

Uniform Coercivity, Pointwise Large-Field Suppression, and Unconditional Closure of the Lattice Yang–Mills Mass Gap at Weak Coupling in $d = 4$

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Abstract

We close the remaining interface gaps in the program [E26I]–[E26VIII] that establishes a uniform log-Sobolev inequality (LSI) and spectral gap for the transfer matrix of lattice $SU(N_c)$ Yang–Mills theory in $d = 4$ at weak coupling. Four technical gaps are identified and resolved: **(G1)** the Balaban small-factor bound for the T -operation is shown to hold *pointwise* for every real background by auditing Balaban’s proof and verifying that it uses only the uniform inductive conditions; **(G2)** we establish a uniform small-field coercivity estimate (Hessian lower bound) for the effective action and use it, together with Balaban’s small-factor mechanism, to control the conditional inequalities in the multiscale entropy decomposition—circumventing the need for a global fiber LSI with constant $O(\beta)$; **(G3)** uniform analyticity of boundary terms is extracted from Balaban’s inductive scheme; **(G4)** a quantitative bootstrap verifies the simultaneous compatibility of all constants for a single choice of β_0 . Combined with [E26I]–[E26VIII], these closures yield an unconditional proof that $\Delta_{\text{phys}}(\beta, L) \geq c(N_c, \beta_0) > 0$ uniformly in the volume L for $\beta \geq \beta_0$.

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1 Introduction

1.1 Context and objective

The existence of a mass gap in pure Yang–Mills theory is one of the outstanding open problems in mathematical physics, listed as a Clay Millennium Problem [11]. On the torus $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ with the Wilson action

$$A(U) := \sum_{p \in \mathcal{P}(\Lambda)} \left(1 - \frac{1}{N_c} \operatorname{Re} \operatorname{Tr} U_p\right), \quad S_\beta(U) := \beta A(U), \quad \beta = \frac{2N_c}{g^2}. \quad (1)$$

the transfer matrix \widehat{T} acts on $L^2(\mu_{\beta, \Sigma})$ where Σ is a spatial slice. The mass gap is $\Delta_{\text{phys}}(\beta, L) = -\log \|\widehat{T}|_{\mathbf{1}^\perp}\|$.

The series [E26I]–[E26VIII] (Papers 68–75) establishes a route

$$\text{Uniform LSI} \implies \text{DLR-LSI} \implies \text{SZ mixing} \implies \text{RP} \implies \Delta_{\text{phys}} > 0.$$

Four interface gaps remain. This paper closes all four.

1.2 The four gaps

- (G1) *Unconditional large-field suppression.* Paper 75 requires $\operatorname{ess\,sup}_{\mathcal{G}_{k+1}} \mu_k(\mathcal{L}_k(B) \mid \mathcal{G}_{k+1}) \leq e^{-c p_0(g_k)}$. The bound must hold for *all* real backgrounds, not only “admissible” ones in Balaban’s sense.
- (G2) *Fiber-level control without exponential collapse.* Holley–Stroock with oscillation $\operatorname{osc}(W_{k,B}) = O(\beta)$ produces $\alpha_{\text{blk}} \sim e^{-O(\beta)}$, which competes destructively with the polymer decay $e^{-p_0(g_k)}$. What is needed is a small-field coercivity estimate giving Hessian control of order β_k , together with a mechanism ensuring that the large-field complement is negligible.
- (G3) *Uniform analyticity of boundary terms.* The claim that $\mathbf{B}^{(k)}(X)$ shares the analyticity radius $\hat{\alpha}_1$ of $\mathbf{R}^{(k)}(X)$ requires explicit extraction from the inductive conditions.
- (G4) *Positivity of α_* .* The Rothaus closure $\alpha_* = (C_E/2 + C_B C_P)^{-1}$ and the simultaneous compatibility of all constants with a single β_0 must be verified without circularity.

1.3 Main result

Theorem 1.1 (Mass gap, lattice, $d = 4$, weak coupling). *For $G = \mathrm{SU}(N_c)$ and $d = 4$, there exists $\beta_0 = \beta_0(N_c) < \infty$ such that for all $\beta \geq \beta_0$ and all even $L \geq 2$:*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq \frac{2}{\alpha_*} \sum_{e \in E(\Lambda)} \int |\nabla_e f|^2 d\mu_\beta, \quad \alpha_* = \alpha_*(N_c, \beta_0) > 0, \quad (2)$$

and the transfer matrix has spectral gap

$$\Delta_{\mathrm{phys}}(\beta, L) \geq c(N_c, \beta_0) > 0 \quad (3)$$

uniformly in L .

The proof assembles §§2–6 of this paper with [E26I]–[E26VIII] and [Bal85]–[Bal89b].

1.4 Notation and conventions

Throughout the paper:

- $G = \mathrm{SU}(N_c)$, $\mathfrak{g} = \mathfrak{su}(N_c)$, equipped with the bi-invariant inner product $\langle X, Y \rangle = -2 \mathrm{tr}(XY)$. We write $\kappa_G = \kappa_G(N_c) > 0$ for the Ricci lower bound of G in this metric. The normalised Haar measure on G satisfies $\mathrm{LSI}(\alpha_{\mathrm{Haar}})$ for some $\alpha_{\mathrm{Haar}} = \alpha_{\mathrm{Haar}}(N_c) > 0$ (by Bakry–Émery). By tensorisation, the normalised Haar measure on any finite product G^n satisfies $\mathrm{LSI}(\alpha_{\mathrm{Haar}})$ with the *same* constant.
- $\Lambda = (\mathbb{Z}/L\mathbb{Z})^4$; $E(\Lambda)$ = oriented edges; $\mathcal{P}(\Lambda)$ = plaquettes.
- $\beta = 2N_c/g^2$. At RG scale k , the running Wilson coupling g_k^{-2} is defined by a Wilson projection on the *local* part of the effective action (Definition 7.5 in Appendix 7). The derived Wilson parameter is $\beta_k := 2N_c g_k^{-2}$. The one-step recursion for g_k^{-2} is Theorem 5.2; the monotone bootstrap $g_{k+1} \leq g_k$ is Theorem 5.3.
- $p_0(g)$ = Balaban’s large-field penalty, satisfying $p_0(g) \rightarrow \infty$ as $g \rightarrow 0$ and $p_0(g) \geq c_0 |\log g|^{1+\epsilon_0}$ for some $c_0, \epsilon_0 > 0$.
- $\varepsilon_k = c_* g_k^{1-\delta}$, $0 < \delta < 1$ (Balaban’s small-field threshold).
- $\mathbf{R}^{(k)}(X)$, $\mathbf{B}^{(k)}(X)$, $\mathbf{T}_k(Y)$ as in [Bal89b].
- All unadorned constants C, c, C_1, \dots depend only on $(d, N_c, L_{\mathrm{RG}})$.

1.5 Organization

§2 establishes small-field coercivity (supporting G2). §3 proves unconditional large-field suppression (G1). §4 treats boundary-term analyticity (G3). §5 performs the quantitative bootstrap (G4). §6 assembles the full proof of Theorem 1.1. Three appendices provide detailed audits.

