{RG}})$ and β_0, independent of Λ' and ω.

Proof. As in the periodic case: Balaban's activities are local functionals of the dynamical variables with bounds uniform in the background field; the frozen links enter as background and do not affect the deterministic constants; $\sup_{\Lambda',\omega}$ is harmless by Lemma 4.4. (Detailed version: [28], Proposition 4.1.) □

4.3 Uniform conditional large-field suppression

Proposition 4.6 (Uniform large-field suppression with frozen boundary)**.** *For $\beta \geq \beta_0$, there exist $c > 0$ and $C_{\mathrm{blk}} < \infty$ depending only on $(d, N_c, L_{\mathrm{RG}})$ and β_0 such that for every scale k (v2: $k \leq k^*(\beta)$), every block B, and every ω,*

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \; \mu_k\big(Z_k(B;\omega) \mid \mathcal{G}_{k+1}, \omega\big) \leq C_{\mathrm{blk}} \exp\!\big(-c\, p_0(g_k)\big). \tag{18}$$

Proof. This is the mechanism verified in [6] (T-operation small factor + uniformity estimate), both uniform in the background; frozen links are background; $\sup_\omega$ harmless. (Detailed version: [28], Proposition 4.2. v2: per [27], the suppression constant is $c = 2/(1+\beta_0)$, so (18) carries quantitative content only under (H-P0).) □

4.4 Uniform cross-scale derivative bounds

Proposition 4.7 (Uniform cross-scale bounds with frozen boundary; v2: under (H-P0))**.** *Assume* (H-P0)*. Under the hypotheses of [5, 7, 6], the integrated cross-scale derivative bounds of [5] hold for $\mu^\omega_{\Lambda'}$ with constants independent of Λ' and ω: for every unit fast direction $v \in E_k$ supported on a single dynamical link,*

$$\sup_{\Lambda',\omega} \; \mathbb{E}_{\mu^\omega_{\Lambda'}}\big[|vV_{<k}|^2\big] \leq D_k, \tag{19}$$

$$\sup_{\Lambda',\omega} \; \operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \; \mathbb{E}\big[|vV_{<k}|^2 \mid \mathcal{G}_{k+1}, \omega\big] \leq D_k, \tag{20}$$

where $D_k := 2C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}$ and $\sum_k D_k < \infty$ depends only on $(d, N_c, L_{\mathrm{RG}})$ and β_0.

Proof. As [5, Theorem 1.1]: small-field contribution by Cauchy bounds and polymer decay (Proposition 4.5); large-field contribution by the pointwise bound of [7] and the suppression factor of Proposition 4.6. (v2: the absorption $M_k^2 C_{\mathrm{blk}} e^{-c\,p_0(g_k)} \leq C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}$ is the F3-type failure point adjudicated in [27, 28]: under the polylog floor it fails at $k = O(1)$ — Appendix A, check 8 — hence the standing hypothesis.) □

Theorem 4.8 (Boundary-uniform RG outputs; v2: structural, quantitative part under (H-P0))**.** *For $d \geq 3$ and $N_c \geq 2$, there exists $\beta_0 < \infty$ such that for all $\beta \geq \beta_0$ and all scales $k \leq k^*(\beta)$, the RG inputs needed to extend the periodic uniform LSI toward the DLR-LSI property hold uniformly for every finite $\Lambda' \Subset \mathbb{Z}^d$ (within the window) and every frozen boundary condition ω; the quantitative absorption requires* (H-P0)*.*

Proof. Combine Propositions 4.5, 4.6, and 4.7. □

Remark 4.9 (v2: what §4 does and does not deliver)*.* The boundary-uniformity ("frozen = slow") content of this section is structural and stands, with the detailed proofs in [28] (§§3–4 there, validated numerically). What §4 does *not* deliver — in v1's silence — is the inter-block Dobrushin condition needed by the fiber assembly in §5; that is (H-DOB-blk), and no worst-case argument can prove it ([28], Remark 3.7).

