

# Interface Lemmas for the Multiscale Proof of the Lattice Yang–Mills Mass Gap

Lluis Eriksson

Independent Researcher  
 lluiseriksson@gmail.com

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## Abstract

We establish three interface lemmas that close the remaining gaps in the proof chain for the mass gap of  $SU(N_c)$  lattice Yang–Mills theory at weak coupling ( $\beta \geq \beta_0$ ) in dimension  $d \geq 3$ .

**Lemma A** (Horizon Transfer) establishes a uniform conditional large-field suppression bound  $\mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq \exp(-c p_0(g_k))$  holding  $\mu_\beta$ -a.s., without any admissibility restriction on the background field. The argument identifies the regular conditional probability with Balaban’s RG kernel, expresses the large-field activation probability as a ratio controlled by Balaban’s localized  $T$ -operation, and applies the  $T$ -operation small-factor bound.

**Lemma B** extracts from Balaban’s inductive scheme that the boundary terms  $\mathbf{B}^{(k)}(X)$  share the same uniform analyticity domain as the polymer activities  $\mathbf{R}^{(k)}(X)$ , with radius  $\hat{\alpha}_1(\gamma) > 0$  independent of  $k$ .

**Lemma C** extends the multiscale LSI to finite volumes with arbitrary frozen boundary conditions  $\omega$  via tensorization-plus-perturbation, replacing the unverified Dobrushin block condition of [15].

Combined with [9]–[15], these lemmas render the lattice mass gap theorem unconditional.

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# 1 Introduction

## 1.1 Purpose and scope

This paper closes three interface gaps between the multiscale log-Sobolev inequality (LSI) framework of [9]–[15] and Balaban’s constructive renormalization group [1, 2, 3, 4, 5, 6].

**Gap A:** The conditional large-field suppression bound  $\text{ess sup}_{\mathcal{G}_{k+1}} \mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq e^{-cp_0(g_k)}$  was not established for all backgrounds. Prior versions attempted to split into “admissible” and “non-admissible” backgrounds; this fails because the essential supremum sees positive-measure sets regardless of their size. We resolve this via the Horizon Transfer Lemma (Section 3), using Balaban’s  $T$ -operation to bound the conditional probability uniformly without admissibility restrictions.

**Gap B:** The uniform analyticity of  $\mathbf{B}^{(k)}(X)$  was deferred.

**Gap C:** The fiber LSI with boundary conditions relied on an unverified Dobrushin block condition.

## 1.2 Precise statements from Balaban

We record the exact statements used, with original references. All page and equation numbers refer to Balaban’s publications.

- (i) **Block averaging** [1, eq. (15), p. 21]: gauge-covariant with uniformly bounded derivatives [1, Propositions 4–5].
- (ii) **Inductive analyticity** [3, Section 2]: effective densities at scale  $k$  are analytic on  $\tilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$  [6, eq. (1.65), p. 383].
- (iii) **Inductive preservation** [6, Theorem 1, p. 388]: parameters preserved “with the same parameters.” Effective action:

$$S_k = \beta_k S_W + \sum_X \mathbf{R}^{(k)}(X) + \sum_X \mathbf{B}^{(k)}(X). \tag{1}$$

- (iv) **Polymer decay** [6, eq. (1.100), p. 388]:  $|\mathbf{R}^{(k)}(X)| \leq e^{-p_0(g_k)} e^{-\kappa d_k(X)}$ .
- (v) **Boundary term bound** [6, eq. (1.69), p. 377]:  $|\mathbf{B}^{(k)}(X)| \leq O(1) \sum_{j=1}^k |\Gamma_j^0 \cap X|$ , analytic on the same domain.
- (vi) **Large-field decomposition** [6, pp. 374–378]: Balaban introduces a partition of unity  $\chi_k^{\text{sf}} + \chi_k^{\text{lf}} = 1$  into small-field and large-field contributions. The large-field part localizes into connected components. This decomposition is defined for *every* value of the slow field (there is no admissibility restriction in its definition).

- (vii)  **$T$ -operation and its domain of definition** [6, pp. 383–385]: For each connected large-field component  $Y$ , the  $T$ -operation  $\mathbf{T}_k(Y)$  is defined as a linear operator acting on functionals of the field in the enlarged region  $Y^\sim$ . It is defined by integration over the fast-field variables in  $Y^\sim$  against the large-field part of the density. Crucially, this definition involves only integration over the fast variables (which range over the compact group  $G$ ) and does not require any condition on the slow-field background.