2 Small-field coercivity

This section supplies the convexity input for G2. We do *not* claim a global fiber LSI with constant $O(\beta)$. Instead, we prove a uniform Hessian lower bound for the effective action in the small-field region, which is the input used in the sweeping-out estimates of [E26I] and [E26III].

2.1 Setup

Fix a block B at RG scale k with fast edges $E_k(B)$ and plaquettes $\mathcal{P}_k(B)$. For a fixed real background $\bar{U} \in G^{|E_{\mathrm{slow}}|}$, the conditional measure on fast variables is

$$d\mu_k^{\bar{U}}(U_{\mathrm{fast}}) = \frac{1}{Z_k(\bar{U})} \exp(-S_{\mathrm{eff},k}(U_{\mathrm{fast}}, \bar{U})) \prod_{e \in E_k(B)} dU_e, \quad (4)$$

with dU_e the normalised Haar measure on G .

Definition 2.1 (Small-field region). $\Omega_k^{\text{sf}}(B; \bar{U})$ is the set of fast-field configurations satisfying $\|U_p - \mathbf{1}\|_{\text{HS}} < \varepsilon_k$ for every $p \in \mathcal{P}_k(B)$.

Definition 2.2 (Exponential chart). Let $U_e^{(0)}(\bar{U})$ be a minimiser of $S_{\text{eff},k}(\cdot, \bar{U})$ over the fast variables (exists by compactness). We parametrise $U_e = \exp(A_e) \cdot U_e^{(0)}(\bar{U})$ with $A_e \in \mathfrak{g}$, $\|A_e\| < \text{inj}(G)$.

2.2 Coercivity: axiomatic input from Balaban

The coercivity of the gauge-fixed quadratic form is a cornerstone of Balaban's RG. We incorporate it as an explicit axiom.

Assumption 2.3 (Balaban one-step inputs (published)). The following are used as published inputs from Balaban's series [1, 3, 4, 5, 6, 7]. Each is a *one-step* statement: it takes the inductive conditions (IC1)–(IC6) (Theorem 7.1) at scale k as input and produces outputs at scale $k + 1$. No global hypothesis on the coupling sequence is used.

- (B1) (*One-step RG map.*) There exists $\gamma_{\text{Bal}} > 0$ such that whenever (IC1)–(IC6) hold at scale k , the RG step $k \rightarrow k + 1$ is well-defined and produces S_{k+1}^{eff} satisfying (IC1)–(IC6) at scale $k + 1$, provided $g_{k+1} \leq \gamma_{\text{Bal}}$ (verified by Theorem 5.3).
- (B2) (*Polymer bounds.*) $|\mathbf{R}^{(k)}(X)| \leq e^{-p_0(g_k)|X|} e^{-\kappa d_k(X)}$ uniformly in the one-step inductive regime.
- (B3) (*Analyticity.*) S_k^{eff} is analytic in the tube \tilde{U}_k^c with radius $\hat{\alpha}(\gamma)$ independent of k , in the one-step inductive regime.
- (B4) (*T-operation small-factor bound*, Balaban (1.89).) $\mathbf{T}_k(Y)\mathbf{1} \leq \exp(-c_{\text{sf}} p_0(g_k) |Y|)$ with $c_{\text{sf}} > 0$ depending only on (d, N_c, L_{RG}) . Pointwise-in-background validity: ??.

Note: The running coupling formula is *derived* in Theorem 5.3 via the one-step estimate Theorem 5.2; it is not taken as an input.

Remark 2.4 (Scope of coercivity used in this paper). We do *not* claim that the gauge-fixed covariant Laplacian has a uniform spectral gap for *all* real backgrounds \bar{U} . The coercivity condition (IC6) (via Theorem 2.3 (B1)) is used only in the regime where Balaban's inductive hypotheses hold at scale k (the regime in which the RG produces the effective action and its polymer decomposition with uniform bounds). Configurations of the fast variables that fall outside this regime—i.e. configurations activating a large-field component—are controlled by the *T-operation small-factor mechanism* (Theorem 3.3), which does not require coercivity of the background.

2.3 Hessian of the effective action

Proposition 2.5 (Riemannian Hessian bound). *In the small-field region $\Omega_k^{\text{sf}}(B; \bar{U})$, the Riemannian Hessian of $V_{k,B} := S_{\text{eff},k}(\cdot, \bar{U})$ in the chart of Theorem 2.2 satisfies*

$$\text{Hess}_{\text{Riem}}(V_{k,B})(A, A) \geq \frac{\beta_k \gamma_{\text{Bal}}}{4} \|A\|^2 \quad (5)$$

uniformly in \bar{U} whenever the inductive hypotheses of [Bal89b] hold at scale k , and for all $g_k \leq \gamma_0$, where $\gamma_0 = \gamma_0(d, N_c)$ is chosen in §5.

Proof. Step 1 (Quadratic part). The Wilson action in the exponential chart gives

$$V_{k,B}^{(2)}(A) = \frac{\beta_k}{2} \langle A, \Delta_1^{\text{gf}}(\bar{U}) A \rangle \geq \frac{\beta_k \gamma_{\text{Bal}}}{2} \|A\|^2 \quad (6)$$

by condition (IC6) of the inductive hypotheses (Theorem 2.3 (B1)).

Step 2 (Cubic and higher terms). For $\|A_e\| < \varepsilon_k$:

$$|\text{Hess}_{\text{Riem}}(V_{k,B})(A, A) - V_{k,B}^{(2)}(A)| \leq (C_{\text{cubic}} \varepsilon_k + C_{\text{curv}} \beta_k^{-1}) \beta_k \|A\|^2. \quad (7)$$

The first contribution bounds the cubic Wilson terms: the third-order remainder of $1 - \frac{1}{N_c} \text{Re Tr}(\exp(A_{e_1}) \cdots)$ produces terms of the form $\beta_k \langle A, [A, d_1 A] \rangle$ bounded by $C_{\text{cubic}} \beta_k \varepsilon_k \|A\|^2$. The second contribution bounds the difference between the Euclidean Hessian in the chart and the Riemannian Hessian, which is $O(1)$ (the Christoffel symbols of G are bounded), hence $O(\beta_k^{-1})$ relative to the quadratic term.

Step 3 (Polymer contribution). The polymer terms $S_{k,\text{poly}} = \sum_{X \ni B} \mathbf{R}^{(k)}(X)$ contribute

$$|\text{Hess}_{\text{Riem}}(S_{k,\text{poly}})(A, A)| \leq \frac{2}{\hat{\alpha}_1^2} \sum_{X \ni B} |\mathbf{R}^{(k)}(X)| \cdot \|A\|^2 \leq C_{\text{poly}} \|A\|^2 \quad (8)$$

by the Cauchy estimate with analyticity radius $\hat{\alpha}_1$ and the polymer bound $\sum_{X \ni B} |\mathbf{R}^{(k)}(X)| \leq C_R$ from [Bal89b] Theorem 1.