5 DLR-LSI: The Boundary Extension

The Stroock–Zegarlinski theorem, in its standard formulation [11, 14, 15], requires the LSI to hold for every finite sub-volume with every boundary condition (DLR-LSI). While the periodic LSI is established in [2] (conditionally, per its v3), the extension to domains with boundary requires the multiscale RG construction to remain uniform in arbitrary fixed boundary data — a non-trivial requirement (an adversarial ω can force large-field configurations on boundary plaquettes). The boundary-uniform RG outputs are Theorem 4.8.

Theorem 5.1 (DLR-LSI; v2: windowed and conditional on (H-DOB-blk)+(H-P0))**.** *Fix $d \geq 3$, $N_c \geq 2$, and $\beta \geq \beta_0$. Assume* (H-DOB-blk) *and* (H-P0)*. Then for every finite $\Lambda' \subset \mathbb{Z}^d$ with $\log_{L_{\mathrm{RG}}} \operatorname{diam}(\Lambda') \leq k^*(\beta)$ and every boundary condition ω, the conditional Gibbs measure $\mu^{\omega}_{\Lambda'}$ satisfies*

$$\operatorname{Ent}_{\mu^{\omega}_{\Lambda'}}(f^2) \leq \frac{2}{\alpha_*} \sum_{e \in E(\Lambda')} \int |\nabla_e f|^2 \, d\mu^{\omega}_{\Lambda'}, \tag{21}$$

with $\alpha_* > 0$ depending only on d, N_c, β_0 (and the constants in the hypotheses). (v1 stated this unconditionally for all finite Λ'.)

Proof. By Theorem 4.8, the three RG inputs hold for $\mu^{\omega}_{\Lambda'}$ with constants independent of Λ' and ω (quantitative absorption under (H-P0)). The remainder follows the periodic argument of [2] — with the following v2 corrections to v1's presentation:

Step 1 (Entropy telescoping): the telescoping identity ([2], Proposition 5.5) is a general property of conditional expectations along a filtration, valid for any probability measure including $\mu^{\omega}_{\Lambda'}$. (Unconditional.)

Step 2 (Fiber LSI): the conditional fiber LSI at each scale uses Bakry–Émery on $\mathrm{SU}(N_c)$ ($\mathrm{LSI}(N_c/4)$ for Haar) perturbed by the effective potential, assembled across blocks. **v2 correction:** v1 asserted that "the effective potential oscillation within a fiber is $O(1)\ldots$ regardless of β or ω". This is *withdrawn*: the conditional fast potential contains the Wilson terms of the plaquettes meeting the fiber, so its per-block oscillation is $C_{\mathrm{fib}} = 2\beta n_{\mathrm{plaq}} + C_{\mathrm{poly}}$ — *linear in β* ([28], Lemma 3.2; reproduced numerically in Appendix A, check 7: $\mathrm{osc}(\beta{=}1, 10, 100) = 3.0, 30.3, 303.1$). The per-block LSI therefore carries a Holley–Stroock degradation $e^{-C_{\mathrm{fib}}}$ (the (H-YGZ)-type weakness of [27]), and the inter-block assembly requires the block Dobrushin condition — *this is where (H-DOB-blk) enters*, exactly as in [28], Lemma 3.5 and Remark 3.7. The claim that the present route "bypasses any explicit Dobrushin contraction estimate" is withdrawn accordingly.

Step 3 (Sweeping-out): the cross-scale bounds (Proposition 4.7) and large-field suppression (Proposition 4.6) enter the sweeping-out relations ([2], Lemma 5.7) with geometric decay $D_k = O(L_{\mathrm{RG}}^{-(d-1)k})$; summability holds under (H-P0), uniformly in ω and $|\Lambda'|$ within the window.

Step 4 (Defective $\to$ tight LSI): the Rothaus lemma converts the defective LSI plus the uniform Poincaré inequality into a tight LSI with α_* depending only on the fixed parameters. □

Remark 5.2 (v2: where the conditionality lives)*.* Step 2 is the load-bearing conditional step: (H-DOB-blk) for the inter-block assembly and, through Proposition 4.6, (H-P0) for the quantitative suppression. Steps 1, 3 (given the inputs), and 4 are mechanical and were validated end-to-end in toy models in the companion suites ([2](v3), [28]). The honest summary: this paper's route does not remove [2]'s Assumption 6.3; it relocates it to the block level, where [28] shows it cannot be discharged by worst-case bounds.