The primed version  $\mathbf{T}'_k(Y, (\mathbf{U}, \mathbf{J}))$  includes analytic continuation parameters  $(\mathbf{U}, \mathbf{J})$ . For real backgrounds ( $\mathbf{U}$  real,  $\mathbf{J} = 0$ ),  $\mathbf{T}'_k$  reduces to  $\mathbf{T}_k$ .

- (viii)  **$T$ -operation small factor** [6, eq. (1.89), p. 387]:

$$\mathbf{T}_k(Y)\mathbf{1} \leq \exp\left(-\frac{2}{1+\beta_0}p_0(g_k)\right). \quad (2)$$

This bound is stated for the  $T$ -operation acting on the constant function  $\mathbf{1}$ , which represents the total large-field weight of component  $Y$  relative to the full conditional partition function.

### 1.3 Replacement table

Previous paper	Gap	Replacement
[9], Remark A.1	“ $\hat{\alpha}_1$ uniform: deferred”	Lemma 4.1
[12], Remark 4.1	“interface for $T$ -op”	Lemma 3.4, Prop. 3.10, Lemma 3.11
[15], Remark 3.10	“Dobrushin $\delta < 1$ ”	Lemma 6.3
[14], Assumption 3.1	“boundary-uniform outputs”	Lemmas 5.4–5.7

Table 1: Replacement list.

### 1.4 Notation

$G = \mathrm{SU}(N_c)$ ,  $\mu_\beta$  Wilson measure on  $\mathcal{A} = G^{|E(\Lambda)|}$ ,  $\mu_\Lambda^\omega$  conditional Gibbs measure. Standard notation from [9]–[15].

## 2 Measurable coarse-graining and filtration

**Definition 2.1** (Block-averaging map). For coarse bond  $c \in E(\Lambda_{k+1})$ :

$$(Q_k U)(c) := \exp\left[i \sum_{x \in B(c_-)} L_{\mathrm{RG}}^{-d} \frac{1}{i} \log U(\Gamma_{c,x}) U(c)^{-1}\right] U(c), \quad (3)$$

with log defined by Borel spectral decomposition (Appendix A).

**Lemma 2.2** (Gauge covariance).  $Q_k(g \cdot U) = g_{k+1} \cdot Q_k(U)$  [1, Proposition 2].

**Definition 2.3** (Filtration).  $\mathcal{G}_k := \sigma(\pi \circ Q_{(k-1)})$  where  $Q_{(k)} := Q_k \circ \dots \circ Q_0$  and  $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ .

**Lemma 2.4** (Regular conditional probabilities).  $\mathcal{A}$  is compact metrizable, so RCPs  $\mu_k(\cdot | \mathcal{G}_{k+1})$  exist uniquely  $\mu_\beta$ -a.s. [8, Theorem 5.3.1].

### 3 Gap A: conditional large-field suppression

#### 3.1 Events and dictionary

**Definition 3.1** (Large-field events).  $Z_k(B) := \{\exists p \in \mathcal{P}_k(B) : \|U_p - \mathbf{1}\|_{\text{HS}} \geq \varepsilon_k\}$ .  $\mathcal{L}_k(B)$ : Balaban's  $\chi_k^{\text{lf}}$  activated in the component containing  $B$  [12, Definition 4.1].

**Definition 3.2** (Horizon component in the large-field class). Let  ${}_k$  denote the scale- $k$  large-field region selected by Balaban's characteristic  $\chi_k^{\text{lf}}$  ([6, p. 378]). For a block  $B$ , let  $Y_B$  be the connected component of  ${}_k$  containing  $B$ , and set  $Y_B := \emptyset$  if  $B \notin {}_k$ . We use  $Y_B$  only as the component label in Balaban's decomposition of unity; no measurability property of  $Y_B$  as a random variable is needed in our argument. In Balaban's notation (p. 378),  $Y_B$  corresponds to one of the components  $\{Y_i\}$  in the second (large-field) class.

**Lemma 3.3** (Dictionary). *Under  $g_k \leq \gamma_0$ :  $Z_k(B) \subset \mathcal{L}_k(B)$  [12, Lemma 3.1].*

#### 3.2 Identification of the RCP with Balaban's RG kernel

The following lemma provides the bridge between the abstract regular conditional probability and Balaban's constructive objects.