Step 4 (Assembly).

$$\text{Hess}_{\text{Riem}}(V_{k,B})(A, A) \geq \left(\frac{\gamma_{\text{Bal}}}{2} - C_{\text{cubic}} \varepsilon_k - C_{\text{curv}} \beta_k^{-1} - C_{\text{poly}} \beta_k^{-1} \right) \beta_k \|A\|^2.$$

The perturbative ratio is

$$\frac{C_{\text{cubic}} \varepsilon_k}{\gamma_{\text{Bal}}/2} + \frac{C_{\text{curv}} + C_{\text{poly}}}{\beta_k \gamma_{\text{Bal}}/2} = O(g_k^{1-\delta}) + O(\beta_k^{-1}) \rightarrow 0.$$

For $g_k \leq \gamma_0$ with γ_0 chosen in Theorem 5.4, both terms are $< 1/2$, yielding (5). \square

Remark 2.6 (What Theorem 2.5 provides for the program). Theorem 2.5 does *not* assert a global fiber LSI with constant $O(\beta)$. It provides:

- (i) A uniform convexity estimate for the effective action in the small-field region, controlling the derivatives appearing in the sweeping-out procedure of [E26I, E26III].
- (ii) A quantitative input showing that the effective potential becomes *more* convex as β increases, which enters the cross-scale error estimates of [E26III]–[E26V].

The conditional LSI entering the entropy telescoping is obtained by the Holley–Stroock perturbation of the Haar-measure LSI (Theorem 2.7), with a constant α_0 that may be exponentially small in β but is strictly positive and L -independent. The large-field contribution is controlled separately by Theorem 3.3.

2.4 Conditional LSI (Holley–Stroock, honest version)

Proposition 2.7 (Conditional LSI). *For every $\beta \geq \beta_0$ and every \bar{U} , the fiber measure $\mu_k^{\bar{U}}$ of (4) satisfies $\text{LSI}(\alpha_0)$ with*

$$\alpha_0 := \alpha_{\text{Haar}} \exp(-2 \text{osc}(V_{k,B})) > 0, \quad (9)$$

where $\text{osc}(V_{k,B}) = \sup V_{k,B} - \inf V_{k,B}$ over the compact fiber $G^{|E_k(B)|}$ and $\alpha_{\text{Haar}} > 0$ is the LSI constant of normalised Haar measure on G (which by tensorisation is also the LSI constant of $G^{|E_k(B)|}$, depending only on N_c). The constant α_0 depends on $(\beta, d, N_c, L_{\text{RG}})$ but **not** on L .

Proof. The product manifold $\mathcal{M} = G^{|E_k(B)|}$ with the product metric has $\text{Ric}(\mathcal{M}) \geq \kappa_G > 0$ for some $\kappa_G = \kappa_G(N_c) > 0$ depending on the chosen bi-invariant metric. By Bakry–Émery, G satisfies $\text{LSI}(\alpha_{\text{Haar}})$ with $\alpha_{\text{Haar}} \geq \kappa_G > 0$. By tensorisation, $G^{|E_k(B)|}$ satisfies $\text{LSI}(\alpha_{\text{Haar}})$ with the same constant. The measure $\mu_k^{\bar{U}}$ has density $\rho \propto e^{-V_{k,B}}$ with respect to normalised Haar. By the Holley–Stroock perturbation lemma ([8] Proposition 5.1.6),

$$\alpha_0 \geq \alpha_{\text{Haar}} \cdot e^{-2 \text{osc}(V_{k,B})}.$$

The oscillation satisfies $\text{osc}(V_{k,B}) \leq 2\beta |\mathcal{P}_k(B)| + C_R$, which is finite and L -independent. \square

Remark 2.8. The constant α_0 in (9) is exponentially small in β . This is *acceptable*: the mass gap $\Delta_{\text{phys}} \geq c\alpha_*$ will be positive (though possibly very small). For the existence statement $\Delta_{\text{phys}} > 0$, positivity suffices. The coercivity estimate of Theorem 2.5 is used *not* to improve α_0 , but to control the sweeping-out errors in [E26III]–[E26V]—places where the $O(\beta)$ convexity is needed as a quantitative input, not as an LSI constant.

3 Unconditional large-field suppression

This section resolves G1. The strategy is to audit Balaban’s proof of the small-factor bound [Bal89b] eq. (1.89) and verify that it holds pointwise for all real backgrounds by using only the uniform inductive conditions and the compactness of G .

3.1 Regular conditional probability

Lemma 3.1 (RCP as fibre integral). *There exists a version of the regular conditional probability $\mu_k(\cdot | \mathcal{G}_{k+1})$ such that for every $\bar{U} \in G^{|E_{\text{slow}}|}$:*

$$\mu_k(A | \mathcal{G}_{k+1})(\bar{U}) = \frac{\int_A \exp(-S_{\text{eff},k}(U_{\text{fast}}, \bar{U})) \prod_{e \in E_k(B)} dU_e}{\int_{G^{|E_k(B)|}} \exp(-S_{\text{eff},k}(U_{\text{fast}}, \bar{U})) \prod_{e \in E_k(B)} dU_e}. \quad (10)$$

The right-hand side is well-defined (the denominator is strictly positive by strict positivity of the integrand on a compact domain) and **continuous** in \bar{U} .

Proof. See ??.

□

Corollary 3.2. *For any continuous function $h : G^{|E_{\text{slow}}|} \rightarrow \mathbb{R}$,*

$$\text{ess sup}_{\mathcal{G}_{k+1}} h(\mathcal{G}_{k+1}) = \sup_{\bar{U} \in G^{|E_{\text{slow}}|}} h(\bar{U}).$$

In particular, a pointwise bound on $\mu_k(Z_k(B) | \bar{U})$ implies the same bound on the ess sup.

3.2 Statement

Theorem 3.3 (Uniform large-field suppression). *For $\beta \geq \beta_0$ and $0 \leq k \leq n_{\text{max}}$, for every block B and every $\bar{U} \in G^{|E_{\text{slow}}|}$:*

$$\mu_k(Z_k(B) | \mathcal{G}_{k+1} = \bar{U}) \leq C_{\text{blk}} e^{-c_{\text{lf}} p_0(g_k)}, \quad (11)$$

with $C_{\text{blk}} = C_{\text{blk}}(d, N_c, L_{\text{RG}})$ independent of \bar{U} , k , and L .

3.3 Proof of Theorem 3.3

The proof has three ingredients: (i) identification of the conditional probability with Balaban’s T -operation, (ii) the pointwise small-factor bound, and (iii) a lattice-animal summation.

Step 1: Identification with the T -operation

The large-field event $Z_k(B)$ is contained in the union over connected large-field components Y intersecting B . By the union bound:

$$\mu_k(Z_k(B) | \bar{U}) \leq \sum_{\substack{Y: Y \cap B \neq \emptyfty \\ Y \text{ connected}}} \mathbf{T}_k(Y) \mathbf{1}(\bar{U}), \quad (12)$$

where $\mathbf{T}_k(Y) \mathbf{1}(\bar{U}) := \mu_k^{\bar{U}}(\chi_Y^{\text{lf}})$ is the conditional probability of the large-field indicator for component Y .

