DLR-Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills at Weak Coupling: a Conditional and Windowed Reduction

(Version 2 of ai.viXra:2602.0053; v1 title: "…and Unconditional Mass Gap…")

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Abstract

We study the passage from the uniform log-Sobolev inequality (LSI) on periodic tori, developed in the companion series, to a *DLR-uniform* LSI for the conditional Gibbs specification of $\mathrm{SU}(N_c)$ lattice Yang–Mills in $d \geq 3$ at weak coupling ($\beta \geq \beta_0$), and from there to a mass gap via Stroock–Zegarlinski and Osterwalder–Seiler reflection positivity. **Version 2 corrects the logical status of the main results after a quantitative audit** (companion numerical suite included; Appendix A). (i) The fiber assembly (Lemma 3.5) *assumes* the block Dobrushin condition $\delta < 1$, which v1's own Remark 3.6 left unverified; the printed influence bound $c_{ij} \leq \tanh(\beta n_\partial/2)$ tends to 1 as $\beta \to \infty$ and yields $\delta < 1$ only for $\beta \lesssim 10^{-2}$ ($d = 3$) or $\beta \lesssim 10^{-3}$ ($d = 4$) — the opposite of the weak-coupling regime. A rotor exhibit shows the genuine worst-case block influence also tends to 1, so no worst-case criterion can close the gap: the condition is now the explicit hypothesis **(H-DOB-blk)**, and v1's claim of removing the Dobrushin-type Assumption 6.3 of [14] is *withdrawn* — the present paper *reduces* that assumption to (H-DOB-blk). (ii) The quantitative absorption in Proposition 4.3 inherits hypothesis **(H-P0)** of ai.viXra:2602.0052(v2): under the polylog penalty floor $p_0(g) \geq c_0|\log g|^{1+\epsilon_0}$ the required inequality $e^{-c\,p_0(g_k)} \leq C\,L_{\mathrm{RG}}^{-(d-1)k}$ fails already at $k = O(1)$. (iii) The proof assumes $g_k \leq \gamma_0$ for all $k \leq n_{\max} \sim \log_{L_{\mathrm{RG}}} \mathrm{diam}(\Lambda')$; with the corrected asymptotic-freedom flow of the series erratum this holds only on the volume window $\log_{L_{\mathrm{RG}}} \mathrm{diam}(\Lambda') \leq k^*(\beta)$, i.e. $\mathrm{diam}(\Lambda') \leq e^{C/g^2+O(1)}$. Theorems 1.1–1.2 are therefore restated as *windowed and conditional on (H-DOB-blk) and (H-P0)*. What survives unconditionally — and is validated numerically — is the boundary-uniformity mechanism itself: the per-plaquette oscillation and gradient bounds (Lemma 3.1; sharp for $N_c = 2$), the "frozen = slow" reduction (Lemma 3.2), the refined dynamical large-field event, the energy-penalty identity $\|U-\mathbf{1}\|_{\mathrm{HS}}^2 = 2N_c(1-\frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,U)$, the TV $\leq \tanh(\mathrm{osc}/4)$ lemma with its two-point equality case, and the Bakry–Émery constant $N_c/4$ in the $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$ convention. The contribution of the paper is thus retagged: a boundary-uniform *reduction* of the DLR-LSI and the mass gap to (H-DOB-blk)+(H-P0) within the volume window — not an unconditional mass gap.

1 Introduction

1.1 Background and motivation

The mass gap problem for four-dimensional Yang–Mills theory is one of the seven Clay Millennium Prize Problems. On the lattice, the question reduces to establishing a strictly positive spectral gap $\Delta_{\rm phys} > 0$ for the transfer matrix of the Wilson action at sufficiently weak coupling ($\beta \gg 1$), uniformly in the lattice volume.

In our earlier work [14, 15, 16, 17, 18, 19] we developed a multiscale strategy based on Balaban's renormalization group (RG) and log-Sobolev inequalities (LSI). References [15, 16, 17, 18] established a *uniform LSI on periodic tori*:

$$\mathrm{Ent}_{\mu_\beta}(f^2) \;\le\; \frac{2}{\alpha_*}\int |\nabla f|^2\, d\mu_\beta, \qquad \alpha_* > 0 \text{ independent of } L,$$

for the Wilson measure μ_β on $(\mathbb{Z}/L\mathbb{Z})^d$. (v2: per the current versions of the companions — in particular [14] = ai.viXra:2602.0041(v3) — the periodic-torus statement itself carries explicit hypotheses and a volume window; see §7.1.)

To pass from the periodic LSI to a mass gap, [14] relied on Assumption 6.3 (a Dobrushin-type contraction condition, tagged (H-DOB) in [14](v3)), and [19] replaced this with Assumption 3.1 (boundary-uniform RG outputs). Both assumptions remained unproved.

The purpose of the present paper was, in v1, to remove these assumptions. **Version 2 corrects this**: the boundary-uniformity content (Assumption 3.1 of [19]) is indeed established here at the structural level (§§3–4), but the Dobrushin-type input is *not* removed — it is reduced to a block-level condition (H-DOB-blk) which the audit shows cannot be verified by any worst-case argument (§3.3), and the quantitative RG inputs inherit (H-P0) and the volume window from ai.viXra:2602.0052(v2) and 2602.0051(v2) [21, 20].

1.2 Main results

(H-DOB-blk) Block Dobrushin contraction. *At every scale $k \le n_{\max}$ and for every frozen boundary configuration ω, the block interdependence matrix $C = (c_{ij})$ of Lemma 3.5 satisfies $\delta = \sup_i \sum_{j\neq i} c_{ij} < 1$, with δ bounded away from 1 uniformly in (k, Λ, ω).* This is v1's display (9), assumed but not verified (v1, Remark 3.6 = Remark 3.6 below). It is the block-level residue of Assumption 6.3 of [14] ((H-DOB) in [14](v3)). The audit of §3.3 and Appendix A shows that worst-case influence bounds cannot establish it at weak coupling; an averaged or transport-based criterion would be needed.

(H-P0) Power-law large-field penalty (imported from ai.viXra:2602.0052(v2) [21]). *Balaban's large-field penalty satisfies $p_0(g) \gtrsim g^{-2}$ (power law), rather than only the polylog floor $p_0(g) \ge c_0|\log g|^{1+\epsilon_0}$; equivalently, the absorption inequalities of Proposition 4.3 hold for all k in the window.*

Theorem 1.1 (DLR-uniform LSI; v2: windowed and conditional)**.** *Assume* (H-DOB-blk) *and* (H-P0)*. Let $d \ge 3$, $G = \mathrm{SU}(N_c)$ with $N_c \ge 2$, and $\beta \ge \beta_0$ where $\beta_0 = \beta_0(d, N_c)$ is the weak-coupling threshold of [14–18]. For every finite subset $\Lambda' \subset \mathbb{Z}^d$ with*

$$\log_{L_{\rm RG}} \mathrm{diam}(\Lambda') \;\le\; k^*(\beta) \;:=\; \frac{g_0^{-2} - \gamma_0^{-2}}{2b_0 \ln L_{\rm RG}}, \qquad g_0 = \beta^{-1/2},$$

and every boundary condition $\omega \in G^{E(\mathbb{Z}^d \setminus \Lambda')}$*, the conditional Gibbs measure* $\mu^{\omega}_{\Lambda'}$ *satisfies*

$$\mathrm{Ent}_{\mu^{\omega}_{\Lambda'}}(f^2) \;\le\; \frac{2}{\alpha_*}\int |\nabla f|^2\, d\mu^{\omega}_{\Lambda'} \tag{1}$$

for all Lipschitz $f : G^{E(\Lambda')} \to \mathbb{R}$*, with* $\alpha_* = \alpha_*(d, N_c, \beta) > 0$ *independent of* Λ' *(within the window) and of* ω*.* (v1 claimed this for *all* finite Λ' and without hypotheses; see the Note added in v2.)

Theorem 1.2 (Mass gap; v2: windowed and conditional)**.** *Under the hypotheses of Theorem 1.1, for all even integers* L *with* $2 \le L \le e^{C/g^2+O(1)}$*, the transfer matrix* $\hat{T}$ *of the Wilson lattice gauge theory on* $(\mathbb{Z}/L\mathbb{Z})^d$ *has a spectral gap*

$$\Delta_{\text{phys}} \;:=\; -\log \|\hat{T}|_{\mathbf{1}^\perp}\|_{L^2(\nu)\to L^2(\nu)} \;\ge\; m(\beta, N_c, d) \;>\; 0 \tag{2}$$

uniformly in L *within the window, where* ν *denotes the stationary measure of a spatial time-slice.* (v1: "Unconditional mass gap", uniformly in all even L.)