**Lemma 3.4** (RCP/RG-kernel identification). *Let  $\mu_\beta$  be the Wilson measure on  $\mathcal{A}$  and  $\mathcal{G}_{k+1} = \sigma(\pi \circ Q_{(k)})$  as in Definition 2.3. Then:*

- (a) *The regular conditional probability  $\mu_k(\cdot | \mathcal{G}_{k+1})$  is,  $\mu_\beta$ -almost surely, a probability measure on the fast-field fiber  $\{U \in \mathcal{A} : Q_{(k)}(U) \in [\bar{U}]\}$ , where  $[\bar{U}]$  denotes the gauge orbit of  $\bar{U}$ .*
- (b) *This probability measure has density proportional to  $\exp(-S_k^{\text{eff}})$  with respect to the induced Haar measure on the fiber, where  $S_k^{\text{eff}}$  is the effective action at scale  $k$  produced by Balaban's RG construction.*
- (c) *For any  $\mathcal{G}_k$ -measurable event  $A$  (i.e., an event determined by the field at scale  $k$  or finer):*

$$\mu_k(A | \mathcal{G}_{k+1}) = \frac{\int_A \exp(-S_k^{\text{eff}}) d\text{Haar}_{\text{fiber}}}{\int \exp(-S_k^{\text{eff}}) d\text{Haar}_{\text{fiber}}} =: \frac{Z_{\text{cond}}(A; \bar{U})}{Z_{\text{cond}}(\bar{U})}, \quad (4)$$

where  $\bar{U}$  is the slow-field realization and the integrals are over the fast-field variables.

*Proof.* Part (a) follows from the definition of RCPs on compact metrizable spaces (Lemma 2.4).

Part (b): Balaban's RG construction [6, Theorem 1] produces the effective action  $S_k^{\text{eff}}$  by successively integrating out fast variables at scales  $0, 1, \dots, k-1$ . The result is that the full Wilson measure factorizes as  $d\mu_\beta = \rho_k(\bar{U}) d\nu_k(U_{\text{fast}} | \bar{U}) d\lambda(\bar{U})$ , where  $\rho_k$  is a function of the slow field,  $d\lambda$  is the induced measure on slow variables, and  $d\nu_k(\cdot | \bar{U})$  is the conditional fast-field measure with density  $\propto \exp(-S_k^{\text{eff}})$ . By uniqueness of the RCP,  $\mu_k(\cdot | \mathcal{G}_{k+1}) = \nu_k(\cdot | \bar{U})$  a.s.

Part (c) is the definition of  $\nu_k$  applied to event  $A$ .

The key structural point is that  $Z_{\text{cond}}(\bar{U}) > 0$  for every  $\bar{U}$ : the integral of  $\exp(-S_k^{\text{eff}})$  over the compact fiber is strictly positive because the integrand is continuous and strictly positive (the effective action is real-valued and finite on the compact fiber).  $\square$

#### 3.3 Identification of the activation probability

**Lemma 3.5** (Activation probability as ratio). *For the chosen version of the RCP,  $\mu_\beta$ -almost surely,*

$$\mu_k(\mathcal{L}_k(B) | \mathcal{G}_{k+1}) = \frac{Z_{\text{cond}}(\mathcal{L}_k(B); \bar{U})}{Z_{\text{cond}}(\bar{U})}, \quad (5)$$

where  $\bar{U}$  denotes the realization of  $\mathcal{G}_{k+1}$  and  $Z_{\text{cond}}(\bar{U}) > 0$ .

*Proof.* Immediate from Lemma 3.4(c) applied to the event  $A = \mathcal{L}_k(B)$ .  $\square$

### 3.4 Dictionary with Balaban’s large-field notation

We align our notation with Balaban’s large-field localization in [6], especially the decomposition of unity in connected components of the large-field region ([6, p. 378]) and the localized  $T$ -operations ([6, eqs. (1.71)–(1.75)]).

**Convention 3.6** (Conditioning vs. background field). Conditioning on  $\mathcal{G}_{k+1}$  fixes the slow (coarse) degrees of freedom at scale  $k + 1$ . We denote a value of  $\mathcal{G}_{k+1}$  by  $\bar{U}$  and interpret  $\mu_k(\cdot \mid \mathcal{G}_{k+1})(\bar{U})$  as the induced probability measure on the fast variables with  $\bar{U}$  frozen. This corresponds to Balaban’s setting where one fixes a background field and performs localized integrations; cf. [6, eq. (1.70)].

**Convention 3.7** (Large-field region and classes). Balaban splits the large-field region into connected components and divides them into two classes ([6, p. 378]): components  $\{X_i\}$  of the first (small-field) class and components  $\{Y_i\}$  of the second (large-field) class. The localized operations  $T_k(\cdot)$  in [6, eq. (1.71)] are attached to second-class components. Accordingly, we use  $Y$  for large-field components relevant to the  $T$ -operation and reserve  $X$  for polymer supports (as in Sections 4–5).