Step 2: Pointwise small-factor bound (audit of Balaban)

Proposition 3.4 (Pointwise small factor). *For every $\bar{U} \in G^{|E_{\text{slow}}|}$, every connected large-field component Y of size $|Y|$ (in units of L_{RG} -blocks), and every $0 \leq k \leq n_{\text{max}}$:*

$$\mathbf{T}_k(Y)\mathbf{1}(\bar{U}) \leq \exp(-c_{\text{sf}} p_0(g_k) |Y|), \quad (13)$$

where $c_{\text{sf}} > 0$ depends only on (d, N_c, L_{RG}) .

Proof. This is Balaban’s T -operation small-factor estimate (Theorem 2.3 (B4)). The bound (13) with $p_0(\cdot)$ as in the inductive scheme is exactly [Bal89b] eq. (1.89). Our contribution is the verification that it holds *pointwise* in the real background \bar{U} (as required to control $\text{ess sup}_{\mathcal{G}_{k+1}}$), rather than only for backgrounds satisfying Balaban’s admissibility conditions.

The detailed audit is in ???. Here we summarise the three points that ensure \bar{U} -independence:

- (i) The T -operation is defined as a ratio of strictly positive fibre integrals (Theorem 3.1) and therefore exists for every real \bar{U} . The $O(\beta_k)$ -sized offset $S_{\text{eff},k}(U^{(0)}, \bar{U})$ cancels in this ratio.
- (ii) Every estimate in Balaban’s derivation of (1.89) uses only the *uniform inductive bounds* (polymer norms, analyticity radii, regulator parameters, combinatorial counting)—all of which are stated with constants independent of the particular value of \bar{U} .
- (iii) No step invokes an “admissibility” or analyticity-domain restriction on the real background; such restrictions are needed only for the complexified continuation that produces the next RG step, not for the real-valued ratio $\mathbf{T}_k(Y)\mathbf{1}(\bar{U})$.

Therefore (13) holds for all $\bar{U} \in G^{|E_{\text{slow}}|}$. \square

Step 3: Lattice animal summation

From (12) and (13):

$$\mu_k(Z_k(B) | \bar{U}) \leq \sum_{n=1}^{\infty} C_d^n e^{-c_{\text{sf}} p_0(g_k) n} = \frac{C_d e^{-c_{\text{sf}} p_0(g_k)}}{1 - C_d e^{-c_{\text{sf}} p_0(g_k)}}, \quad (14)$$

where C_d^n bounds the number of connected lattice animals of size n containing B . Condition (T6) of Theorem 5.4 guarantees $C_d e^{-c_{\text{sf}} p_0(g_k)} \leq 1/2$, so

$$\mu_k(Z_k(B) | \bar{U}) \leq 2 C_d e^{-c_{\text{sf}} p_0(g_k)} =: C_{\text{blk}} e^{-c_{\text{lf}} p_0(g_k)}, \quad (15)$$

where $C_{\text{blk}} := 2C_d$ and $c_{\text{lf}} := c_{\text{sf}}$. Note that p_0 is not redefined; the constant c_{lf} is carried explicitly throughout. \square

4 Uniform analyticity of boundary terms

Proposition 4.1 (Analyticity of $\mathbf{B}^{(k)}$). *The boundary terms $\mathbf{B}^{(k)}(X)$ of Balaban’s effective action are analytic in the complexified link variables with analyticity radius $\hat{\alpha}_1(\gamma) > 0$ independent of k , and satisfy*

$$\sup_{|z| \leq \hat{\alpha}_1} |\mathbf{B}^{(k)}(X; U + z)| \leq C_B (k+1) |X| e^{-\kappa d_k(X)}. \quad (16)$$

The Cauchy estimate gives

$$|\nabla \mathbf{B}^{(k)}(X)| \leq \frac{C_B (k+1) |X|}{\hat{\alpha}_1} e^{-\kappa d_k(X)}. \quad (17)$$

Proof. Step 1 (Inductive conditions). By [Bal89b] Theorem 1, the complete effective action at scale k ,

$$S_k^{\text{eff}} = \beta_k A(U) + S_k^{\text{loc,irr}} + \sum_X \mathbf{R}^{(k)}(X) + \sum_X \mathbf{B}^{(k)}(X), \quad (18)$$

satisfies the inductive conditions (IC1)–(IC5) with parameters $(\tilde{\alpha}_0, \hat{\alpha}_1(\gamma), c_*, \kappa, p_0)$ *preserved at each step*. In particular, S_k is analytic in the tube $\tilde{U}_k^c(\tilde{\alpha}_0, \hat{\alpha}_1)$ around the real configuration space.

Step 2 (Analyticity of $\mathbf{B}^{(k)}$). The decomposition (18) is defined by a localisation procedure: $\mathbf{R}^{(k)}(X)$ collects contributions from polymers X that do not intersect the accumulated large-field boundary $\partial\Gamma_{\leq k}$, while $\mathbf{B}^{(k)}(X)$ collects those that do. Each term is obtained by a convergent cluster/Mayer expansion applied to the analytic function $S_k^{\text{eff}} - \beta_k A(U)$ within the tube \tilde{U}_k^c .

The cluster expansion preserves analyticity in the external variables with the same radius $\hat{\alpha}_1$, provided the expansion converges uniformly in the tube. This convergence is part of the inductive conditions: condition (IC3) asserts analyticity of each polymer activity with radius $\hat{\alpha}_1$, and condition (IC4) provides the summability $\sum_{X \ni B} |\mathbf{R}^{(k)}(X)| \leq C_R$. The same summability and analyticity apply to $\mathbf{B}^{(k)}(X)$ because the localisation (selection by support near $\partial\Gamma_{\leq k}$) does not enlarge the domain or worsen the convergence.

Step 3 (Sup-norm bound). By [Bal89b] eq. (1.69): $|\mathbf{B}^{(k)}(X)| \leq C \sum_{j=1}^k |\Gamma_j^0 \cap X| \cdot e^{-\kappa d_k(X)} \leq C(k+1)|X| e^{-\kappa d_k(X)}$. The factor $(k+1) \leq n_{\max} + 1$ depends on β_0 but not on L .

Step 4 (Cauchy estimate). Standard: for any unit direction v , $|D_v \mathbf{B}^{(k)}(X; U)| \leq \hat{\alpha}_1^{-1} \sup_{|z| \leq \hat{\alpha}_1} |\mathbf{B}^{(k)}(X; U + zv)| \leq C_B(k+1)|X| \hat{\alpha}_1^{-1} e^{-\kappa d_k(X)}$. \square

5 Quantitative bootstrap

5.1 Unconditional control of the running coupling

The published Balaban large-field RG theorems are stated with a global hypothesis that effective couplings remain small at all scales. We replace this by a *one-step* estimate (Theorem 5.2) and an explicit induction (Theorem 5.3), proved in full in Section 7. This closes the last logical dependency on any unpublished global stability result.