6 The Stroock–Zegarlinski Theorem: From Uniform LSI to Mixing and Clustering

6.1 Statement of the theorem

We invoke the Stroock–Zegarlinski theorem for compact-spin finite-range lattice models; see [11, 10], Martinelli [14], and the extensions in [12, 13].

Theorem 6.1 (Stroock–Zegarlinski: DLR-LSI implies DS mixing [11, 14])**.** *Consider a lattice spin system on $\mathbb{Z}^d$ with:*

(H1) ***Compact single-spin space:*** (Ω_0, d_{Ω_0}) *a compact Riemannian manifold.*

(H2) ***Reference LSI:*** *the single-spin measure μ_0 satisfies* $\mathrm{LSI}(\alpha_0)$*, $\alpha_0 > 0$.*

(H3) ***Finite-range interaction:*** $\Phi = \{\Phi_X\}$ *with $\Phi_X = 0$ for* $\mathrm{diam}(X) > R$*.*

If the specification satisfies DLR-LSI(α_*), then the system satisfies the Dobrushin–Shlosman mixing condition (complete analyticity) and exponential decay of truncated correlations: there exist $C > 0$ and $\xi = \xi(\alpha_*, \Phi) < \infty$ such that for every finite Λ, every ω, and all local $\mathcal{O}_A, \mathcal{O}_B$:

$$|\mathrm{Cov}_{\mu_\Lambda^\omega}(\mathcal{O}_A, \mathcal{O}_B)| \le C\, \|\mathcal{O}_A\|_\infty \|\mathcal{O}_B\|_\infty\, e^{-\mathrm{dist}(A,B)/\xi}. \tag{22}$$

In particular, on periodic tori with even L, clustering holds with constants independent of L.

Proof. Standard: [11] proved DLR-LSI ⇒ complete analyticity in the sense of [17]; see [15, Theorem 8.8]; complete analyticity implies exponential clustering by Dobrushin–Shlosman theory, see [14], Theorems 3.3 and 3.5; compact-manifold extensions in [12, 13]. □

6.2 Verification of hypotheses for lattice Yang–Mills

Proposition 6.2 (Hypotheses verified)**.** *For $\beta \ge \beta_0$, the* $\mathrm{SU}(N_c)$ *lattice Yang–Mills model satisfies (H1)–(H3) of Theorem 6.1. Moreover, the DLR-LSI input holds by Theorem 5.1 — (v2) on the window, under (H-DOB-blk)+(H-P0).*

Proof. (H1): $\Omega_0 = \mathrm{SU}(N_c)$, compact of dimension $N_c^2 - 1$. (H2): Haar satisfies $\mathrm{LSI}(N_c/4)$ by Bakry–Émery [18] with $\mathrm{Ric} \ge \frac{N_c}{4} g$ (see [3]; verified numerically in the companion suites). (H3): the Wilson action is a plaquette interaction of range 2. DLR-LSI: Theorem 5.1, with its v2 hypotheses and window, which propagate to every consequence below. □

6.3 Conclusion: exponential clustering

Corollary 6.3 (Exponential clustering for lattice Yang–Mills; v2: windowed, conditional)**.** *For $\beta \ge \beta_0$, under* (H-DOB-blk)+(H-P0)*, there exist $C > 0$ and $\xi = \xi(\beta) < \infty$ such that for every even L in the window, $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$, and all local $\mathcal{O}_A, \mathcal{O}_B$:*

$$|\mathrm{Cov}_{\mu_{\beta,\Lambda}}(\mathcal{O}_A, \mathcal{O}_B)| \le C\, \|\mathcal{O}_A\|_\infty \|\mathcal{O}_B\|_\infty\, e^{-\mathrm{dist}(A,B)/\xi}, \tag{23}$$

with C, ξ independent of L within the window.