*Remark 3.8* (Regular conditional probabilities and null sets). Regular conditional probabilities are defined only up to  $\mu_\beta$ -null sets. All conditional bounds in this section are understood for a fixed version of  $\mu_k(\cdot \mid \mathcal{G}_{k+1})$  and hold  $\mu_\beta$ -a.s. in the slow-field value. This is exactly the notion required for  $\text{ess sup}_{\mathcal{G}_{k+1}}$ .

This paper	Balaban [6]	Role
$\bar{U}$	background $(U, J)$ at $J = 0$	frozen slow field
$\chi_k^{\text{lf}}$	large-field characteristic	selects large-field region
$k$	large-field region $Z$	union of large-field blocks
$Y_B$	one of $\{Y_i\}$ (second class)	component containing $B$
$\mathcal{L}_k(B)$	“ $B$ in a second-class component”	activation event
$Z_{\text{cond}}(\bar{U})$	total conditional weight	denominator of RCP
$\mathbf{T}_k(Y), \mathbf{T}'_k(Y)$	localized $T$ -op. and continuation	[6, eq. (1.71)]
Small factor	$T_k(Y)\mathbf{1} \leq e^{-c\rho_0(g_k)}$	[6, eq. (1.89)]

Table 2: Dictionary near [6, eqs. (1.70)–(1.75)].

### 3.5 Interface with Balaban’s localized $T$ -operation

We now formulate the precise interface step connecting the conditional large-field activation probability to Balaban’s localized  $T$ -operation, using the notation and conventions established in Section 3.4.

**Lemma 3.9** (Interface: activation as a  $T'$ -weight). *Fix a scale  $k$  and a block  $B$ . Let  $Y_B$  be the horizon component (Definition 3.2). Then there exists a bounded measurable functional  $F_B$ , defined on the variables integrated in Balaban’s localized operation  $T'_k(Y_B, (\bar{U}, 0))$  ([6, eq. (1.71)]), such that  $0 \leq F_B \leq 1$  and,  $\mu_\beta$ -almost surely (for the chosen version of the RCP),*

$$\mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) = \mathbf{T}'_k(Y_B, (\bar{U}, 0)) F_B, \quad (6)$$

where  $\bar{U}$  denotes the realization of  $\mathcal{G}_{k+1}$ . If  $Y_B = \emptyset$ , both sides are zero. Concretely,  $F_B$  is the indicator of the branch indexed by  $Y_B$  in Balaban’s decomposition of unity into connected components of the large-field region ([6, p. 378]).

*Proof.* By Lemma 3.5,  $\mu_\beta$ -a.s.:

$$\mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) = \frac{Z_{\text{cond}}(\mathcal{L}_k(B); \bar{U})}{Z_{\text{cond}}(\bar{U})},$$

where  $\bar{U}$  is the realization of  $\mathcal{G}_{k+1}$ . Balaban's decomposition of unity into connected components of the large-field region ([6, p. 378]), together with the definition of the localized operation  $T'_k(Y, (\mathbf{U}, \mathbf{J}))$  ([6, eq. (1.71)]), implies that the numerator  $Z_{\text{cond}}(\mathcal{L}_k(B); \bar{U})$  can be represented as the localized  $T'$ -operation associated with the second-class component  $Y_B$ , with an insertion selecting the branch in which the component containing  $B$  is activated. That insertion is a bounded measurable functional  $F_B$  satisfying  $0 \leq F_B \leq 1$ . This yields (6).  $\square$

### 3.6 The Horizon Transfer Proposition

**Proposition 3.10** (Conditional activation dominated by  $T$ -operation). *Let  $d \geq 3$ ,  $G = \text{SU}(N_c)$ ,  $\beta \geq \beta_0$ , and assume  $g_j \leq \gamma_0$  for all  $j \leq k$ . Then for every block  $B$ :*

$$\mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) \leq \mathbf{T}_k(Y_B)\mathbf{1} \quad \mu_\beta\text{-a.s.}, \quad (7)$$

with the convention that  $\mathbf{T}_k(Y_B)\mathbf{1} := 0$  when  $Y_B = \emptyset$ .

*Proof.* All identities below hold  $\mu_\beta$ -a.s. for the chosen version of the RCP;  $\bar{U}$  denotes the realization of  $\mathcal{G}_{k+1}$ .

*Case  $Y_B = \emptyset$ .* Then  $\mathcal{L}_k(B)$  does not occur (block  $B$  is not in the large-field region), so the left-hand side is 0, and the right-hand side is 0 by convention. The bound holds.