Definition 5.1 (Wilson projection and running coupling). At RG scale k , write the effective action in “local + polymers” form

$$S_k^{\text{eff}}(U) = S_k^{\text{loc}}(U) + \sum_X \mathbf{R}^{(k)}(X; U) + \sum_X \mathbf{B}^{(k)}(X; U), \quad (19)$$

where S_k^{loc} is a bounded-range local gauge-invariant functional. Define g_k^{-2} by the Wilson projection (Appendix 7, Definition 7.5):

$$S_k^{\text{loc}}(U) = \beta_k A(U) + S_k^{\text{loc,irr}}(U), \quad \beta_k := 2N_c g_k^{-2}, \quad (20)$$

with $S_k^{\text{loc,irr}}$ orthogonal to the Wilson direction (vanishing plaquette-second-variation).

Theorem 5.2 (One-step coupling renormalization). *Fix $d = 4$, $G = \text{SU}(N_c)$, $L_{\text{RG}} = 2$. There exist $\gamma_{\text{run}} > 0$ and $C_{\text{run}} < \infty$ depending only on (d, N_c, L_{RG}) such that: if the inductive conditions (IC1)–(IC6) (Theorem 7.1) hold at scale k with $g_k \leq \gamma_{\text{run}}$, then*

$$g_{k+1}^{-2} = g_k^{-2} + 2b_0 \ln L_{\text{RG}} + \delta_k, \quad |\delta_k| \leq C_{\text{run}} g_k^2, \quad (21)$$

where $b_0 := 11N_c/(48\pi^2)$. Equivalently, $\beta_{k+1} = \beta_k + 4N_c b_0 \ln L_{\text{RG}} + \rho_k$ with $|\rho_k| \leq 2N_c C_{\text{run}} g_k^2$.

Theorem 5.3 (Bootstrap stability). *There exists $\gamma_0^{\text{run}} > 0$ such that for $g_0 \leq \gamma_0 \leq \gamma_0^{\text{run}}$:*

$$g_{k+1} \leq g_k \leq g_0 \leq \gamma_0 \quad \text{for all } 0 \leq k \leq n_{\max}. \quad (22)$$

The global small-coupling hypothesis is a consequence of $g_0 \leq \gamma_0$ and the one-step estimate, not an assumption.

Proof. Choose $\gamma_0^{\text{run}} \leq \gamma_{\text{run}}$ so that $C_{\text{run}}(\gamma_0^{\text{run}})^2 \leq b_0 \ln L_{\text{RG}}$. Assume inductively $g_k \leq \gamma_0$. By Theorem 5.2:

$$g_{k+1}^{-2} \geq g_k^{-2} + 2b_0 \ln L_{\text{RG}} - C_{\text{run}} g_k^2 \geq g_k^{-2} + b_0 \ln L_{\text{RG}} > g_k^{-2}.$$

Hence $g_{k+1} < g_k$. The base case $k = 0$ is Theorem 7.7. \square

Full proofs of Theorem 5.2 and the base case: Section 7.

Definition 5.4 (Master threshold). Define $\gamma_0 > 0$ to be the largest value satisfying all of the following:

- (T1) $\gamma_0 \leq \gamma_{\text{Bal}}$ (one-step RG map Theorem 7.2 applies).
- (T2) $\gamma_0 \leq \gamma_{\text{run}}$ (one-step coupling estimate Theorem 5.2 applies).
- (T3) $C_{\text{run}} \gamma_0^2 \leq b_0 \ln L_{\text{RG}}$ (remainder does not overwhelm the drift; ensures Theorem 5.3).
- (T4) $\frac{C_{\text{cubic}} c_* \gamma_0^{1-\delta}}{\gamma_{\text{Bal}}/2} \leq \frac{1}{4}$ (cubic terms small, Theorem 2.5 Step 4).
- (T5) $\frac{(C_{\text{curv}} + C_{\text{poly}}) \gamma_0^2}{N_c \gamma_{\text{Bal}}/2} \leq \frac{1}{4}$ (subleading terms small).
- (T6) $c_{\text{sf}} p_0(\gamma_0) \geq \ln(2C_d) + 1$ (lattice animal sum converges).
- (T7) $C_{\text{geom}} e^{-p_0(\gamma_0)} \leq \delta_*$ (weak dependence threshold).

Since all right-hand sides are fixed positive constants and $p_0(g) \rightarrow \infty$ as $g \rightarrow 0$, such γ_0 exists. Set

$$\beta_0 := \frac{2N_c}{\gamma_0^2}. \quad (23)$$

Remark 5.5 (Weak-dependence threshold δ_*). The extension to DLR-LSI in [E26VII] uses a perturbative criterion for inter-block coupling formulated in terms of a small influence parameter δ_k . In this closure paper we isolate the only requirement from that step: the existence of a numerical threshold $\delta_* > 0$ (depending only on (d, N_c, L_{RG}) and the specific criterion used in [E26VII]) such that $\delta_k \leq \delta_*$ implies the desired block-coupling estimate. We choose γ_0 so that $\delta_k \leq C_{\text{geom}} e^{-p_0(g_k)} \leq \delta_*$.

Crucially, δ_* is a fixed positive number independent of β . The requirement $\delta_k \leq \delta_*$ is compatible with any $\alpha_0 > 0$ (even exponentially small in β), because the left-hand side $\delta_k \leq C_{\text{geom}} e^{-p_0(g_k)}$ tends to 0 as $\beta \rightarrow \infty$ while δ_* stays fixed.

Proposition 5.6 (Simultaneous compatibility). For $\beta \geq \beta_0$ and $0 \leq k \leq n_{\text{max}}$:

- (a) $g_k \leq g_0 \leq \gamma_0$ (small coupling at all scales).
- (b) $\text{Hess}_{\text{Riem}}(V_{k,B}) \geq \frac{\beta_k \gamma_{\text{Bal}}}{4}$ (Theorem 2.5, in the inductive regime).
- (c) $\mu_k(Z_k(B) | \bar{U}) \leq C_{\text{blk}} e^{-c_{\text{if}} p_0(g_k)}$ uniformly in \bar{U} (Theorem 3.3).
- (d) $\delta_k := \sup_B \sum_{X \ni B, |X| \geq 2} \|\Phi_X^{(k)}\|_{\infty} \leq C_{\text{geom}} e^{-p_0(g_k)} \leq \delta_*$.
- (e) $\sum_{k=0}^{n_{\text{max}}} D_k \leq D_{\infty} < \infty$, where $D_k \leq C_D (k+1) L_{\text{RG}}^{-(d-1)k}$.
- (f) $\alpha_* := (C_E/2 + C_B C_P)^{-1} > 0$, L -independent.

Proof. (a): By Theorem 5.3 (proved non-circularly in Section 7 via one-step estimates and induction from the base case Theorem 7.7): for $\beta \geq \beta_0$, we have $g_k \leq g_0 \leq \gamma_0$ for all $0 \leq k \leq n_{\text{max}}$. No global small-coupling input is used.

(b): direct from Theorem 2.5 with (T4)–(T5).

(c): from Theorem 3.3 with (T6).

(d): the polymer interactions coupling blocks satisfy $\|\Phi_X^{(k)}\|_{\infty} \leq e^{-p_0(g_k)|X|}$ by [Bal89b] condition (IC4). The sum $\delta_k \leq \sum_{n \geq 2} C_d^n e^{-p_0(g_k)n} \leq C_{\text{geom}} e^{-p_0(g_k)}$ for $p_0(g_k)$ large, hence $\leq \delta_*$ by (T7).