1.3 Logical structure of the proof

The corrected proof chain is:

$$\boxed{\begin{array}{c} \text{Uniform fiber oscillation (§3)} \\ + \text{ Uniform RG outputs with frozen boundary (§4), } \textit{quantitative part under} \text{ (H-P0)} \\ + \text{ (H-DOB-blk) (§3.3)} \\ \Downarrow \\ \text{DLR-LSI on the window (§5)} \xRightarrow{\text{Stroock–Zegarlinski}} \text{DS mixing / exp. clustering (§6)} \\ \xRightarrow{\text{Reflection Positivity}} \text{Mass gap on the window (§6)} \end{array}}$$

The genuinely new mathematical content is in §3 (the uniform fiber oscillation lemma and the refined large-field event) and it is *unconditional*. All other sections assemble existing results — from [14–19] and the standard literature — with the observation that the relevant constants are independent of boundary conditions; v2 makes explicit which steps are conditional.

1.4 Notation and conventions

We collect notation used throughout.

- $G = \mathrm{SU}(N_c)$ with the bi-invariant metric $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$ on $\mathfrak{su}(N_c)$. The induced Hilbert–Schmidt norm is $\|A\|^2_{\mathrm{HS}} = -2\,\mathrm{tr}(A^2)$ for $A \in \mathfrak{su}(N_c)$ and $\|M\|^2_{\mathrm{HS}} = \mathrm{tr}(M^*M)$ for $M \in \mathrm{Mat}_{N_c}(\mathbb{C})$. (v2: in this convention $\mathrm{Ric} = N_c/4$; the series convention triangle is $N_c/4$ for $-2\,\mathrm{tr}$, $N_c/2$ for $-\mathrm{tr}$, $1/2$ for $-N_c\,\mathrm{tr}$; verified in Appendix A, check 8.)
- For a lattice $\Lambda \subset \mathbb{Z}^d$, $E(\Lambda)$ denotes its oriented edges and $\mathcal{P}(\Lambda)$ its plaquettes.
- The Wilson action is $S_W(U) = \sum_{p\in\mathcal{P}(\Lambda)}(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,U_p)$ where U_p is the ordered product of link variables around p.
- The Wilson measure at coupling β is $d\mu_\beta(U) = Z^{-1}\exp(-\beta S_W(U))\prod_{e\in E(\Lambda)} dU_e$ where dU_e is the Haar probability measure on G.

- $L_{\mathrm{RG}} \geq 2$ is the RG blocking factor, and $n_{\max}$ the number of RG steps.
- For $\phi : \mathcal{M} \to \mathbb{R}$, $\operatorname{osc}(\phi) := \sup \phi - \inf \phi$.
- $\operatorname{Ent}_\mu(h) = \int h \log h \, d\mu - (\int h \, d\mu) \log(\int h \, d\mu)$ for $h \geq 0$.
- $\mathcal{E}(f, f) = \int |\nabla f|^2 \, d\mu$ where ∇ is the Riemannian gradient on $G^{|E|}$.
- (v2) The coupling flow is $1/g_k^2 = 1/g_0^2 - 2b_0 \, k \ln L_{\mathrm{RG}}$ with $b_0 = 11N_c/(48\pi^2)$: g_k *increases* toward the infrared (series erratum; [20, 21]). The small-field regime $g_k \leq \gamma_0$ therefore holds precisely for $k \leq k^*(\beta)$.

2 Lattice gauge theory with boundary conditions

2.1 The DLR specification

Let $\Lambda' \subset \mathbb{Z}^d$ be a finite connected subset. A *boundary condition* is a configuration $\omega = (\omega_e)_{e \in E(\mathbb{Z}^d \setminus \Lambda')} \in G^{E(\mathbb{Z}^d \setminus \Lambda')}$. The conditional Gibbs measure on Λ' with boundary condition ω is

$$d\mu^\omega_{\Lambda'}(U) \;=\; \frac{1}{Z_{\Lambda'}(\omega)} \exp\Big(-\beta \sum_{p \in \mathcal{P}^*_{\Lambda'}} \big(1 - \tfrac{1}{N_c} \operatorname{Re} \operatorname{tr} U_p^\omega\big)\Big) \prod_{e \in E(\Lambda')} dU_e, \tag{3}$$

where $\mathcal{P}^*_{\Lambda'} := \{p \in \mathcal{P}(\mathbb{Z}^d) : p \cap E(\Lambda') \neq \emptyset\}$ is the set of plaquettes involving at least one link in Λ', and U_p^ω denotes the plaquette holonomy computed using U_e for $e \in E(\Lambda')$ and ω_e for $e \notin E(\Lambda')$. The collection $\{\mu^\omega_{\Lambda'}\}_{\Lambda', \omega}$ forms a *Gibbs specification* in the sense of Dobrushin–Lanford–Ruelle (DLR).

Definition 2.1 (DLR-uniform LSI)**.** *A Gibbs specification satisfies the* DLR-uniform LSI *with constant $\alpha_* > 0$ if, for every finite $\Lambda' \subset \mathbb{Z}^d$ and every boundary condition ω,*

$$\operatorname{Ent}_{\mu^\omega_{\Lambda'}}(f^2) \;\leq\; \frac{2}{\alpha_*} \, \mathcal{E}_{\Lambda'}(f, f) \qquad \forall f \in \operatorname{Lip}(G^{E(\Lambda')}),$$

where $\mathcal{E}_{\Lambda'}(f, f) = \int_{G^{E(\Lambda')}} |\nabla f|^2 \, d\mu^\omega_{\Lambda'}$. (v2: Theorem 1.1 establishes this on the volume window *and under (H-DOB-blk)+(H-P0).)*

2.2 The embedding lemma

To apply Balaban's RG (which requires a rectangular block structure), we embed arbitrary domains into boxes.

Lemma 2.2 (Embedding)**.** *For any finite connected $\Lambda' \subset \mathbb{Z}^d$ and any boundary condition ω, there exists a rectangular box $\Lambda'' = \prod_{i=1}^d [0, L_i)$ with each L_i a multiple of $L_{\mathrm{RG}}^{n_{\max}}$, $n_{\max} = \lfloor \log_{L_{\mathrm{RG}}} \operatorname{diam}(\Lambda') \rfloor + 1$, and an extended configuration $\tilde{\omega} \in G^{E(\Lambda'' \setminus \Lambda')}$ such that:*

(a) $\Lambda' \subset \Lambda''$;

(b) $\mu^\omega_{\Lambda'}$ *is the marginal on $E(\Lambda')$ of the measure $\mu^{\tilde{\omega}}_{\Lambda''}$, where $\tilde{\omega}$ extends ω to $E(\Lambda'' \setminus \Lambda')$ by setting $\tilde{\omega}_e = \mathbf{1}_G$ for links $e \in E(\Lambda'' \setminus \Lambda') \setminus E(\mathbb{Z}^d \setminus \Lambda')$;*

(c) the terminal lattice $\Lambda''_{n_{\max}}$ has $O(1)$ sites (depending only on d and L_{RG}).

Proof. Choose Λ'' as the smallest d-dimensional box containing Λ' whose side lengths are multiples of $L_{\mathrm{RG}}^{n_{\max}}$. The links in $E(\Lambda'') \setminus E(\Lambda')$ are frozen to $\tilde{\omega}$ values: those in $E(\mathbb{Z}^d \setminus \Lambda')$ are set to ω, and any remaining links are set to the identity $\mathbf{1}_G$. The marginal property follows because integrating over $E(\Lambda')$ with $E(\Lambda'' \setminus \Lambda')$ frozen to $\tilde{\omega}$ reproduces the conditional measure (3): plaquettes not meeting $E(\Lambda')$ contribute ω-dependent constants that cancel in the normalization. □

Remark 2.3. Henceforth, we work on the rectangular box Λ'' with frozen external links $\tilde{\omega}$. To simplify notation, we write ω for $\tilde{\omega}$ and Λ for Λ''. The links of Λ are partitioned into *free links* $E_{\mathrm{free}} := E(\Lambda')$ (integration variables) and *frozen links* $E_{\mathrm{frz}} := E(\Lambda) \setminus E(\Lambda')$ (fixed to ω). (v2) Note $n_{\max} \sim \log_{L_{\mathrm{RG}}} \operatorname{diam}(\Lambda')$: the requirement $g_k \leq \gamma_0$ for all $k \leq n_{\max}$ is exactly what produces the volume window of Theorem 1.1.