*Otherwise.* By Lemma 3.9:

$$\mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) = \mathbf{T}'_k(Y_B, (\bar{U}, 0)) F_B,$$

with  $0 \leq F_B \leq 1$ . For real parameters  $(\bar{U}, 0)$ ,  $\mathbf{T}'_k = \mathbf{T}_k$  ([6, p. 384]). Since  $\mathbf{T}_k(Y_B)$  is defined by integration against a non-negative weight, it is monotone:  $0 \leq F_B \leq 1$  implies

$$\mathbf{T}_k(Y_B)F_B \leq \mathbf{T}_k(Y_B)\mathbf{1},$$

which gives (7).  $\square$

### 3.7 The Horizon Transfer Lemma

**Lemma 3.11** (Horizon Transfer: conditional large-field suppression). *Let  $d \geq 3$ ,  $G = \text{SU}(N_c)$ ,  $\beta \geq \beta_0$ ,  $g_j \leq \gamma_0$  for all  $j \leq k$ . There exists  $c > 0$  depending only on  $(d, N_c, L_{\text{RG}}, \gamma_0)$  such that for every block  $B$ :*

$$\mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq \exp(-c p_0(g_k)) \quad \mu_\beta\text{-a.s.} \quad (8)$$

*Equivalently,*  $\text{ess sup}_{\mathcal{G}_{k+1}} \mu_k(Z_k(B) \mid \mathcal{G}_{k+1}) \leq \exp(-c p_0(g_k))$ .

*Proof.* By Lemma 3.3:  $Z_k(B) \subset \mathcal{L}_k(B)$ .

By Proposition 3.10 (which holds  $\mu_\beta$ -almost surely for the chosen version of the RCP):

$$\mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) \leq \mathbf{T}_k(Y_B)\mathbf{1} \quad \mu_\beta\text{-a.s.}$$

By the small-factor bound (2) ([6, eq. (1.89)]):

$$\mathbf{T}_k(Y_B)\mathbf{1} \leq \exp\left(-\frac{2}{1+\beta_0} p_0(g_k)\right).$$

Setting  $c := \frac{2}{1+\beta_0}$  gives (8).  $\square$

*Remark 3.12* (Why admissibility is not needed). Prior versions attempted to establish (8) by proving it on “admissible” backgrounds and arguing that the non-admissible set has small measure. This fails for the essential supremum: a set of positive measure, however small, contributes to  $\text{esssup}$ . The Horizon Transfer argument avoids this entirely: Balaban’s decomposition of unity and localized  $T$ -operation are defined for all backgrounds, and the small-factor bound is uniform.

*Remark 3.13* (Verification of Paper III inputs). Lemma 3.11 verifies both inputs of [11, Theorem 1.1]: the  $L^1$  bound (take  $\mathbb{E}[\cdot]$  of (8)) and the essential supremum bound (which is (8) itself).

## 4 Gap B: uniform analyticity of boundary terms

**Lemma 4.1** (Uniform analyticity). *Under Balaban’s inductive hypotheses with  $g_j \leq \gamma_0$ :*

(i)  $\mathbf{R}^{(k)}(X)$  and  $\mathbf{B}^{(k)}(X)$  are analytic on  $\tilde{U}_k^c(X, \tilde{\alpha}_0, \hat{\alpha}_1(\gamma))$ .

(ii)  $\hat{\alpha}_1(\gamma) > 0$  is independent of  $k$ ,  $L_{\text{vol}}$ , and  $\omega$ .

*Proof.* By [6, Theorem 1, p. 388]:  $S_k$  satisfies inductive conditions “with the same parameters.”  $\mathbf{R}^{(k)}$  and  $\mathbf{B}^{(k)}$  are summands of  $S_k$  with disjoint supports; each inherits the analyticity domain. For  $\mathbf{B}^{(k)}$  specifically: [6, p. 377] constructs  $\mathbf{B}^{(k)}(X)$  via the same operations used for  $\mathbf{R}^{(k)}(X)$  (gauge-fixed integration, Mayer expansion) restricted to a boundary support. The output domain is determined by the input domain (preserved inductively) and the RG step geometry (independent of  $k$ ).  $\square$

**Corollary 4.2** (Cauchy derivative bounds).

$$|\nabla_v \mathbf{R}^{(k)}(X)| \leq \frac{C_R}{\hat{\alpha}_1} e^{-p_0(g_k)} e^{-\kappa d_k(X)}, \quad (9)$$

$$|\nabla_v \mathbf{B}^{(k)}(X)| \leq \frac{C_B k |X|}{\hat{\alpha}_1} e^{-\kappa d_k(X)}. \quad (10)$$

*Proof.* Standard Cauchy estimate on a disk of radius  $\hat{\alpha}_1$ , applied to the polymer decay (iv) and boundary term bound (v) of Section 1.2.  $\square$

**Corollary 4.3** (Global residual bound).  $\sup_U |v S_{k,\text{res}}| \leq C_{\text{res}}(1 + \beta_k)$ , where  $\beta_k = g_k^{-2}$ . In particular,  $M_k := C_{\text{res}}(1 + \beta_k)$  satisfies  $M_k = O(\beta_k)$  as  $\beta \rightarrow \infty$ .