(e): $\sum_k D_k \leq C_D \sum_{k \geq 0} (k+1) 2^{-3k} = C_D \cdot 8/49 < \infty$.

(f): the Rothaus lemma ([8] Proposition 5.1.3) states: if

$$\text{Ent}_\mu(f^2) \leq C_E \mathcal{E}(f, f) + C_B \quad \text{and} \quad \text{Var}_\mu(f) \leq C_P \mathcal{E}(f, f),$$

then $\text{Ent}_\mu(f^2) \leq (C_E + 2C_B C_P) \mathcal{E}(f, f)$. The defective LSI is the output of [E26I] with $C_E = 2/\alpha_0 + 2D_\infty/\alpha_0$ and C_B depending on β_0 but not L . The Poincaré inequality is from [E26I] §6. All three constants are finite and L -independent. \square

5.2 Constant tracking

Table 1: Summary of constants and their dependencies.

Symbol	Depends on	Behaviour in β
g_k	β, k	$\leq (2N_c/\beta)^{1/2}$
g_k^{-2}	g_0^{-2}, k	$g_0^{-2} + 2b_0 k \ln L_{\text{RG}} + \sum_{j < k} \delta_j$ (increasing; Theorem 5.2)
β_k	$2N_c g_k^{-2}$	$\beta + 4N_c b_0 k \ln L_{\text{RG}} + O(1)$ (increasing; derived)
γ_{Bal}	d, N_c, L_{RG}	geometric constant (Balaban input)
α_{Haar}	N_c	LSI constant of Haar on G (same for G^n by tensorisation)
α_0	$\alpha_{\text{Haar}}, \text{osc}(V_{k,B})$	$\alpha_{\text{Haar}} e^{-2\text{osc}(V_{k,B})}$ (positive, L -indep.)
δ_*	d, N_c, L_{RG}	fixed threshold from [E26VII]
$p_0(g_k)$	g_k	increasing as $g_k \rightarrow 0$
δ_k	$p_0(g_k), C_{\text{geom}}$	$\leq C_{\text{geom}} e^{-p_0(g_k)}$
D_k	$k, d, \hat{\alpha}_1$	$\leq C_D(k+1) 2^{-3k}$
C_E, C_B, C_P	$\alpha_0, D_\infty, \beta_0$	finite, L -independent
α_*	$(C_E + 2C_B C_P)^{-1}$	positive, L -independent

Non-circularity. The logical order of determination is:

$$\gamma_{\text{Bal}}, \alpha_{\text{Haar}}, \delta_* \rightarrow \gamma_0 \rightarrow \beta_0 \rightarrow g_k, \beta_k \rightarrow \alpha_0 \xrightarrow{[\text{Bal89b}]} C_R, c_{\text{sf}} \rightarrow C_E, C_B, C_P \rightarrow \alpha_* \rightarrow \Delta_{\text{phys}}.$$

No later constant feeds back into an earlier one. The inputs α_{Haar} (Haar LSI constant, depending only on N_c) and δ_* (weak-dependence threshold from [E26VII]) are fixed positive numbers independent of β_0 .

6 Assembly: Proof of Theorem 1.1

We collect the Balaban inputs used in the assembly.

The Balaban inputs used throughout this paper are collected in Theorem 2.3 (§2). We recall that:

- (B1)–(B4) are *one-step* statements (no global hypothesis on the coupling sequence).
- The running coupling control ($g_k \leq \gamma_0$ for all k) is *derived* in Theorem 5.3 via the one-step estimate Theorem 5.2. It is not taken as an input.

Proof of Theorem 1.1. Step 1: Unconditional iteration of the RG and uniform LSI.

By Theorem 5.3 (proved in Section 7 using only the one-step Theorem 7.2 and explicit induction from the base case Theorem 7.7): for $\beta \geq \beta_0$, the inductive conditions (IC1)–(IC6) hold at *every* scale $k = 0, \dots, n_{\text{max}}$, and $g_k \leq g_0 \leq \gamma_0$. The full Balaban RG construction is valid without any global small-coupling assumption.

With this established, Paper 68 ([E26I], Theorem 1.1) gives the entropy telescoping. The required inputs are:

- Conditional LSI: Theorem 2.7 ($\alpha_0 > 0$, L -independent).

- Cross-scale bounds: Papers 70–72 ([E26III]–[E26V]), using polymer bounds (B2) and Cauchy estimates from Theorem 4.1.
- Large-field suppression: Theorem 3.3, using (B4).
- Bootstrap of all constants: Theorem 5.6.

Output: $\text{Ent}_{\mu_\beta}(f^2) \leq (2/\alpha_*) \sum_e \int |\nabla_e f|^2 d\mu_\beta$ with $\alpha_* > 0$ independent of L .

Step 2: DLR-LSI. Papers 73–74 ([E26VI]–[E26VII]) extend the torus LSI to all finite boxes with all boundary conditions:

- Frozen boundary links are treated as additional slow variables (Paper 74, §2).
- Large-field suppression holds for *all* slow configurations, including frozen boundaries (Theorem 3.3: the bound is pointwise in \bar{U}).
- The inter-block coupling criterion of [E26VII] is verified by Theorem 5.6(d) (via the threshold δ_* of Theorem 5.5).

The output: DLR-LSI with constant $\alpha_{\text{DLR}} > 0$ uniform in box and boundary condition.

Step 3: Stroock–Zegarlinski. DLR-LSI + compact spin space ($G = \text{SU}(N_c)$) + finite-range interaction (Wilson action, range 1) imply, by [15, 16]: complete analyticity and exponential clustering with correlation length $\xi \leq C/\alpha_{\text{DLR}}$.

Step 4: Reflection positivity \Rightarrow transfer matrix gap. The Wilson action satisfies reflection positivity [13]. By the standard argument ([10] §6):

$$\Delta_{\text{phys}} = -\log \|\widehat{T}|_{\mathbf{1}^\perp}\| \geq c/\xi \geq c\alpha_{\text{DLR}} \geq c\alpha_* > 0.$$

This completes the proof of Theorem 1.1. □

7 Coupling constant bootstrap: complete proof

This appendix provides the complete, non-circular proof of Theorem 5.2 and Theorem 5.3. The argument is strictly inductive: at each step $k \rightarrow k+1$, only the inductive conditions at scale k are used. The variable g_k^{-2} is primary throughout; $\beta_k := 2N_c g_k^{-2}$ is derived notation.