2.3 Interior and boundary plaquettes

Definition 2.4 (Plaquette classification at scale k)**.** *Let B be a scale-k block and let $E_k(B)$ denote the set of* fast links integrated at scale k within B. *Consider a plaquette p such that p contains at least one link of $E_k(B)$.*

(i) p is dynamical *for B if **none** of its four links lies in E_{frz}. Equivalently, all links of p are free links (integration variables), though some may be slow variables in $\mathcal{G}_{k+1}$. Write $\mathcal{P}_k^{\mathrm{dyn}}(B)$ for the set of such plaquettes.*

(ii) p is mixed *if at least one link of p lies in E_{frz}. Mixed plaquettes contribute to the conditional potential $V_{k,B}$ (hence to oscillation bounds), but are excluded from the large-field event, since their holonomies depend on fixed parameters ω.*

(iii) p is fully frozen *if all links of p lie in E_{frz}. Such plaquettes contribute only constants and are ignored.*

Definition 2.5 (Refined large-field event)**.** *The large-field event at block B and scale k is*

$$Z_k^{\mathrm{dyn}}(B) \; := \; \big\{\exists\, p \in \mathcal{P}_k^{\mathrm{dyn}}(B) \;:\; \|U_p - \mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon_k\big\}, \tag{4}$$

where $\varepsilon_k = g_k^{1-\delta}$ is the scale-k small-field threshold from Balaban's construction. The complementary small-field region is $\Omega_k^{\mathrm{sf}}(B) := (Z_k^{\mathrm{dyn}}(B))^c$.

Remark 2.6. Frozen-boundary plaquettes are *not* included in the large-field event. Their holonomies involve frozen links ω_e and may be far from the identity, but this is irrelevant: the event $Z_k^{\mathrm{dyn}}(B)$ concerns only the random (free) link variables, and the energy suppression of large fields operates on these random variables alone.

3 Uniform fiber oscillation and fiber LSI

This section contains the main new technical content of the paper. (v2: §§3.1–3.2 are unconditional and numerically validated; §3.3 is where the conditional structure enters.)

3.1 Per-plaquette oscillation

Lemma 3.1 (Per-plaquette oscillation bound). *For any $M \in G = \mathrm{SU}(N_c)$, define $\phi_M : G \to \mathbb{R}$ by*

$$\phi_M(U) \ := \ 1 - \tfrac{1}{N_c} \operatorname{Re}\operatorname{tr}(UM). \tag{5}$$

Then:

(a) ***Oscillation bound:*** $\operatorname{osc}(\phi_M) \le 2$*, independently of* M*.*

(b) ***Gradient bound:*** *for any unit tangent vector* $v \in T_U G$ *with* $|v| = 1$*,*

$$|v(\phi_M)| \ \le \ \frac{1}{\sqrt{2N_c}}, \tag{6}$$

independently of M *and* U*.*

Proof. **Part (a).** Since $M \in \mathrm{SU}(N_c)$, all singular values of M equal 1. For any $U \in \mathrm{SU}(N_c)$, $\operatorname{Re}\operatorname{tr}(UM) \le |\operatorname{tr}(UM)| \le \|U\|_{\mathrm{HS}}\|M\|_{\mathrm{HS}} = N_c$; the maximum N_c is achieved at $U = M^{-1} = M^*$. For a lower bound: since $UM \in \mathrm{SU}(N_c)$, its eigenvalues are $e^{i\theta_1}, \ldots, e^{i\theta_{N_c}}$ with $\sum_j \theta_j \in 2\pi\mathbb{Z}$, whence $\operatorname{Re}\operatorname{tr}(UM) = \sum_j \cos\theta_j \ge -N_c$. Therefore $\operatorname{osc}(\phi_M) \le \frac{1}{N_c}(N_c - (-N_c)) = 2$.

Part (b). A unit tangent vector $v \in T_U\mathrm{SU}(N_c)$ satisfies $v = U \cdot X$ for some $X \in \mathfrak{su}(N_c)$ with $|X|^2 = -2\operatorname{tr}(X^2) = 1$, i.e. $\|X\|_{\mathrm{HS}} = 1/\sqrt{2}$. Then

$$|v(\phi_M)| = \tfrac{1}{N_c}|\operatorname{Re}\operatorname{tr}(UXM)| \le \tfrac{1}{N_c}\|UX\|_{\mathrm{HS}}\|M\|_{\mathrm{HS}} = \tfrac{1}{N_c} \cdot \tfrac{1}{\sqrt{2}} \cdot \sqrt{N_c} = \tfrac{1}{\sqrt{2N_c}}. \qquad \square$$

(v2) *Numerical adjudication* (Appendix A, checks 1–2): for $N_c = 2$ the constants are *sharp* — osc $= 2$ is attained at $U = -M^*$ and the gradient bound is attained exactly at $M = 2(UX)^* \in \mathrm{SU}(2)$; for $N_c = 3$ the exact oscillation is $3/2$ (attained at $U = e^{2\pi i/3}M^*$) and the sampled gradient maximum is $\approx 0.31 < 1/\sqrt{6} \approx 0.408$.

3.2 Per-block fiber oscillation

Lemma 3.2 (Uniform per-block oscillation). *Fix $d \ge 2$, $N_c \ge 2$, $L_{\mathrm{RG}} \ge 2$, and $\beta > 0$. Let B be a scale-k block in the RG decomposition of Λ, and let ω be any frozen configuration on E_{frz}. Let $E_k(B)$ denote the set of free fast-variable links at scale k within B. The conditional fast potential $V_{k,B}(\cdot \mid \mathcal{G}_{k+1}, \omega)$, viewed as a function of the fast variables $(U_e)_{e \in E_k(B)}$ with the slow variables $\mathcal{G}_{k+1}$ and frozen links ω held fixed, satisfies*

$$\operatorname{osc}_{E_k(B)}(V_{k,B}) \ \le \ C_{\mathrm{fib}}, \tag{7}$$

where

$$C_{\mathrm{fib}} \ := \ 2\beta \cdot n_{\mathrm{plaq}} + C_{\mathrm{poly}}, \tag{8}$$

with $n_{\mathrm{plaq}} \le d(d-1)L_{\mathrm{RG}}^d + 2d(d-1)L_{\mathrm{RG}}^{d-1}$ the maximum number of plaquettes involving at least one link in $E_k(B)$, and $C_{\mathrm{poly}} < \infty$ accounting for polymer residuals, obtained by lattice-animal counting from the exponential polymer decay.

The constant C_{fib} is independent of the block position (interior or boundary), the domain Λ, the boundary condition ω, the scale k, and the volume $|\Lambda'|$.

Proof. The potential $V_{k,B}$ receives contributions from two sources.

1. Wilson plaquette terms. Each plaquette p involving at least one link $e \in E_k(B)$ contributes a term $\beta\,\phi_{M_p}(U_e)$ where $M_p \in G$ is the product of the remaining links of p (which are either slow variables in $\mathcal{G}_{k+1}$ or frozen links in ω — in both cases, fixed parameters from the perspective of the fast fiber). By Lemma 3.1(a), $\mathrm{osc}_{U_e}(\beta\phi_{M_p}) \le 2\beta$, independently of M_p (hence of ω). The number of such plaquettes is at most n_{plaq}: each free link $e \in E_k(B)$ belongs to at most $2d(d-1)$ plaquettes, and $|E_k(B)| \le C_d L_{\mathrm{RG}}^d$. The bound is the same for interior and boundary blocks because it counts plaquettes by their intersection with $E_k(B)$ — a local geometric quantity.

2. Polymer residuals. By [18] (Theorem 1.1 and Counting Lemma 3.3), the polymer activities satisfy $|R^{(k)}(X)| \le C_R e^{-\kappa d_k(X)}$ for all polymers X and all background fields (including frozen boundary values), where C_R and κ depend on $(d, N_c, \gamma_0, L_{\mathrm{RG}})$ only (cf. Proposition 4.1 below). The total contribution to the oscillation from polymers containing B is bounded by

$$2\sum_{X\ni B} |R^{(k)}(X)| \;\le\; 2C_R \sum_{n=0}^{\infty} \big|\{X \ni B : d_k(X) = n\}\big| e^{-\kappa n} \;\le\; C_{\mathrm{poly}},$$

where $C_{\mathrm{poly}} = C_{\mathrm{poly}}(d, N_c, \gamma_0, L_{\mathrm{RG}}, \kappa) < \infty$ follows from standard lattice-animal counting. C_{poly} is independent of ω, k, and the block position. Combining gives (7). $\square$

(v2) Note the size of C_{fib}: it is *linear in* β and extensive in the block. Wherever C_{fib} enters an exponential (Holley–Stroock, worst-case total variation), the resulting constant degrades like $e^{-2\beta n_{\mathrm{plaq}}}$ — this is the quantitative weakness adjudicated as F3/(H-YGZ) in [21] and it is what blocks a worst-case verification of (H-DOB-blk) below.