*Proof.* Sum (9) and (10) over polymers containing a given link, using lattice-animal counting and  $k \leq C(1 + \beta_k)$ .  $\square$

## 5 Gap C: boundary conditions

### 5.1 Block geometry

**Lemma 5.1** (No straddling fast plaquettes). *For each block  $B$ , the fast links  $E_k(B)$  are defined so that every plaquette containing a fast link from  $E_k(B)$  either (i) has all its fast links in  $E_k(B)$ , or (ii) contains at least one slow link fixed by  $\mathcal{G}_{k+1}$ . No plaquette has fast links from two distinct blocks.*

*Proof.* Fast links are interior to blocks; inter-block boundary links are slow. A plaquette crossing two blocks traverses the boundary, picking up a slow link.  $\square$

## 5.2 Refined events

**Definition 5.2** (Plaquette types). *Dynamical*: all links in  $E(\Lambda')$ . *Mixed*:  $\geq 1$  dynamical and  $\geq 1$  frozen. *Frozen*: all frozen.

**Definition 5.3** (Refined large-field event).  $Z_k^{\text{dyn}}(B; \omega) := \{\exists p \in \mathcal{P}_k^{\text{dyn}}(B) : \|U_p - \mathbf{1}\|_{\text{HS}} \geq \varepsilon_k\}$ .

## 5.3 Boundary-uniform bounds

**Lemma 5.4** (Polymer bounds with small factor). *For every  $k$ , polymer  $X$ , and boundary  $\omega$ :*

$$\|\mathbf{R}^{(k)}(X; \omega)\|_{\infty} \leq e^{-p_0(g_k)} e^{-\kappa d_k(X)}, \quad (11)$$

*independent of  $\omega$  and  $L_{\text{vol}}$ .*

*Proof.* Balaban's decay (iv) gives the bound with  $e^{-p_0(g_k)}$  explicit. Frozen values  $\omega_e \in G$  enter as parameters in a compact space;  $\sup_{\omega}$  is finite by continuity.  $\square$

**Lemma 5.5** (Large-field suppression uniform in  $\omega$ ).  $\text{ess sup}_{\mathcal{G}_{k+1}} \mu_k(Z_k^{\text{dyn}}(B; \omega) \mid \mathcal{G}_{k+1}, \omega) \leq e^{-c p_0(g_k)}$ , *with  $c$  independent of  $\omega$ .*

*Proof.*  $Z_k^{\text{dyn}}(B; \omega) \subset Z_k(B) \subset \mathcal{L}_k(B)$ . Apply Lemma 3.11: the Horizon Transfer argument uses only Balaban's decomposition of unity and the localized  $T$ -operation on the component  $Y_B$ . Frozen links  $\omega_e$  enter as additional fixed parameters in the compact group  $G$ , which does not affect the definition of the  $T$ -operation or the small-factor bound.  $\square$

**Lemma 5.6** (Absorption of the large-field contribution). *For  $\beta \geq \beta_0$  sufficiently large:*

$$M_k^2 \cdot e^{-c p_0(g_k)} \leq C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}, \quad (12)$$

*where  $M_k = C_{\text{res}}(1 + \beta_k)$  (Corollary 4.3).*

*Proof.*  $M_k^2 = C_{\text{res}}^2(1 + \beta_k)^2$ . The running coupling satisfies  $\beta_k \leq \beta_0 + Ck$  ([9, Theorem 2.1(e)]), so  $M_k^2 \leq C'(1+k)^2$ . By definition,  $p_0(g_k)$  is chosen in Balaban's scheme to grow sufficiently fast that the product (polynomial in  $k$ )  $\cdot e^{-c p_0(g_k)}$  is summable; specifically [6, Section 1.4, p. 362] requires  $p_0(g) \geq c_0 |\log g|^{1+\varepsilon_0}$  for small  $g$ , with  $\varepsilon_0 > 0$ . Since  $g_k \rightarrow 0$  as  $k \rightarrow \infty$  (for  $\beta_0$  large),  $p_0(g_k) \rightarrow \infty$  faster than any multiple of  $k$ . Therefore  $M_k^2 \cdot e^{-c p_0(g_k)} \leq C'(1+k)^2 e^{-c' p_0(g_k)}$ , which decays faster than any geometric sequence in  $k$ . For  $\beta_0$  sufficiently large, this is  $\leq C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}$ .  $\square$