Proof. Proposition 6.2 verifies (H1)–(H3) and the (conditional) DLR-LSI input; Theorem 6.1 gives the clustering. □

7 From Exponential Clustering to Mass Gap

7.1 Temporal clustering

Corollary 6.3 gives exponential clustering in *all* directions, including time. For a gauge-invariant $\mathcal{O}$ supported on the slice $\Lambda_s \times \{0\}$ and its translate $\mathcal{O}(t)$:

$$|\mathrm{Cov}_{\mu_{\beta,\Lambda}}(\mathcal{O}(0), \mathcal{O}(t))| \le C\|\mathcal{O}\|_\infty^2 \, e^{-|t|/\xi}, \tag{24}$$

since $A = \Lambda_s \times \{0\}$ and $B = \Lambda_s \times \{t\}$ have $\mathrm{dist}(A, B) \ge |t|$.

7.2 Proof of the main theorem

Proof of Theorem 1.1. Combine the three ingredients (all quantifiers within the volume window, under (H-DOB-blk)+(H-P0)):

Step 1 (Uniform LSI): Proposition 3.1 ([2](v3)).

Step 2 (DLR-LSI and exponential clustering): Theorem 5.1, Theorem 6.1, Corollary 6.3: clustering (23) with $\xi(\beta) < \infty$ uniformly in even L within the window.

Step 3 (RP $\to$ gap): Theorem 2.6 and Lemma 2.3: $\widehat{T}$ is a positive self-adjoint contraction (kernel form per Remark 2.8). By Proposition 2.4 and (24):

$$\Delta_{\mathrm{phys}} = -\log\|\widehat{T}|_{\mathbf{1}^\perp}\| \;\ge\; \frac{1}{\xi(\beta)} \;>\; 0. \tag{25}$$

Since $\xi(\beta)$ depends only on β, N_c, d:

$$\Delta_{\mathrm{phys}}(\beta, L) \;\ge\; \frac{1}{\xi(\beta)} \;>\; 0 \qquad \text{for all even } L \text{ in the window.} \tag{26}$$

This step is unconditional given the clustering input; the conditionality of Theorem 1.1 is inherited entirely from Step 2. $\square$

8 Adaptation to Gauge Theory: Technical Details

8.1 The physical Hilbert space

The transfer matrix acts on $L^2(\nu)$; the physical observables are gauge-invariant, and the relevant gap is that of $\widehat{T}$ restricted to

$$\mathcal{H}_{\mathrm{phys}} = \{f \in L^2(\nu) : f \text{ is gauge-invariant}\}. \tag{27}$$

Lemma 8.1 (Gauge invariance of $\widehat{T}$)**.** *$\widehat{T}$ preserves $\mathcal{H}_{\mathrm{phys}}$.*

Proof. Gauge transformations act independently at each time-slice; the Wilson action is gauge-invariant, so the conditional expectation defining $\widehat{T}$ preserves gauge invariance. (Validated exactly in Appendix A, check 6: $[P, \widehat{T}] = 0$ for the gauge-averaging projector in the toy.) $\square$

Corollary 8.2.

$$\mathrm{gap}(\widehat{T}|_{\mathcal{H}_{\mathrm{phys}}}) \;\ge\; \mathrm{gap}(\widehat{T}|_{\mathbf{1}^\perp}) \;\ge\; \frac{1}{\xi(\beta)}. \tag{28}$$

Proof. $\mathcal{H}_{\mathrm{phys}} \cap \mathbf{1}^\perp$ is a subspace of $\mathbf{1}^\perp$. $\square$

8.2 Compatibility of RP with gauge invariance

Lemma 8.3. *Time reflection θ commutes with gauge transformations: $\theta \circ g = g \circ \theta$.*

Proof. Gauge transformations act by conjugation at endpoints; θ changes time coordinates of endpoints but preserves the conjugation structure: $\theta(g_s U_e g_t^{-1}) = g_{s'}(\theta U_e) g_{t'}^{-1}$ with $s' = \theta(s)$, $t' = \theta(t)$. □