3.3 Fiber LSI: per-block Bakry–Émery and quasi-factorization

We aim at a log-Sobolev inequality for the *scale-k fast fiber* conditional measure, uniformly in the frozen boundary condition ω: (i) a per-block LSI from Balaban's coercive quadratic form on the small-field region and Holley–Stroock patching; (ii) an assembly from per-block LSI to the full fiber LSI via a quasi-factorization criterion.

Theorem 3.3 (Fiber LSI at scale k, uniform in ω; v2: conditional on (H-DOB-blk))**.** *Assume Balaban's RG hypotheses at weak coupling (as in [14]), in particular $g_k \le \gamma_0$ for all $0 \le k \le n_{\max}$ (v2: this restricts to the window $k \le k^*(\beta)$), and assume* (H-DOB-blk)*. Fix a scale k and a frozen configuration $\omega \in G^{E_{\mathrm{frz}}}$. Then for μ-a.e. realization of the slow σ-algebra $\mathcal{G}_{k+1}$, the conditional fast measure $\mu_k(\cdot \mid \mathcal{G}_{k+1}, \omega)$ on the fast links E_k satisfies* $\mathrm{LSI}(\alpha_{\mathrm{fiber}})$ *with $\alpha_{\mathrm{fiber}} > 0$ depending only on $(d, N_c, \beta, L_{\mathrm{RG}}, \gamma_0, \kappa, \delta)$ and independent of Λ, k, the volume, and ω.*

Lemma 3.4 (Per-block LSI)**.** *Fix a scale k and a scale-k block B. Condition on all link variables outside the fast set $E_k(B)$ (slow variables in $\mathcal{G}_{k+1}$, fast variables in other blocks $B' \ne B$ at scale k, and the frozen links ω). The resulting single-block conditional measure on $(U_e)_{e\in E_k(B)}$ satisfies* $\mathrm{LSI}(\alpha_{\mathrm{blk}})$ *for some $\alpha_{\mathrm{blk}} > 0$ depending only on $(d, N_c, \beta, L_{\mathrm{RG}}, \gamma_0, \kappa)$ and independent of the block position and ω.* (v2: by Bakry–Émery on $G^{|E_k(B)|}$ (Ric $\ge N_c/4$) and Holley–Stroock with Lemma 3.2, $\alpha_{\mathrm{blk}} \ge (N_c/4)e^{-C_{\mathrm{fib}}}$: strictly positive but exponentially degraded in βn_{plaq}; cf. F3/(H-YGZ) of [21].)

Lemma 3.5 (Quasi-factorization assembly at scale k)**.** *Let $\{B_1, \ldots, B_N\}$ be the scale-k blocks in Λ and write $\mu_k(\cdot \mid \mathcal{G}_{k+1}, \omega)$ for the conditional fast measure on the full fast fiber $E_k = \bigcup_{i=1}^N E_k(B_i)$. Assume:*

(i) (Uniform per-block LSI) Each single-block conditional measure satisfies $\mathrm{LSI}(\alpha_{\mathrm{blk}})$ *with the same constant* α_{blk}.

(ii) (Weak inter-block dependence) Let $C = (c_{ij})_{1\le i,j\le N}$ *be the Dobrushin interdependence matrix on the block lattice,*

$$c_{ij} := \sup_{\tau,\tau'} \big\| \mu_{E_k(B_i)}(\cdot \mid \tau) - \mu_{E_k(B_i)}(\cdot \mid \tau') \big\|_{\mathrm{TV}},$$

the supremum over pairs τ, τ' *of configurations on* $E_k \setminus E_k(B_i)$ *differing only on* $E_k(B_j)$. *Assume the Dobrushin condition*

$$\delta \ := \ \sup_{1\le i\le N} \sum_{j\ne i} c_{ij} \ < \ 1. \tag{9}$$

Then the full conditional fast measure at scale k *satisfies* $\mathrm{LSI}(\alpha_{\mathrm{fiber}})$ *with* $\alpha_{\mathrm{fiber}} = \alpha_{\mathrm{fiber}}(\alpha_{\mathrm{blk}}, \delta) > 0$ *independent of* N *and hence of the volume.*

Remark 3.6. (v1's Remark 3.6, retained verbatim in substance.) The verification of the Dobrushin condition (9) cannot rely on a worst-case oscillation bound on the log-density, since such bounds scale like β and therefore do not yield $\delta < 1$ in the weak-coupling regime $\beta \to \infty$. To close the argument one needs an additional weak-dependence input exploiting small-field coercivity and the large-field suppression mechanism (cf. Proposition 4.2). *We leave this verification to a separate proposition once an appropriate influence criterion is fixed* (e.g. an averaged or transport-based Dobrushin criterion).

Remark 3.7 (v2: quantitative adjudication of Remark 3.6; birth of (H-DOB-blk)). Version 1 stated Remark 3.6 and then *assumed* (9) in the proof of Lemma 3.5, while the abstract and §7 claimed an unconditional theorem. The audit (Appendix A, checks 5–6) quantifies the obstruction:

(i) *The printed bound trivializes.* v1's estimate for adjacent blocks was $c_{ij} \le \tanh(\beta n_\partial/2)$ with $n_\partial \le 2d(d-1)L_{\mathrm{RG}}^{d-1}$ (from $\mathrm{osc} \le 2\beta n_\partial$ and $\|\nu - \nu'\|_{\mathrm{TV}} \le \tanh(\mathrm{osc}/4)$; both ingredients are correct and validated, Appendix A, check 4). With $2d$ block neighbours, this gives $\delta < 1$ only for $\beta < (2/n_\partial)\,\mathrm{artanh}(1/2d)$: numerically $\beta \lesssim 7.0 \times 10^{-3}$ ($d = 3$, $L_{\mathrm{RG}} = 2$) and $\beta \lesssim 1.3 \times 10^{-3}$ ($d = 4$) — the *opposite* of the regime $\beta \ge \beta_0 \gg 1$.

(ii) *The failure is not an artifact of the bound.* In a one-rotor conditional toy $\mu_\tau \propto e^{\beta\cos(x-\tau)}$ the genuine worst-case influence $\sup_{\tau,\tau'} \|\mu_\tau - \mu_{\tau'}\|_{\mathrm{TV}}$ is 0.561, 0.985, 1.000 at $\beta = 1, 4, 16$: the supremum over boundary pairs really tends to 1 at weak coupling. Hence *no* worst-case (sup over τ, τ') criterion can verify (9); only an averaged, entropic, or transport-based criterion — which would use that *typical* boundary pairs are close under the conditional slow measure — could. None is proved here.

Accordingly, (9) is promoted to the standing hypothesis (H-DOB-blk), Theorem 3.3 is conditional on it, and v1's claim that this paper removes Assumption 6.3 of [14] is withdrawn: the correct statement is that Assumption 6.3 of [14] (site-level, (H-DOB) in [14](v3)) is *reduced* to the block-level (H-DOB-blk), with the boundary-uniformity of all other inputs established in §4.

Verification of the hypotheses of Lemma 3.5. Hypothesis (i) is Lemma 3.4. For hypothesis (ii), the conditional measure $\mu_{E_k(B_i)}(\cdot \mid \tau)$ depends on τ only through plaquette terms and polymer activities involving at least one link in $E_k(B_i)$. Changing τ on $E_k(B_j)$ ($j \ne i$) affects only:

- *Plaquette terms spanning B_i and B_j*: only for adjacent blocks; each contributes oscillation $\leq 2\beta$ to the log-density ratio (Lemma 3.1), with at most $n_\partial \leq 2d(d-1)L_{\mathrm{RG}}^{d-1}$ such plaquettes. If $\operatorname{osc}(\log(d\nu/d\nu')) \leq r$ then $\|\nu - \nu'\|_{\mathrm{TV}} \leq \tanh(r/4)$; with $r = 2\beta n_\partial$,

$$c_{ij} \;\leq\; \tanh\Big(\frac{\beta n_\partial}{2}\Big) \qquad \text{if } \operatorname{dist}_{\mathrm{block}}(B_i, B_j) = 1.$$

- *Polymer activities $R^{(k)}(X)$ for X intersecting both blocks*: by Proposition 4.1 and lattice-animal counting, total log-density oscillation $\leq C_{\mathrm{poly}} e^{-\kappa \operatorname{dist}_{\mathrm{block}}(B_i,B_j)}$.

(v2) These estimates are correct but, per Remark 3.7, insufficient: (9) is *assumed* as (H-DOB-blk). Under (H-DOB-blk), the conclusion follows from [12], Theorem 3.1 (equivalently [9], Theorem 3.2 and Corollary 3.4): uniform single-block LSI together with $\delta < 1$ implies LSI(α_{fiber}) for the full fast fiber.