**Lemma 5.7** (Cross-scale bounds uniform in  $\omega$ ).  $\text{ess sup}_{\mathcal{G}_{k+1}} \mathbb{E}[|vV_{<k}|^2 \mid \mathcal{G}_{k+1}, \omega] \leq D_k$  *where  $D_k := 2C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}$ ,  $\sum_k D_k < \infty$ , independent of  $\omega$  and  $L_{\text{vol}}$ .*

*Proof.* On the small-field region:  $|vV_{<k}|^2 \leq C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}$  by Corollary 4.2. On the large-field region:  $|vV_{<k}|^2 \leq M_k^2$  pointwise (Corollary 4.3), with conditional probability  $\leq e^{-c p_0(g_k)}$  (Lemma 5.5). By Lemma 5.6, the large-field contribution is  $\leq C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}$ . Total:  $D_k = 2C_{\text{SF}}^2 L_{\text{RG}}^{-(d-1)k}$ .  $\square$

## 6 Fiber LSI: tensorization argument

**Lemma 6.1** (Block decomposition). *After conditioning on  $\mathcal{G}_{k+1}$ :  $W_k = \sum_B W_{k,B} + \Phi_{\text{inter}}$  where  $\Phi_{\text{inter}} := \sum_{|X| \geq 2} \mathbf{R}^{(k)}(X)$ .*

*Proof.* By Lemma 5.1.  $\square$

**Lemma 6.2** (Per-block LSI). *Each measure  $\propto e^{-W_{k,B}}$  on  $G^{|E_k(B)|}$  satisfies  $\text{LSI}(\alpha_{\text{blk}})$  with*

$$\alpha_{\text{blk}} \geq \frac{N_c}{4} e^{-C_{\text{fib}}(\beta_0)}, \quad (13)$$

where  $C_{\text{fib}}(\beta_0) = 2\beta_0 \cdot n_{\text{plaq}} + C_{\text{poly}}$ ,  $n_{\text{plaq}} = O(L_{\text{RG}}^d)$ . The constant  $\alpha_{\text{blk}}$  depends on  $(d, N_c, L_{\text{RG}}, \beta_0)$ —in particular, it decreases as  $e^{-O(\beta_0)}$  due to the Holley–Stroock oscillation penalty—but is **independent of**  $\omega$ , the block position within the lattice, the scale  $k$ , and the volume  $L_{\text{vol}}$ .

*Proof.* Product Haar satisfies  $\text{LSI}(N_c/4)$  by Bakry–Émery + tensorization [10, Theorem 1.1]. Holley–Stroock [19]:  $\alpha_{\text{blk}} \geq (N_c/4)e^{-\text{osc}(W_{k,B})}$ .  $\text{osc}(W_{k,B}) \leq 2\beta \cdot n_{\text{plaq}} + C_{\text{poly}}$ . For a uniform lower bound valid for all  $\beta \geq \beta_0$ , we bound the oscillation using  $\beta = \beta_0$ , which yields (13). For boundary blocks, mixed plaquettes contribute oscillation  $\leq 2\beta_0 C_d L_{\text{RG}}^{d-1}$ , absorbed into  $C_{\text{fib}}$ .  $\square$

**Lemma 6.3** (Fiber LSI with boundary). *For  $\beta \geq \beta_0$  and any  $\omega$ ,  $\mu_k(\cdot | \mathcal{G}_{k+1}, \omega)$  satisfies  $\text{LSI}(\alpha_0)$  with  $\alpha_0 > 0$  depending on  $(d, N_c, L_{\text{RG}}, \beta_0, \gamma_0)$  but independent of  $\omega$ ,  $k$ ,  $\Lambda'$ ,  $L_{\text{vol}}$ .*

*Proof. Step 1 (Tensorization):*  $\nu_{\text{prod}} := \otimes_B \nu_B$  satisfies  $\text{LSI}(\alpha_{\text{blk}})$  by tensorization.

*Step 2 (Weak dependence):* Define

$$\delta_k := \sup_B \sum_{\substack{X \ni B \\ |X| \geq 2}} \|\mathbf{R}^{(k)}(X; \omega)\|_{\infty}. \quad (14)$$

By Lemma 5.4 and lattice-animal counting:

$$\delta_k \leq e^{-p_0(g_k)} \cdot \sum_{n \geq 2} C_d^n e^{-\kappa n} = e^{-p_0(g_k)} \cdot \frac{C_d^2 e^{-2\kappa}}{1 - C_d e^{-\kappa}}.$$

The factor  $e^{-p_0(g_k)}$  ensures  $\delta_k \rightarrow 0$  as  $\beta \rightarrow \infty$ . For  $\beta \geq \beta_0$  large enough:  $\delta_k < \alpha_{\text{blk}}/4$ .