8.3 From lattice correlations to transfer matrix spectrum

Proposition 8.4. *Let $\Lambda = \Lambda_s \times \{0, \dots, L-1\}$ with periodic time, and let $\mathcal{O}$ be gauge-invariant on one slice with $\langle \mathcal{O} \rangle_\nu = 0$. Then,* in kernel form *(v2; Remark 2.8):*

$$\mathbb{E}_{\mu_{\beta,\Lambda}}\big[\mathcal{O}(0)\,\mathcal{O}(n)\big] = \frac{\operatorname{tr}\big(K^{L-n}\,\mathcal{O}\,K^n\,\mathcal{O}\big)}{\operatorname{tr}(K^L)} \tag{29}$$

for $0 \le n \le L$, where $\mathcal{O}$ acts as a multiplication operator. (v1 printed (29) with $\widehat{T}$ in place of K; exact only when D is constant — adjudicated in Appendix A, checks 2–3.)

Proof. $Z = \operatorname{tr}(K^L)$ by the slab algebra (Definition 2.1, validated in check 1); inserting $\mathcal{O}$ at times 0 and n and using cyclicity gives (29); see [9], Chapter 6. □

For $L \gg \xi$ and $n \ll L$ this reduces to

$$\operatorname{Cov}_{\mu_{\beta,\Lambda}}(\mathcal{O}(0), \mathcal{O}(n)) \approx \int \lambda^n \, d\rho_{\mathcal{O}}(\lambda), \tag{30}$$

exactly in the infinite-temporal-length limit (spectral measure of the symmetrized kernel; same spectrum as $\widehat{T}$). The exponential clustering (24) then implies $\|\widehat{T}|_{\mathbf{1}^\perp}\| \le e^{-1/\xi}$ as in Proposition 2.4.

9 Explicit Estimates for SU(2) in $d = 4$

To illustrate the quantitative content of Theorem 1.1, we provide explicit (non-optimal) estimates for $G = \mathrm{SU}(2)$, $d = 4$.

9.1 Parameter values

$N_c = 2$, $\dim \mathrm{SU}(2) = 3$; $\alpha_0 = N_c/4 = 1/2$ (Haar LSI constant); $d = 4$: each link belongs to $2 \cdot 4 \cdot 3 = 24$ plaquettes in the oriented-counting convention ($2(d-1) = 6$ unoriented; both verified combinatorially, Appendix A, check 8); $L_{\mathrm{RG}} = 2$; $r = L_{\mathrm{RG}}^{-(d-1)/2} = 2^{-3/2} \approx 0.354$.

9.2 Single-site estimate (naive)

The naive Holley–Stroock single-site estimate gives

$$\alpha_{\mathrm{ss}} = \frac{1}{2} e^{-4\cdot 3\cdot 2\cdot \beta} = \frac{1}{2} e^{-24\beta}. \tag{31}$$

For $\beta = 2.3$ (a typical weak-coupling value in lattice QCD simulations [24]), this gives $\alpha_{\mathrm{ss}} \approx 5.3 \times 10^{-25}$, an extremely poor bound. (v2: v1 printed 1.2×10^{-24}, which is $e^{-24\beta}$ without the prefactor $\frac{1}{2}$; corrected.)

9.3 Multiscale estimate

The multiscale RG approach of [2] targets a dramatically better constant. From Lemma 6.2 of [2]:

$$\mathfrak{D}_{\text{row}} \leq C_\Gamma\, e^{-\kappa}\, \frac{r}{(1-r)^2}, \tag{32}$$

where for SU(2) at $\beta \geq \beta_0$: $r = 2^{-3/2}$, so $r/(1-r)^2 \approx 0.846$; $\kappa \geq c_0\gamma_0^{-1}$ with $\gamma_0 = O(1/\sqrt{\beta})$; $C_\Gamma = O(1)$. For $\beta \geq 3$, $\kappa \geq 5$, $C_\Gamma \leq 2$:

$$\mathfrak{D}_{\text{row}} \leq 2 \cdot e^{-5} \cdot 0.846 \approx 0.0114 \ll 1. \checkmark \tag{33}$$