4 Uniform RG outputs with frozen boundary

In this section we verify that the three main inputs to the multiscale LSI proof — polymer bounds, large-field suppression, and cross-scale derivative bounds — extend to the setting with frozen boundary links, with constants independent of ω. (v2: the boundary-uniformity is the genuine content and stands; the *quantitative* absorption in §4.3 additionally requires (H-P0).)

4.1 Polymer bounds

Proposition 4.1 (Polymer bounds, uniform in ω)**.** *Under Balaban's RG at weak coupling ($g_k \leq \gamma_0$), for every scale $k \leq n_{\max}$, every polymer X at scale k, and every frozen configuration $\omega \in G^{E_{\mathrm{frz}}}$:*

$$|R^{(k)}(X; U, \omega)| \;\leq\; C_R e^{-\kappa d_k(X)}, \tag{10}$$

with $C_R = C_R(d, N_c, \gamma_0, L_{\mathrm{RG}}) > 0$ and $\kappa = \kappa(d, L_{\mathrm{RG}}) > 0$ independent of ω, Λ, and L_{vol}.

Proof. The polymer activity $R^{(k)}(X)$ is produced by Balaban's cluster/polymer expansion [18, 4]. The frozen boundary values ω_e enter the expansion as fixed parameters in the background field. The three ingredients controlling convergence are: (i) *small-coupling condition* $g_k \leq \gamma_0$, which depends only on $g_0 = 1/\sqrt{\beta}$ and k — not on ω (v2: and which restricts k to the window, §1.4); (ii) *analyticity radius* $\hat{\alpha}_1 = \hat{\alpha}_1(\gamma_0, L_{\mathrm{RG}})$, determined by block geometry and the coercivity of the quadratic form, a property of the lattice Laplacian on a single block, independent of boundary values; (iii) *Cauchy estimates on free variables*: the activities are analytic in the free link variables on a fixed-radius domain, and the frozen links enter only as fixed parameters in coefficients; by compactness of $\mathrm{SU}(N_c)$, $\sup_{\omega \in G^{E_{\mathrm{frz}}}}$ produces a finite bound absorbed into C_R. The lattice-animal counting giving $e^{-\kappa d_k(X)}$ is purely combinatorial, hence ω-independent. □

4.2 Large-field suppression

Proposition 4.2 (Large-field suppression, uniform in ω)**.** *For every scale k, every block B, and every frozen configuration ω:*

$$\operatorname*{ess\,sup}_{\mathcal{G}_{k+1}} \; \mu_k\big(Z_k^{\mathrm{dyn}}(B) \;\big|\; \mathcal{G}_{k+1}, \omega\big) \;\leq\; C_{\mathrm{blk}}\, e^{-c\, p_0(g_k)}, \tag{11}$$

where $Z_k^{\mathrm{dyn}}(B)$ is the refined large-field event of Definition 2.5, and $C_{\mathrm{blk}}, c > 0$ depend on $(d, N_c, \beta_0, L_{\mathrm{RG}})$ only.

Proof. The proof follows [17] (which formalizes Balaban's T-operation [4], §1.3). We highlight the modifications for the boundary setting.

1. Energy cost of dynamical large fields. For a dynamical plaquette $p \in \mathcal{P}_k^{\mathrm{dyn}}(B)$, the Wilson energy penalty is

$$\beta\big(1 - \tfrac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,U_p\big) \;\geq\; \frac{\beta\,\varepsilon_k^2}{2N_c}$$

on the event $\|U_p - \mathbf{1}\|_{\mathrm{HS}} \geq \varepsilon_k$. This involves only free link variables and is *independent of* ω. (v2: the inequality is in fact an exact identity, $\|U-\mathbf{1}\|_{\mathrm{HS}}^2 = 2N_c(1-\frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,U)$; Appendix A, check 3.)

2. The T-operation. Balaban's T-operation (CMP **122**, eq. (1.75)–(1.89)) absorbs large-field factors into the effective action. The key uniformity estimate bounds the prefactor by $e^{3\sup|\sigma|}$, where $\sup|\sigma|$ is the supremum of the slow-field norm. For boundary blocks, the "slow field" includes frozen values, but $\sup|\sigma| \leq C(\gamma_0)$ by the same analyticity estimates as in the periodic case (the supremum is over the compact group and the analyticity domain, both ω-independent).

3. Refined event. The event $Z_k^{\mathrm{dyn}}(B)$ involves only dynamical plaquettes. Frozen-boundary plaquettes — which could have arbitrarily large $\|U_p - \mathbf{1}\|_{\mathrm{HS}}$ if ω is far from the identity — are excluded. This ensures $\mu_k(Z_k^{\mathrm{dyn}}(B))$ is a genuine probability (not forced to 1) and is suppressed by the Wilson energy penalty on dynamical plaquettes. □

(v2) The constant c and (H-P0). The suppression constant produced by the T-operation route is $c = c_{\mathrm{sf}} = 2/(1+\beta_0) \sim \gamma_0^2/N_c$ (adjudicated in [21], F2): with only the polylog floor $p_0(g) \geq c_0|\log g|^{1+\epsilon_0}$, the exponent $c\,p_0(g_k) \to 0$ in the regime where the companion thresholds force γ_0 small, and (11) carries no smallness. Every quantitative use of (11) in this paper (notably Proposition 4.3 and the defect summability in §5) is therefore made under (H-P0).

4.3 Cross-scale derivative bounds

Proposition 4.3 (Cross-scale bounds, uniform in ω; v2: under (H-P0))**.** *Assume* (H-P0)*. For every unit fast direction v supported on a single link $e \in E_k(B)$ and every frozen configuration ω:*

$$\mathbb{E}\big[|vV_{<k}|^2 \;\big|\; \mathcal{G}_{k+1}, \omega\big] \;\leq\; D_k \;:=\; 2C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}, \tag{12}$$

μ-a.s. in $\mathcal{G}_{k+1}$, with $C_{\mathrm{SF}} = C_{\mathrm{SF}}(d, N_c, \beta, L_{\mathrm{RG}})$ independent of ω and L_{vol}. Moreover $\sum_{k=0}^{n_{\max}} D_k < D_\infty(d, L_{\mathrm{RG}}) < \infty$.

Proof. This is [16], Theorem 1.1, with the inputs replaced by their ω-uniform versions:

- *Small-field (analyticity) contribution:* on $\Omega_k^{\mathrm{sf}}(B)$, the derivative $vV_{<k}$ is bounded by Cauchy estimates on the analyticity domain of the polymer activities; by Proposition 4.1 the radius and activity bounds are ω-independent. The block-averaging adjoint $\|Q_{(k)}^*\| = O(L_{\mathrm{RG}}^{-(d-1)k/2})$ is geometric and ω-independent.

- *Large-field contribution:* on $Z_k^{\mathrm{dyn}}(B)$, the pointwise bound is $|vV_{<k}|^2 \leq M_k^2$, with M_k^2 depending on (d, N_c, β) and the number of plaquettes per link. By Proposition 4.2, the absorption condition

$$M_k^2 \cdot \mu_k(Z_k^{\mathrm{dyn}}(B) \mid \mathcal{G}_{k+1}, \omega) \;\leq\; M_k^2\, C_{\mathrm{blk}}\, e^{-c\,p_0(g_k)} \;\leq\; C_{\mathrm{SF}}^2 L_{\mathrm{RG}}^{-(d-1)k}$$

  is required for $\beta \geq \beta_0$ (identical to [16], eq. (15)).

Summing the two contributions gives (12). $\square$

Remark 4.4 (v2: the absorption step is exactly the audited failure point). The displayed absorption inequality is the step adjudicated in [21] (F3-type) and inherited here: it demands $c\,p_0(g_k) \geq (d-1)k \ln L_{\mathrm{RG}} + O(1)$ for all $k \leq n_{\max}$, with a left side that is *bounded and decreasing* along the corrected flow ($p_0(g_k)$ decreases since g_k increases toward the infrared) and a right side growing linearly in k. Under the polylog floor it fails already at $k = O(1)$: with $g_0 = 0.1$, $d = 4$, $L_{\mathrm{RG}} = 2$, $c = 1$ generously, the failure scale is $k = 2$, while the small-field window itself extends to $k^* \approx 994$ (Appendix A, check 7). Proposition 4.3 is therefore stated under (H-P0); without it, only finitely many scales are controlled and the defect sum in §5 acquires an uncontrolled tail.

5 DLR-uniform log-Sobolev inequality

5.1 Assembly of the multiscale proof

We now reproduce the assembly of [14] with the uniform constants established in §§3–4. (v2: under (H-DOB-blk) and (H-P0), on the volume window.)

Proof of Theorem 1.1. Fix a finite connected $\Lambda' \subset \mathbb{Z}^d$ with $\log_{L_{\mathrm{RG}}} \operatorname{diam}(\Lambda') \leq k^*(\beta)$ and a boundary condition ω.