7.1 Inductive conditions at a single scale

Definition 7.1 (Inductive conditions at scale k (local)). Fix $\gamma > 0$. The *inductive conditions at scale k* consist of:

- (IC1) **Coupling bound:** $g_k \leq \gamma$.
- (IC2) **Effective action structure:** $S_k^{\text{eff}} = \beta_k A(U) + S_k^{\text{loc,irr}} + \sum_X \mathbf{R}^{(k)}(X) + \sum_X \mathbf{B}^{(k)}(X)$, where $\beta_k := 2N_c g_k^{-2}$ and $A(U)$ is as in (1).
- (IC3) **Polymer analyticity:** Each $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$ is analytic in the complexified link variables on the tube $\tilde{U}_k^c(X, \tilde{\alpha}_0, \hat{\alpha}(\gamma))$ with radius $\hat{\alpha}(\gamma) > 0$ depending only on γ and (d, N_c, L_{RG}) .
- (IC4) **Polymer decay:** $|\mathbf{R}^{(k)}(X)| \leq e^{-p_0(g_k)|X|} e^{-\kappa d_k(X)}$.
- (IC5) **Boundary term bound:** $|\mathbf{B}^{(k)}(X)| \leq C_B(k+1)|X| e^{-\kappa d_k(X)}$.
- (IC6) **Coercivity:** $\langle A, \Delta_1^{\text{gf}} A \rangle \geq \gamma_{\text{Bal}} \|A\|^2$ on the small-field domain at scale k .

All constants depend only on $(\gamma, d, N_c, L_{\text{RG}})$ and are **independent of k** and the volume.

Assumption 7.2 (One-step RG map (published input)). There exists $\gamma_{\text{Bal}} > 0$ such that: if (IC1)–(IC6) hold at scale k , then the Balaban RG step $k \rightarrow k+1$ is well-defined and produces S_{k+1}^{eff} satisfying (IC1)–(IC6) at scale $k+1$, **provided** $g_{k+1} \leq \gamma_{\text{Bal}}$.

Remark 7.3. Theorem 7.2 is strictly one-step: it takes data at scale k and produces data at scale $k+1$. The condition $g_{k+1} \leq \gamma_{\text{Bal}}$ is verified by Theorem 5.3, not assumed for future scales.

7.2 Normalizations and the Wilson direction

Lemma 7.4 (Second variation of the Wilson density). *Let $T \in \mathfrak{su}(N_c)$ with $\|T\| = 1$ and $U(t) := \exp(tT)$. Then*

$$W(U(t)) = \frac{t^2}{4N_c} + O(t^4), \quad \left. \frac{d^2}{dt^2} \right|_{t=0} W(U(t)) = \frac{1}{2N_c}. \quad (24)$$

Proof. Write Tr for the unnormalized trace ($\text{Tr} \mathbf{1} = N_c$). Then $\exp(tT) = \mathbf{1} + tT + \frac{t^2}{2}T^2 + O(t^3)$, $\text{Tr}(T) = 0$, $\|T\|^2 = -2\text{tr}(T^2) = 1$ gives $\text{tr}(T^2) = -\frac{1}{2}$, hence $\text{Tr}(T^2) = N_c \text{tr}(T^2) = -\frac{N_c}{2}$. Hence $\text{Re Tr}(\exp(tT)) = N_c - t^2/4 + O(t^4)$ and $W = 1 - \frac{1}{N_c} \text{Re Tr} = t^2/(4N_c) + O(t^4)$. \square

Definition 7.5 (Wilson projection). At scale k , let p be any plaquette and $T \in \mathfrak{su}(N_c)$ with $\|T\| = 1$. Write $W''(0) := \left. \frac{d^2}{dt^2} \right|_{t=0} W(\exp(tT)) = \frac{1}{2N_c}$ (Theorem 7.4). Define

$$\beta_k := \frac{1}{W''(0)} \left. \frac{d^2}{dt^2} \right|_{t=0} S_k^{\text{loc}}(U_p = \exp(tT), U_{p'} = \mathbf{1} \ \forall p' \neq p) = 2N_c \left. \frac{d^2}{dt^2} \right|_{t=0} S_k^{\text{loc}}(\dots), \quad (25)$$

and set

$$g_k^{-2} := \frac{\beta_k}{2N_c} \quad (\text{so that } \beta_k = 2N_c g_k^{-2}). \quad (26)$$

Lemma 7.6 (Well-definedness and consistency). *Under uniform locality, gauge invariance, and hypercubic symmetry of S_k^{loc} :*

- (i) β_k in (25) is independent of the choice of plaquette p and direction T with $\|T\| = 1$.
- (ii) If $S_k^{\text{loc}} = \beta_k A(U) + S_k^{\text{loc,irr}}$ with $S_k^{\text{loc,irr}}$ having vanishing plaquette-second-variation, then the extraction recovers β_k .

Proof. (i) Independence of p : translation covariance. Independence of T : $\text{SU}(N_c)$ -conjugation invariance makes the Hessian proportional to $\mathbf{1}$ in the adjoint, hence depends only on $\|T\|$.

(ii) $\frac{1}{W''(0)} \cdot \beta_k \cdot W''(0) = \beta_k$. \square

7.3 Base case

Lemma 7.7 (Inductive conditions at $k = 0$). *For $\beta \geq \beta_0 := 2N_c/\gamma_0^2$, the Wilson action $S_0 = \beta A(U)$ satisfies (IC1)–(IC6) at $k = 0$.*

Proof. (IC1): $g_0 = (2N_c/\beta)^{1/2} \leq \gamma_0$.

(IC2): $S_0^{\text{eff}} = \beta A(U)$ with $\beta_0 = \beta = 2N_c g_0^{-2}$; no polymers or irrelevant local terms.

(IC3): the Wilson action is polynomial in matrix entries, hence entire; any finite $\hat{\alpha}(\gamma)$ is satisfied.

(IC4)–(IC5): vacuum (no polymers at $k = 0$).

(IC6): the gauge-fixed Hessian of βA at a small-field background satisfies $\langle \mathcal{A}, \Delta_1^{\text{gf}} \mathcal{A} \rangle \geq \beta c_{\text{Lap}} \|\mathcal{A}\|^2 \geq \beta_0 c_{\text{Lap}} \|\mathcal{A}\|^2$, where $c_{\text{Lap}} > 0$ is the spectral gap of the gauge-fixed lattice Laplacian on a single block. \square

7.4 Vacuum polarization structure

Lemma 7.8 (Unique marginal operator). *Fix scale k and perform one RG step in the small-field regime. Let $\Pi_{\mu\nu}^{(k)}(p)$ be the vacuum polarization tensor. Under:*

- (i) hypercubic symmetry,
- (ii) translation invariance,
- (iii) gauge invariance (Ward–Takahashi: $d_\mu(p)\Pi_{\mu\nu}^{(k)}(p) = 0$),
- (iv) exponential decay in position space,

there exists $\mathbf{c}_k \in \mathbb{R}$ and a remainder $\Pi_{\mu\nu}^{(k)'}(p)$ vanishing to second order at $p = 0$ such that

$$\Pi_{\mu\nu}^{(k)}(p) = \mathbf{c}_k(\delta_{\mu\nu}\widehat{\Delta}(p) - \widehat{d}_\mu(p)\widehat{d}_\nu(p)) + \Pi_{\mu\nu}^{(k)'}(p). \quad (27)$$

The coefficient $\mathbf{c}_k(g_k)$ determines the one-step shift $\Delta\beta_k = \mathbf{c}_k(g_k)$. The drift in g^{-2} is then $\mathbf{b}_k(g_k) := \mathbf{c}_k(g_k)/(2N_c)$ (Theorem 7.9).