(v2) Two caveats. First, per the erratum below (Remark 10.1(b), which stands), $\mathfrak{D}_{\text{row}} < 1$ does *not* translate to a site-level Dobrushin condition uniformly in the volume. Second, per [2](v3), C_Γ is in fact $\propto n_{\max}(\beta)$ through the polymer-counting constants, which is what produced the β-window there. The display (33) is retained as v1's illustrative arithmetic (numerically correct as arithmetic; Appendix A, check 8), not as a proof ingredient.

9.4 Correlation length estimate

The SZ/DS theory yields $\xi(\beta) < \infty$ depending on α_* and a finite-range interaction strength, e.g.

$$J(\beta) := \sup_{e\in E(\Lambda)} \sum_{p\ni e} \beta\, \|\mathrm{Re}\,\mathrm{tr}(U_p)\|_\infty = 2d(d-1)\,\beta\, N_c, \tag{34}$$

(oriented counting). See [14], Theorem 4.1, for explicit (non-optimized) dependence. The key point is qualitative: for each $\beta \geq \beta_0$, $\xi(\beta) < \infty$ — (v2) under the hypotheses and window of Theorem 1.1. For comparison, lattice Monte Carlo [25] for SU(2) in $d = 4$ gives $m_{\text{glueball}} \cdot a \approx 1.2$ at $\beta = 2.5$, i.e. $\Delta_{\text{phys}} \approx 1.2$ in lattice units.

10 Erratum for [2]

Remark 10.1 (Erratum for Assumption 6.3 and Lemma 6.4 of [2]; v2: corrected).

(a) **v1 claimed: "Assumption 6.3 is removed." Withdrawn.** The route through DLR-LSI does not bypass the Dobrushin-type input: the fiber assembly inside the proof of Theorem 5.1 (Step 2) requires the block-level Dobrushin condition (H-DOB-blk), made explicit in [28], which also shows that no worst-case influence bound can verify it at weak coupling. The correct statement: Assumption 6.3 of [2] ((H-DOB) in [2](v3)) is *reduced* to (H-DOB-blk) — genuine progress (block-level, boundary-uniform, all other inputs discharged structurally), but a Dobrushin-type hypothesis remains.

(b) **Lemma 6.4 of [2] is withdrawn. (v1 item; correct, stands.)** Lemma 6.4 of [2] attempted to translate the multiscale contraction $\mathfrak{D}_{\text{row}} < 1$ into a site-to-site Dobrushin condition $\sup_x \sum_y C_{xy} < 1$. The translation involved a factor $(MR^{n_{\max}})^d$ growing with the number of RG scales, rendering the bound non-uniform in the volume. This withdrawal is consistent with (and was the origin of) the (H-DOB) analysis of [2](v3), Remark 6.6.

(c) **v1 claimed: "Theorem 1.1(ii) of [2] is now unconditional." Withdrawn.** Theorem 1.1(ii) of [2] is windowed and conditional on (H-DOB-blk)+(H-P0), via Theorem 1.1.

11 Discussion

11.1 Summary of the logical structure (v2)

The chain (3) stands architecturally. Its status by segment: *periodic LSI* — conditional and windowed per [2](v3); *boundary-uniform RG outputs* — structural content unconditional (detailed in [28]), quantitative absorption under (H-P0); *DLR-LSI* — additionally requires (H-DOB-blk) (Step 2 of Theorem 5.1); *SZ* $\to$ *clustering* $\to$ *RP* $\to$ *gap* — unconditional given the inputs, and validated end-to-end in exact toy models (Appendix A): slab algebra, self-adjointness, spectral clustering-to-gap, reflection positivity, gauge invariance.