Case 1: Small volumes. If $|\Lambda'| \leq V_0 := L_{\mathrm{RG}}^d$, then $|E(\Lambda')| \leq C_d L_{\mathrm{RG}}^d$. The configuration space is $G^{|E(\Lambda')|}$, a product of copies of $\mathrm{SU}(N_c)$ with Ricci curvature $\geq N_c/4$ per factor (Appendix A, check 8). The conditional measure has density $e^{-\beta S_W^\omega}/Z$ against Haar with total oscillation $\operatorname{osc}(\beta S_W^\omega) \leq 2\beta|\mathcal{P}_{\Lambda'}^*| \leq 2\beta C_d L_{\mathrm{RG}}^d$. By Holley–Stroock:

$$\alpha_{\mathrm{small}} \;=\; \frac{N_c}{4} \exp\bigl(-2\beta C_d L_{\mathrm{RG}}^d\bigr) \;>\; 0, \tag{13}$$

independent of ω (the oscillation bound uses only $|\mathcal{P}_{\Lambda'}^*| \leq C_d L_{\mathrm{RG}}^d$, ω-independent by plaquette counting).

Case 2: Large volumes ($|\Lambda'| > V_0$). Apply the embedding (Lemma 2.2) to work on a rectangular box Λ with frozen external links; by the window assumption, $n_{\max} \leq k^*(\beta) + O(1)$ and $g_k \leq \gamma_0$ holds at every scale used. We run the proof of [14]:

Step 2a: Entropy telescoping. The scale filtration $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{n_{\max}}$ gives ([14], Proposition 5.5):

$$\operatorname{Ent}_\mu(f^2) \;=\; \sum_{k=0}^{n_{\max}-1} \mathbb{E}\bigl[\operatorname{Ent}(f_k^2 \mid \mathcal{G}_{k+1})\bigr] + \operatorname{Ent}_\mu(f_{n_{\max}}^2), \tag{14}$$

where $f_k = \mathbb{E}[f \mid \mathcal{G}_k]$. This identity is purely measure-theoretic.

Step 2b: Conditional entropy bounds (fiber LSI). By Theorem 3.3 (under (H-DOB-blk)), each conditional entropy term satisfies $\mathbb{E}[\operatorname{Ent}(f_k^2 \mid \mathcal{G}_{k+1})] \leq \frac{2}{\alpha_{\mathrm{fiber}}}\mathbb{E}[|\nabla_{E_k} f_k|^2]$, with α_{fiber} independent of ω.

Step 2c: Energy-only sweeping-out without $n_{\max}$. The sum over scales of the fast-gradient energies does not produce a factor $n_{\max}$: the energy-only sweeping-out estimate (cf. [14], §5) yields $C_{\mathrm{sw}}, C_{\mathrm{def}} < \infty$ (depending only on $(d, N_c, \beta, L_{\mathrm{RG}})$) such that

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}\bigl[|\nabla_{E_k} f_k|^2\bigr] \;\leq\; C_{\mathrm{sw}}\,\mathcal{E}(f,f) + C_{\mathrm{def}}\,\|f\|_\infty^2, \tag{15}$$

uniformly in the volume (within the window) and in ω. Here C_{def} collects the cross-scale defect terms, whose summability is ensured by the geometric decay of Proposition 4.3 and the large-field suppression of Proposition 4.2 — (v2) i.e. precisely under (H-P0), per Remark 4.4.

Step 2d: Defective LSI (volume-independent). Combining (14), Step 2b, and (15):

$$\mathrm{Ent}_\mu(f^2) \;\le\; \frac{2C_{\mathrm{sw}}}{\alpha_{\mathrm{fiber}}}\,\mathcal{E}(f,f) + \frac{2C_{\mathrm{def}}}{\alpha_{\mathrm{fiber}}}\,\|f\|_\infty^2 + \mathrm{Ent}_\mu(f_{n_{\max}}^2), \tag{16}$$

with no $n_{\max}$ dependence.

Step 2e: Terminal LSI. The terminal measure lives on $G^{|E(\Lambda_{n_{\max}})|}$ with $|E(\Lambda_{n_{\max}})| = O(1)$. By Holley–Stroock (as in Case 1): $\mathrm{Ent}_\mu(f_{n_{\max}}^2) \le (2/\alpha_{\mathrm{term}})\mathcal{E}_{n_{\max}}(f_{n_{\max}}, f_{n_{\max}}) + C_{n_{\max}}\|f\|_\infty^2$, with $\alpha_{\mathrm{term}} > 0$ independent of ω.

Step 2f: Variance telescoping and uniform Poincaré. By the same mechanism (conditional Poincaré from fiber LSI + energy-only sweeping-out, [14], Lemma 5.10): $\mathrm{Var}_\mu(f) \le C_P\,\mathcal{E}(f,f)$ with C_P independent of ω.

Step 2g: Rothaus closure. By Rothaus' lemma ([14], Lemma 5.12), the defective LSI (16) combined with the uniform Poincaré inequality gives a tight LSI, $\mathrm{Ent}_\mu(f^2) \le \frac{2}{\alpha_{\mathrm{large}}}\mathcal{E}(f,f)$, with $\alpha_{\mathrm{large}} > 0$ depending on $(\alpha_{\mathrm{fiber}}, C_{\mathrm{sw}}, C_{\mathrm{def}}, \alpha_{\mathrm{term}}, C_P)$ — all ω-independent.

Conclusion. Setting $\alpha_* := \min(\alpha_{\mathrm{small}}, \alpha_{\mathrm{large}}) > 0$ gives DLR-LSI for the specification, on the window and under (H-DOB-blk)+(H-P0). □

5.2 Verification of DLR compatibility

Remark 5.1. The DLR-LSI constant α_* produced above is independent of the choice of Λ' (within the window) and ω because every constant in the proof chain depends only on the fixed parameters $(d, N_c, \beta, L_{\mathrm{RG}}, \gamma_0, \kappa, \delta)$. For small volumes, α_{small} from (13) is a fixed positive number. For large volumes, α_{large} is assembled from ω-independent ingredients. Thus the specification satisfies DLR-LSI with constant α_* in the sense of Definition 2.1, restricted to the window.

Remark 5.2 (v2: the window is a boundary of the method, not of the model)*.* The restriction $\log_{L_{\mathrm{RG}}} \mathrm{diam}(\Lambda') \le k^*(\beta)$, i.e. $\mathrm{diam}(\Lambda') \le e^{C/g^2+O(1)}$ with $C = 24\pi^2/(11N_c)$, is inherited from the corrected coupling flow (series erratum; [20, 21]): beyond k^* the cascade exits the small-field regime $g_k \le \gamma_0$ and Balaban's weak-coupling RG outputs (§4) are no longer available. This is the same common boundary as in ai.viXra:2602.0032/0033/0051/0052 (v2). Removing it would require the strong-coupling handoff (H-XOVER) of [20]. Note the window is doubly exponentially large in practice ($k^* \approx 994$ at $g_0 = 0.1$, i.e. diam $\lesssim 2^{994}$): the limitation is conceptual (the DLR statement quantifies over *all* finite Λ'), not practical.

6 From DLR-LSI to mass gap

6.1 Stroock–Zegarlinski equivalence

We apply the Stroock–Zegarlinski theorem in the formulation of Martinelli [9].

Theorem 6.1 (Stroock–Zegarlinski)**.** *Let $\{\mu^\omega_{\Lambda'}\}$ be a Gibbs specification on a countable lattice with compact single-spin space Ω_0 satisfying: (i) Ω_0 carries a reference measure with* $\mathrm{LSI}(\alpha_{\mathrm{ref}})$*; (ii) the interaction has finite range; (iii) the specification satisfies DLR-LSI(α_*). Then the specification satisfies Dobrushin–Shlosman complete analyticity: there exist $C, \xi < \infty$ such that for all finite Λ', all ω, and all local observables f, g supported on $A, B \subset \Lambda'$:*

$$\left|\mathrm{Cov}_{\mu^\omega_{\Lambda'}}(f,g)\right| \;\le\; C\,\|f\|_\infty \|g\|_\infty\,|A||B|\,e^{-\mathrm{dist}(A,B)/\xi}. \tag{17}$$

Application to Wilson YM. We verify the hypotheses: (i) the single-spin space is $\Omega_0 = \mathrm{SU}(N_c)$ with Haar measure; by Bakry–Émery (Ric $\geq N_c/4$), Haar satisfies LSI($N_c/4$) ✓; (ii) the Wilson action is a nearest-neighbor (plaquette) interaction with range 2 on the link lattice ✓; (iii) DLR-LSI(α_*) is Theorem 1.1 ✓ — (v2) *on the window and under (H-DOB-blk)+(H-P0)*; the Stroock–Zegarlinski equivalence is formulated for the full specification, so its output below is likewise windowed: it applies to volumes and observable separations within the window, which is what the transfer-matrix argument of §6.3 uses. Therefore the Wilson YM specification satisfies DS mixing with correlation length $\xi = \xi(\beta, N_c, d) < \infty$, in the stated conditional/windowed sense.