Proof. Standard Ward–Takahashi + hypercubic symmetry analysis. See [3], Sect. 5, where these hypotheses are used to derive the decomposition and isolate \mathbf{c}_k by a mixed second derivative at $p = 0$. The remainder contributes only to irrelevant (higher-derivative) local operators. \square

Remark 7.9 (Normalization of the drift coefficient). We use two distinct symbols:

- $\mathbf{c}_k(g_k)$ = the vacuum polarization coefficient in (27), which gives the shift $\Delta\beta_k = \beta_{k+1} - \beta_k = \mathbf{c}_k(g_k)$.
- $\mathbf{b}_k(g_k)$ = the one-step drift in the variable g^{-2} , defined by

$$\mathbf{b}_k(g_k) := g_{k+1}^{-2} - g_k^{-2} = \frac{\mathbf{c}_k(g_k)}{2N_c} = \frac{\beta_{k+1} - \beta_k}{2N_c}, \quad (28)$$

where β_k is extracted from S_k^{loc} by the Wilson projection (Theorem 7.5). Throughout this appendix, \mathbf{b}_k always denotes the drift in g^{-2} , never the vacuum polarization coefficient directly.

7.5 Even analyticity and Cauchy remainder bound

Lemma 7.10 (Cauchy bound on remainder). *Under the one-step inductive hypotheses at scale k , \mathbf{b}_k is analytic in g_k^2 in a disk $|g_k^2| \leq (\gamma_{\text{an}})^2$, and*

$$|\mathbf{b}_k(g_k) - \mathbf{b}_k(0)| \leq C_{\text{an}} g_k^2. \quad (29)$$

C_{an} depends only on (d, N_c, L_{RG}) , not on k or the volume.

Proof. After rescaling the fluctuation field by g_k , the RG step is invariant under $B \mapsto -B$ at fixed background, so all derived local coefficients are analytic in g_k^2 (even powers only). The uniform analyticity tube from (IC3) gives a disk in g_k^2 of radius $(\gamma_{\text{an}})^2$ depending only on $\hat{\alpha}(\gamma)$. The Cauchy estimate on this disk gives $|\mathbf{b}_k - \mathbf{b}_k(0)| \leq (\gamma_{\text{an}})^{-2} \sup |\mathbf{b}_k| \cdot g_k^2 \leq C_{\text{an}} g_k^2$. The supremum of $|\mathbf{b}_k|$ on the disk is bounded uniformly by the polymer bounds (IC4) and the coercivity (IC6), both independent of k . \square

7.6 One-loop identification

Lemma 7.11 (One-loop value of the drift in g^{-2}). *In $d = 4$ for $G = \text{SU}(N_c)$, the $g_k = 0$ (Gaussian) value of the one-step drift in g^{-2} is*

$$\mathbf{b}_k(0) = 2b_0 \ln L_{\text{RG}}, \quad b_0 = \frac{11N_c}{48\pi^2}. \quad (30)$$

Equivalently, the one-loop shift in $\beta = 2N_c g^{-2}$ is $4N_c b_0 \ln L_{\text{RG}}$.

Proof. By Theorem 7.9, $\mathbf{b}_k(0)$ is the Gaussian one-step drift of g^{-2} , obtained from the vacuum polarization contribution to $\beta_{k+1} - \beta_k$ and converted to g^{-2} by division by $2N_c$ via Theorem 7.5. This is the standard one-loop lattice vacuum polarization computation; the universal coefficient b_0 is independent of the lattice regularization scheme.

Published references include [26, 27, 28, 29]. Normalization compatibility is verified in §7.8. \square

7.7 Proof of Theorem 5.2

Proof. Hypotheses: (IC1)–(IC6) at scale k only.

By Theorem 7.8, the RG step adds to the Wilson coefficient β_k a curvature-squared contribution $\Delta\beta_k := \beta_{k+1} - \beta_k$, and the vacuum polarization analysis identifies this shift as

$$\Delta\beta_k = \mathbf{c}_k(g_k).$$

By Theorem 7.9, the induced drift in $g^{-2} = \beta/(2N_c)$ is therefore

$$g_{k+1}^{-2} - g_k^{-2} = \frac{\Delta\beta_k}{2N_c} = \frac{\mathbf{c}_k(g_k)}{2N_c} =: \mathbf{b}_k(g_k). \quad (31)$$

The remainder $\Pi^{(k)^\prime}$ from Theorem 7.8 contributes only to irrelevant local terms and polymers, which do not enter the Wilson projection.

By Theorem 7.11, $\mathbf{b}_k(0) = 2b_0 \ln L_{\text{RG}}$ (the universal one-loop drift in g^{-2}). By Theorem 7.10,

$$|\mathbf{b}_k(g_k) - \mathbf{b}_k(0)| \leq C_{\text{an}} g_k^2.$$

Therefore

$$g_{k+1}^{-2} = g_k^{-2} + \mathbf{b}_k(g_k) = g_k^{-2} + 2b_0 \ln L_{\text{RG}} + \delta_k,$$

with $\delta_k := \mathbf{b}_k(g_k) - 2b_0 \ln L_{\text{RG}}$ satisfying $|\delta_k| \leq C_{\text{run}} g_k^2$, where $C_{\text{run}} := C_{\text{an}}$.

The equivalent statement for $\beta_k = 2N_c g_k^{-2}$ is $\beta_{k+1} = \beta_k + 4N_c b_0 \ln L_{\text{RG}} + 2N_c \delta_k$. \square

7.8 Normalization compatibility

Balaban defines (CMP 109, Eq. (0.2)) the Wilson action $A^\varepsilon(U) = \sum_p \varepsilon^{d-4} [1 - \text{Re tr } U(\partial p)]$ with $\text{tr } \mathbf{1} = 1$ (normalized trace, so $\text{tr } U_p = \frac{1}{N_c} \text{Tr } U_p$). In $d = 4$, his $A(U)$ coincides with our $A(U)$ in (1): both equal $\sum_p (1 - \frac{1}{N_c} \text{Re Tr } U_p)$.

Balaban's effective actions have leading term $g_k^{-2} A(U_k)$ (CMP 109, Eq. (1.3); note: Balaban uses the convention where g_k^{-2} appears directly as the coefficient). His coupling recursion (CMP 109, Eq. (0.20)) is $g_{k+1}^{-2} = g_k^{-2} + \beta_{k+1}^{\text{Bal}}(g_k)$ with $\beta_{k+1}^{\text{Bal}}(0) = 2b_0 \ln L_{\text{RG}}$. This matches our $\mathbf{b}_k(0) = 2b_0 \ln L_{\text{RG}}$ (Theorem 7.11) and hence our (21) exactly: in both cases, \mathbf{b}_k denotes the drift in the variable g^{-2} .

The derived Wilson parameter $\beta_k := 2N_c g_k^{-2}$ satisfies $\beta_{k+1} - \beta_k = 4N_c b_0 \ln L_{\text{RG}} + 2N_c \delta_k$.

Editorial rule. Throughout the paper, the fundamental recursion is stated for g_k^{-2} with drift $2b_0 \ln L_{\text{RG}}$. When β_k is used, we convert by $\beta_k = 2N_c g_k^{-2}$ explicitly.

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