11.2 Relation to the Clay Millennium Problem

Theorem 1.1 bears on the mass gap for *lattice* Yang–Mills at weak coupling, in the windowed, conditional sense stated. The Clay problem [1] asks for the continuum theory; beyond discharging (H-DOB-blk) and (H-P0) and removing the window ((H-XOVER) of [26]), the continuum limit requires: controlling $\xi(\beta)$ as $\beta \to \infty$ (asymptotic freedom predicts $\xi \sim Ce^{c\beta}$); constructing the continuum limit of the Gibbs measures; and verifying the Osterwalder–Schrader axioms. These remain beyond the scope of the present program.

11.3 Alternative approach: direct Wasserstein contraction

One can attempt a Wasserstein-type contraction for time-slice conditional kernels:

$$W_1\big(\mu_{\beta,\Lambda}(\cdot|\eta), \mu_{\beta,\Lambda}(\cdot|\eta')\big) \le q \cdot d(\eta, \eta'), \qquad q < 1, \tag{35}$$

which would give a more constructive route. However, the translation of block-level $\Gamma_{k\ell}$ bounds to a site-level W_1 contraction faces the volume-factor issue of Remark 10.1(b) — and, (v2), the same weak-coupling obstruction as (H-DOB-blk): the worst-case contraction coefficient of a directly coupled conditional kernel tends to 1 as $\beta \to \infty$ ([28], Remark 3.7(ii)). An *averaged* W_1 criterion remains the natural candidate; we defer this to future work.

Note added in v2 (July 2026)

This version replaces v1 (February 2026) after a quantitative audit; a deterministic verification suite (`verify_2602_0054.py`, 8 checks, Appendix A) accompanies the paper in the repository `THE-ERIKSSON-PROGRAMME` under `verification/2602-0054/`. Changes:

(1) **Title and main claim retagged.** The unconditional gap claim becomes a *conditional and windowed assembly*: Theorem 1.1 and Theorem 5.1 carry (H-DOB-blk), (H-P0) and the volume window $L \le e^{C/g^2+O(1)}$ explicitly.

(2) **The "Dobrushin bypass" is withdrawn.** v1 claimed the SZ route bypasses any Dobrushin contraction estimate; in fact the fiber assembly inside Theorem 5.1 requires (H-DOB-blk) ([28]). v1's supporting assertion that the fiber oscillation is "$O(1)$ regardless of β" is withdrawn: osc $= 2\beta n_{\text{plaq}} + C_{\text{poly}}$, linear in β (check 7).

(3) **Erratum for [2] corrected.** Items (a) and (c) of Remark 10.1 (removal of Assumption 6.3; unconditionality of [2], Theorem 1.1(ii)) are withdrawn; item (b) (withdrawal of [2], Lemma 6.4 — the $(MR^{n_{\max}})^d$ volume factor) was correct and stands.

(4) **New finding (Remark 2.8):** the row-normalized $\widehat{T}$ of Definition 2.2 satisfies the correlation identities (12)/(29) exactly only when $D(\sigma)$ is constant; for $d \geq 3$ it is not ($\operatorname{osc}(\log D) = 2.09$ in a toy; the $\widehat{T}$-form of (29) deviates from the exact path integral by 0.30), and the identities hold in kernel form. The spectrum, hence Δ_{phys}, is unaffected (similarity; equal to 4×10^{-16} numerically). Proposition 8.4 is restated in kernel form.

(5) **§9 arithmetic.** α_{ss} at $\beta = 2.3$ corrected to 5.3×10^{-25} (v1: 1.2×10^{-24}, missing the factor $\frac{1}{2}$); the $\mathfrak{D}_{\text{row}}$ display retained as arithmetic with two caveats (window through $C_\Gamma \propto n_{\max}$; no site-level translation).

(6) **Companion citation.** v1 did not cite the detailed DLR-extension companion; v2 adds ai.viXra:2602.0053(v2) [28] and delegates the detailed §4–5 proofs to it, together with ai.viXra:2602.0051(v2) [26] and 2602.0052(v2) [27].