6.2 Exponential clustering

Corollary 6.2 (Exponential clustering)**.** *Under the hypotheses of Theorem 1.1, for all gauge-invariant local observables $\mathcal{O}_A, \mathcal{O}_B$ supported on $A, B \subset \Lambda$ and all volumes $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ with $L \leq e^{C/g^2+O(1)}$:*

$$\big|\langle \mathcal{O}_A; \mathcal{O}_B\rangle_{\mu_\beta}\big| \;\leq\; C\,\|\mathcal{O}_A\|_\infty \|\mathcal{O}_B\|_\infty\, e^{-\mathrm{dist}(A,B)/\xi}, \tag{18}$$

with C, ξ independent of L within the window. In particular, temporal correlations satisfy $|\langle \mathcal{O}(0)\,\mathcal{O}(t)\rangle_c| \leq C\|\mathcal{O}\|_\infty^2 e^{-|t|/\xi}$.

Proof. DS mixing (17) applied on the torus gives spatial exponential clustering. Since the Wilson action treats all lattice directions symmetrically, the same clustering holds in the temporal direction with the same correlation length. □

6.3 Reflection positivity and spectral gap

Proof of Theorem 1.2. **Step 1: Transfer matrix.** The Wilson action on $(\mathbb{Z}/L\mathbb{Z})^d$ with one distinguished "temporal" direction defines a transfer matrix $\hat{T}$ acting on $L^2(\nu)$, where ν is the stationary measure on a spatial time-slice ([19], §2). Temporal correlations are expressed as

$$\langle \mathcal{O}(0)\,\mathcal{O}(t)\rangle_c \;=\; \langle \mathcal{O}, \hat{T}^{|t|}\mathcal{O}\rangle_\nu - \langle \mathcal{O}\rangle_\nu^2. \tag{19}$$

Step 2: Reflection positivity. The Wilson action satisfies Osterwalder–Seiler reflection positivity (cf. [10]), which ensures $\hat{T} \geq 0$ on $L^2(\nu)$. The spectral theorem gives $\langle \mathcal{O}, \hat{T}^t\mathcal{O}\rangle_\nu = \int_0^{\|\hat{T}\|} \lambda^t \, d\rho_{\mathcal{O}}(\lambda)$ for a positive spectral measure $\rho_{\mathcal{O}}$.

Step 3: From clustering to gap. By Corollary 6.2, temporal correlations decay as $Ce^{-t/\xi}$. This forces $\rho_{\mathcal{O}}([e^{-1/\xi}, \|\hat{T}\|]) = 0$ for all gauge-invariant $\mathcal{O} \perp \mathbf{1}$. Therefore $\|\hat{T}|_{\mathbf{1}^\perp}\| \leq e^{-1/\xi}$ and

$$\Delta_{\mathrm{phys}} = -\log\|\hat{T}|_{\mathbf{1}^\perp}\| \;\geq\; \frac{1}{\xi} \;=:\; m(\beta, N_c, d) \;>\; 0. \qquad \square$$

7 Discussion

7.1 Summary of what has been achieved (v2)

Combining the present paper with [14–19] *as of their current versions*:

(I) **Uniform LSI on periodic tori** [15–18]: per [14](v3) and the series erratum, the assembled statement carries explicit hypotheses (in particular (H-DOB) at site level in [14](v3)) and the volume window. (v1 called this "unconditional".)

(II) **DLR-uniform LSI** (Theorem 1.1): *windowed and conditional on (H-DOB-blk)+(H-P0).* The boundary-uniformity content of Assumption 3.1 of [19] is established at the structural level (§§3–4); its quantitative absorption inherits (H-P0).

(III) **Mass gap** (Theorem 1.2): windowed and conditional as above. Assumption 6.3 of [14] is *not* removed; it is reduced to (H-DOB-blk).

The key structural insight of the paper survives intact and is unconditional: *frozen boundary links are indistinguishable from slow variables from the perspective of the fast-variable fiber* — both are fixed parameters contributing bounded oscillation to the conditional potential, thanks to compactness of $\mathrm{SU}(N_c)$ and the locality of Balaban's polymer expansion (Lemmas 3.1–3.2, validated numerically).

7.2 Erratum for earlier papers in this series (v2: corrected)

- **[14], Assumption 6.3** ((H-DOB) in [14](v3)): v1 claimed this assumption "is no longer needed". **Withdrawn.** The present paper reduces it to the block-level (H-DOB-blk) (Remark 3.7); the reduction is genuine progress (block-level, boundary-uniform, with all other inputs discharged), but a Dobrushin-type hypothesis remains.
- **[19], Assumption 3.1** (boundary-uniform RG outputs): the structural (boundary-uniformity) content is verified as Theorem 3.3, Propositions 4.1, 4.2 and 4.3; the *quantitative* content of Proposition 4.3 additionally requires (H-P0) (Remark 4.4).
- **[14], Theorem 1.1(ii) and [19], Theorem 1.1**: v1 claimed both become unconditional. **Withdrawn**; both are windowed and conditional on (H-DOB-blk)+(H-P0) via Theorem 1.2.
- (v2) **Downstream**: ai.viXra:2602.0054/0055 cite the present paper's v1 conclusion; per the audit of [21] and the present v2, any such use inherits (H-DOB-blk), (H-P0) and the window until independently discharged.

7.3 Relation to the Clay Millennium Problem

The Clay problem asks for (a) construction of a continuum Yang–Mills theory on $\mathbb{R}^4$ satisfying the Wightman (or Osterwalder–Schrader) axioms, and (b) proof that this theory has a strictly positive mass gap. The present work bears on (b) *on the lattice* at weak coupling, and — v2 — only in the windowed, conditional sense of Theorem 1.2. The outstanding challenges are unchanged and are now strictly larger than v1 stated: the hypotheses (H-DOB-blk) and (H-P0); the volume window / strong-coupling handoff (H-XOVER) [20]; the continuum limit; and the $d = 4$ renormalization specifics.

7.4 Possible extensions

1. **Averaged Dobrushin criteria:** the audit (Remark 3.7) shows worst-case influence bounds cannot give (H-DOB-blk), but does not obstruct averaged, entropic, or transport-based criteria (e.g. Dobrushin–Shlosman averaged conditions, or coupling arguments using small-field coercivity); this is the natural next target, and any proof would simultaneously discharge (H-DOB) of [14](v3) on the window.
2. **Other compact gauge groups:** the boundary-uniformity mechanism extends verbatim to any compact Lie group with $\mathrm{Ric} \geq K > 0$.

3. **Matter fields:** coupling to matter may be tractable if the matter sector satisfies its own LSI and the gauge-matter coupling is weak.

4. **Improved correlation length bounds:** the correlation length $\xi(\beta)$ produced by the method grows polynomially in β; lattice perturbation theory suggests $\xi \sim \beta^{1/2}$.

Note added in v2 (July 2026)

This version replaces v1 (February 2026) after a quantitative audit; a deterministic verification suite (`verify_2602_0053.py`, 8 checks, Appendix A) accompanies the paper in the repository `THE-ERIKSSON-PROGRAMME` under `verification/2602-0053/`. Changes:

(1) **Title and main claims retagged.** "Unconditional Mass Gap" → "a Conditional and Windowed Reduction". Theorems 1.1–1.2 now carry (H-DOB-blk), (H-P0) and the volume window explicitly.

(2) **(H-DOB-blk).** v1 assumed the block Dobrushin condition (display (9)) inside the proof of Lemma 3.5 while conceding in Remark 3.6 that it was unverified, yet claimed an unconditional theorem. The condition is now an explicit hypothesis; Remark 3.7 quantifies why no worst-case criterion can verify it (printed bound trivializes for $\beta \gtrsim 10^{-2}$; genuine worst-case influence $\to 1$ in a rotor exhibit).

(3) **(H-P0) inheritance.** The absorption step of Proposition 4.3 is the F3-type failure point adjudicated in ai.viXra:2602.0052(v2): under the polylog floor it fails at $k = O(1)$ (Remark 4.4).

(4) **Volume window.** v1's proof used $g_k \le \gamma_0$ for all $k \le n_{\max} \sim \log \operatorname{diam}(\Lambda')$, which on the corrected asymptotic-freedom flow (series erratum) holds only for $k \le k^*(\beta)$; the DLR statement is restricted accordingly (Remark 5.2).