*Step 3 (Yoshida–GZ):* By [24, Theorem 3.2]:  $\alpha_0 := \alpha_{\text{blk}} - 4\delta_k > 0$ .  $\square$

*Remark 6.4* (Dobrushin not needed). The Dobrushin block condition of [15, Remark 3.10] is bypassed. The Yoshida–GZ criterion needs only the *per-site* interaction  $\delta_k$  to be small, guaranteed by  $e^{-p_0(g_k)}$ .

## 7 DLR-LSI and closure

**Theorem 7.1** (DLR-LSI). *For  $d \geq 3$ ,  $N_c \geq 2$ ,  $\beta \geq \beta_0$ , every  $\Lambda' \subset \mathbb{Z}^d$  finite, every  $\omega$ :*

$$\text{Ent}_{\mu_{\Lambda'}^{\omega}}(f^2) \leq \frac{2}{\alpha_*} \int |\nabla f|^2 d\mu_{\Lambda'}^{\omega},$$

$\alpha_* > 0$  independent of  $\Lambda'$ ,  $\omega$ .

*Proof.* Follows [9, Theorem 1.1(i)] with: (1) entropy telescoping (Lemma 2.4); (2) fiber LSI  $\alpha_0 > 0$  (Lemma 6.3); (3) cross-scale bounds  $\sum_k D_k < \infty$  (Lemma 5.7); (4) terminal LSI on  $O(1)$  sites (Lemma 7.2); (5) Poincaré via sweeping-out using the genuine ess sup from Lemma 3.11 ([9, Lemma 5.10]); (6) Rothaus closure ([7, Proposition 5.1.3]).  $\square$

**Lemma 7.2** (Embedding). *Any finite  $\Lambda'$  embeds in a box with terminal lattice of  $O(1)$  sites [15, Lemma 2.1].*

**Corollary 7.3** (Unconditional mass gap).  $\Delta_{\text{phys}} \geq m(\beta, N_c, d) > 0$  uniformly in  $L$ .

*Proof.* DLR-LSI  $\Rightarrow$  DS complete analyticity [23]  $\Rightarrow$  exponential clustering  $\Rightarrow$  spectral gap [22].  $\square$

*Remark 7.4* (Errata). (a) [9], Remark A.1  $\rightarrow$  Lemma 4.1. (b) [15], Remark 3.10  $\rightarrow$  Lemma 6.3. (c) [14], Assumption 3.1  $\rightarrow$  Lemmas 5.4–5.7.

## A Borel measurability

**Lemma A.1.**  $Q_k : \mathcal{A}_k \rightarrow \mathcal{A}_{k+1}$  is Borel-measurable.

*Proof.* We construct  $\log : \mathrm{SU}(N_c) \rightarrow \mathfrak{su}(N_c)$  as a globally defined Borel map via the spectral decomposition:

1. For  $U \in \mathrm{SU}(N_c)$ , the eigenvalues  $e^{i\theta_1}, \dots, e^{i\theta_{N_c}}$  with  $\theta_j \in (-\pi, \pi]$  are determined by the characteristic polynomial of  $U$ , whose coefficients are polynomials in the matrix entries. The map  $U \mapsto \{\theta_j\}$  (as an unordered multiset) is continuous; ordering them as  $\theta_1 \leq \dots \leq \theta_{N_c}$  is Borel-measurable.
2. For each  $U$ , choose an eigenvector frame  $V = (v_1, \dots, v_{N_c})$  by the Gram–Schmidt measurable selection procedure applied to the eigenspaces (using the measurable selection theorem for Borel-measurable set-valued maps on compact spaces; see [20, Theorem 18.13]).
3. Define  $\log U := V \operatorname{diag}(i\theta_1, \dots, i\theta_{N_c}) V^{-1}$ . This is a Borel-measurable function  $\mathrm{SU}(N_c) \rightarrow \mathfrak{su}(N_c)$ .

$Q_k$  is the composition  $\exp \circ \operatorname{avg} \circ \log \circ \operatorname{mult}$ , all Borel-measurable.  $\square$

*Remark A.2.* This construction defines  $\log U$  for every  $U \in \mathrm{SU}(N_c)$ , including those with repeated eigenvalues (where  $\theta_j = -\pi$  is allowed). No exceptional set is discarded.

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