(7) **Numbering.** All v1 numbered items (Theorem 1.1, Remark 1.2, items 2.1–2.7, 3.1–3.3, 4.1–4.8, 5.1, 6.1–6.3, 8.1–8.4, Remark 10.1, displays (1)–(35)) are preserved in count and position; new v2 material appears as the two hypothesis boxes, Remarks 1.3, 2.8, 3.4, 4.9, 5.2, and Appendix A.

What is *not* changed: the transfer-matrix and reflection-positivity formalism of §2 (up to the kernel-form reading of Remark 2.8), the SZ statement and hypothesis verification of §6, the clustering-to-gap argument of §7, and the gauge-invariance lemmas of §8 — all validated numerically.

A Numerical verification

The companion suite `verify_2602_0054.py` (NumPy, fixed seed, <1 s, exit code 0) validates the back end in exact toy models and adjudicates the v2 corrections. All 8 checks pass:

1. **Slab composition (Definition 2.1):** in a $d = 3$ $\mathbb{Z}_2$ toy (2×2 spatial torus), the $\beta/2$ splitting reproduces the full Wilson action exactly ($|\text{full} - \sum \text{slabs}| = 0$ over 50 random configurations).
2. **Exact $d = 2$ toy:** K symmetric, $\widehat{T}\mathbf{1} = \mathbf{1}$, detailed balance to 10^{-15}, spectrum in $[-1, 1]$; the trace formula (29) matches the *direct path integral* (full enumeration of 2^{12} configurations, $L = 3$) to 10^{-14}; here D is constant and the $\widehat{T}$- and K-forms coincide.
3. **F-TN (Remark 2.8):** $d = 3$ toy: $\operatorname{osc}(\log D) = 2.09$ (non-constant); $\operatorname{spec}(D^{-1}K) = \operatorname{spec}(D^{-1/2}KD^{-1/2})$ to 4×10^{-16} (gap unaffected); the $\widehat{T}$-form of (29) deviates from the kernel form by 3.0×10^{-1} on $\mathbb{E}[\mathcal{O}(0)\mathcal{O}(2)]$.
4. **Proposition 2.4:** measured temporal covariance decay rate $= -\log(\lambda_1/\lambda_0) = 1.007145$ to 10^{-6}; $\|\widehat{T}|_{\mathbf{1}^\perp}\| = e^{-1/\xi}$.
5. **Theorem 2.6 (RP):** OS Gram matrix PSD by full 2^{16} enumeration ($L = 4$ even; min eigenvalue -6×10^{-17}); Peter–Weyl kernels $\operatorname{Re}\operatorname{tr}(UV^{-1})$ and $e^{2.3 \operatorname{Re}\operatorname{tr}(UV^{-1})}$ PSD on 40 Haar samples of SU(2).
6. **Lemma 8.1/Corollary 8.2:** $[P, \widehat{T}] = 0$ exactly for the gauge-averaging projector; $\operatorname{gap}(\widehat{T}|_{\mathcal{H}_{\text{phys}}}) \geq \operatorname{gap}(\widehat{T}|_{\mathbf{1}^\perp})$.

7. **Fiber oscillation linear in β (v2, Step 2 of Theorem 5.1):** single SU(2) link with $n_p = 6$ background plaquettes: $\mathrm{osc}(\beta{=}1, 10, 100) = 3.03, 30.26, 303.12$ (ratios 10.0 per decade; per-plaquette oscillation $= 2\beta$ exactly at $U = \pm M^*$), refuting "$O(1)$ regardless of β" and matching C_{fib} of [28], Lemma 3.2.

8. **§9 arithmetic + window:** $\alpha_{\mathrm{ss}} = 5.32 \times 10^{-25}$ (v1 printed 1.2×10^{-24}); $r/(1-r)^2 = 0.846$, $\mathfrak{D}_{\mathrm{row}} = 0.0114$ confirmed; unoriented plaquettes per link in $\mathbb{Z}^4 = 2(d-1) = 6$ (24 in the oriented convention as printed); corrected flow: the absorption of Proposition 4.7 fails at $k = 2$ under the polylog floor while $k^* = 994$ at $g_0 = 0.1 \Rightarrow$ (H-P0) + volume window.

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