(5) **§7 rewritten.** The erratum claims of v1 §7.2 (removal of Assumption 6.3 of [14]; unconditionality of [14]/[19] main theorems) are withdrawn and replaced by the reduction statements above.

(6) **Cross-reference hygiene.** v1's prose cited several lemmas/definitions/propositions under the wrong environment names (e.g. "Theorem 3.1(a)" for Lemma 3.1(a), "Theorem 2.5" for Definition 2.5, "Theorem 4.1–4.3" for Propositions 4.1–4.3); corrected throughout.

(7) **Numbering.** All v1 numbered items (Theorems 1.1–1.2, Definitions 2.1/2.4/2.5, Lemmas 2.2/3.1/3.2/3.4/3.5, Remarks 2.3/2.6/3.6/5.1, Theorems 3.3/6.1, Propositions 4.1–4.3, Corollary 6.2, displays (1)–(19)) are preserved in count and position; new v2 material appears as unnumbered hypothesis boxes, Remarks numbered after the v1 items of each section (3.7, 4.4, 5.2), an unnumbered v2 paragraph in §4.2, and Appendix A.

What is *not* changed: the DLR specification and embedding (§2), the classification of plaquettes and the refined large-field event, the statements and proofs of Lemmas 3.1–3.2 and Propositions 4.1–4.2, and the Stroock–Zegarlinski/reflection-positivity pipeline of §6, which are correct as stated (conditionally quantified as above).

A Numerical verification

The companion suite `verify_2602_0053.py` (NumPy, fixed seed, ~2 s, exit code 0) adjudicates every quantitative claim of this paper that is checkable in finite terms. All 8 checks pass:

1. **Lemma 3.1(a).** $\mathrm{osc}(\phi_M) \le 2$ with exact extremes: $\mathrm{Re}\,\mathrm{tr}(UM) = N_c$ at $U = M^*$; minimum $-N_c$ attained for $N_c = 2$ at $U = -M^*$ (bound sharp), and $-N_c/2$ for $N_c = 3$ at $U = e^{2\pi i/3}M^*$ (exact oscillation 3/2). 4000 Haar samples per group stay within range.

2. **Lemma 3.1(b).** Gradient bound $1/\sqrt{2N_c}$: sampled maxima 0.4997 ($N_c = 2$), 0.3102 ($N_c = 3$); for $N_c = 2$ the bound is attained *exactly* at $M = 2(UX)^* \in \mathrm{SU}(2)$ (unitarity and $\det = 1$ verified to 10^{-12}).

3. **Energy-penalty identity** (Proposition 4.2, step 1): $\|U - \mathbf{1}\|_{\mathrm{HS}}^2 = 2N_c(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,U)$ to 3.6×10^{-15} over 4000 samples; the Wilson penalty on the large-field event is $\beta\varepsilon_k^2/(2N_c)$ *exactly*.

4. **TV–oscillation lemma:** $\|\nu - \nu'\|_{\mathrm{TV}} \le \tanh(\mathrm{osc}(\log d\nu/d\nu')/4)$ over random 40-point measures at $r = 0.5, 2, 8$; the symmetric two-point case attains equality ($\mathrm{TV} = \tanh(r/4)$ to 10^{-12}).

5. **F-DOB (printed bound trivializes):** with $n_\partial = 2d(d-1)L_{\mathrm{RG}}^{d-1}$ and $2d$ block neighbours, $\delta_{\mathrm{printed}} < 1$ iff $\beta < (2/n_\partial)\mathrm{artanh}(1/2d)$: 7.0×10^{-3} ($d = 3$), 1.3×10^{-3} ($d = 4$); at $\beta = 1$, $\delta_{\mathrm{printed}} = 6.0$ resp. 8.0.

6. **Genuine worst-case influence $\to$ 1:** rotor conditional $\mu_\tau \propto e^{\beta\cos(x-\tau)}$, $\sup_{\tau,\tau'}\mathrm{TV} = 0.5610, 0.9849, 1.0000, 1.0000$ at $\beta = 1, 4, 16, 64$ (grid $N = 4096$): worst-case Dobrushin genuinely fails at weak coupling; (H-DOB-blk) requires an averaged/transport criterion.

7. **Window and absorption (F1+F3 inherited):** on the corrected flow $1/g_k^2 = 1/g_0^2 - 2b_0 k\ln 2$ ($N_c = 3$, $g_0 = 0.1$, $\gamma_0 = 0.5$): g_k increases $0.100 \to 0.499$, $p_0(g_k) = |\log g_k|^{3/2}$ decreases $3.49 \to 0.58$; the absorption $e^{-c\,p_0(g_k)} \le 2^{-3k}$ (Proposition 4.3, $c = 1$, $d = 4$) fails at $k = 2$, while $k^* = 994$.

8. **Ricci convention:** orthonormal $\mathfrak{su}(N_c)$ basis under $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$ (Gram matrix $= I$ to 10^{-12}); $\mathrm{Ric}(X,X) = \frac{1}{4}\sum_a \|[X, T_a]\|^2 = N_c/4$ to 10^{-10} for $N_c = 2, 3$ — the constant used in (13) and Theorem 6.1(i).

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References

[1] T. Balaban, *Averaging operations for lattice gauge theories*, Commun. Math. Phys. **98** (1985), 17–51.

[2] T. Balaban, *Propagators and renormalization transformations for lattice gauge theories. I*, Commun. Math. Phys. **95** (1984), 17–40.

[3] T. Balaban, *Renormalization group approach to lattice gauge field theories. I*, Commun. Math. Phys. **109** (1987), 249–301.

[4] T. Balaban, *Large field renormalization. II. Localization, exponentiation, and bounds for the R operation*, Commun. Math. Phys. **122** (1989), 355–392.

[5] D. Bakry and M. Émery, *Diffusions hypercontractives*, Sém. de Probab. XIX, Lecture Notes in Math. **1123** (1985), 177–206.

[6] R. L. Dobrushin and S. B. Shlosman, *Completely analytical interactions: constructive description*, J. Stat. Phys. **46** (1987), 983–1014.

[7] A. Guionnet and B. Zegarlinski, *Lectures on Logarithmic Sobolev Inequalities*, Séminaire de Probabilités XXXVI, Lecture Notes in Math. **1801** (2002), 1–134.

[8] R. Holley and D. Stroock, *Logarithmic Sobolev inequalities and stochastic Ising models*, J. Stat. Phys. **46** (1987), 1159–1194.

[9] F. Martinelli, *Lectures on Glauber dynamics for discrete spin models*, Saint-Flour 1997, Lecture Notes in Math. **1717** (1999), 93–191.

[10] K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Ann. Phys. **110** (1978), 440–471.

[11] D. Stroock and B. Zegarlinski, *The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition*, Commun. Math. Phys. **144** (1992), 303–323.

[12] D. Stroock and B. Zegarlinski, *The logarithmic Sobolev inequality for continuous spin systems on a lattice*, J. Funct. Anal. **104** (1992), 299–326.

[13] A. Blanca, P. Caputo, Z. Chen, D. Parisi, D. Štefankovič, and E. Vigoda, *On mixing of Markov chains: coupling, spectral independence, and entropy factorization*, Proc. 63rd IEEE FOCS (2022), 1150–1160.

[14] L. Eriksson, *Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills Theory*, ai.viXra:2602.0041 (2026), v3. `https://ai.vixra.org/abs/2602.0041`

[15] L. Eriksson, *Ricci Curvature of the Orbit Space of Lattice Gauge Theory and Single-Scale Log-Sobolev Inequalities*, ai.vixra.org, 2026.

[16] L. Eriksson, *Integrated cross-scale derivative bounds for Wilson lattice gauge theory: closing the log-Sobolev gap*, ai.vixra.org, 2026.

[17] L. Eriksson, *Large-field suppression for lattice gauge theories: From Balaban's renormalization group to conditional concentration*, ai.vixra.org, 2026.

[18] L. Eriksson, *Unconditional uniform log-Sobolev inequality for* $\mathrm{SU}(N_c)$ *lattice Yang–Mills at weak coupling*, ai.vixra.org, 2026.

[19] L. Eriksson, *From Uniform Log-Sobolev Inequality to Mass Gap for Lattice Yang–Mills at Weak Coupling*, ai.vixra.org, 2026.

[20] L. Eriksson, *A Conditional Closure of the Multiscale Log-Sobolev Program for the Lattice Yang–Mills Mass Gap*, ai.viXra:2602.0051 (2026), v2.

[21] L. Eriksson, *Interface Lemmas for the Multiscale Proof of the Lattice Yang–Mills Mass Gap*, ai.viXra:2602.0052 (2026), v2.