Ricci Curvature of the Orbit Space of Lattice Gauge Theory and Single-Scale Log-Sobolev Inequalities

Lluis Eriksson
Independent Researcher
lluiseriksson@gmail.com

February 2026. Version 3: July 2026.

Abstract

We establish that the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of $\mathrm{SU}(N_c)$ lattice gauge theory satisfies the Riemannian curvature-dimension condition $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$; in particular, it satisfies $CD(N_c/4, \infty)$ in the sense of Lott–Villani–Sturm. The proof shows that the configuration space $\mathcal{A} = \mathrm{SU}(N_c)^{|B_1(\Lambda)|}$, with the bi-invariant product metric $\langle X, Y\rangle = -2\,\mathrm{tr}(XY)$, is Einstein with $\mathrm{Ric}_{\mathcal{A}} = (N_c/4)g_{\mathcal{A}}$ (Proposition 2.2), and applies the stability of RCD^* under quotients by compact groups of measure-preserving isometries (Galaz-García–Kell–Mondino–Sosa). This bypasses O'Neill computations and handles the singular stratum automatically. As a consequence we derive a conditional log-Sobolev inequality for measures $d\mu = e^{-\Phi}d\nu/Z$ with constant $\alpha = (N_c/4)e^{-\operatorname{osc}(\Phi)}$. All constants are computed explicitly for SU(2) and SU(3). This provides the geometric input in a program aiming at a volume-uniform LSI for lattice Yang–Mills at weak coupling; the complementary analytic input is developed in the companion papers cited in §6.1. **Note added (v3).** Version 3 accompanies the paper with a numerical verification suite (full Einstein tensor for SU(2/3/4); the convention triangle $N_c/4 \leftrightarrow N_c/2 \leftrightarrow \frac{1}{2}$ closing the series' Ricci bookkeeping; the horizontal characterization at machine precision; the Bakry–Émery convention on the Gaussian via the exact Gross optimizers; Holley–Stroock in LSI form; and the exact energy/entropy correspondence for a $\mathbb{Z}_2$ quotient toy). It corrects a sign in §5.1 ($[T^a, T^b] = +\varepsilon^{abc}T^c$ requires $T^a = -\frac{i}{2}\sigma^a$), repairs the attributions in §1.4 (the O'Neill-sketch Ricci statement lives in ai.viXra:2602.0036, whose v2 sharpened the $-\,\mathrm{tr}$-convention value to $N_c/2$, consistent with $N_c/4$ here), adds the measured sectional-curvature range of SU(3) to §5.2, and fills in the companion identifiers in §6.1 with their honest status. No statement of v2 is refuted.

1 Introduction

1.1 Context and motivation

The Yang–Mills mass gap problem [1] asks for a rigorous continuum construction with a mass gap. A natural strategy since Wilson [2]: (i) volume-uniform spectral gap on the lattice; (ii) continuum limit; (iii) reconstruction. This paper addresses the geometric foundations of step (i): a self-contained proof that the orbit space has positive Ricci curvature in a synthetic sense, and the resulting LSI for fiber-conditioned measures.

1.2 The orbit space and its geometry

Let $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$, $d \geq 2$, $G = \mathrm{SU}(N_c)$, $N_c \geq 2$;

$$\mathcal{A} = \prod_{b \in B_1(\Lambda)} G, \tag{1}$$

$$(\mathcal{G} \ni g): \; U_b \mapsto g_{b_+} U_b g_{b_-}^{-1}, \tag{2}$$

$$\mathcal{B} = \mathcal{A}/\mathcal{G}. \tag{3}$$

We equip G with

$$\langle X, Y \rangle_G = -2 \operatorname{tr}(XY), \qquad X, Y \in \mathfrak{g} = \mathfrak{su}(N_c), \tag{4}$$

the normalization in which $B(X,Y) = 2N_c \operatorname{tr}(XY)$ and the Einstein constant is $N_c/4$ (Proposition 2.2).

1.3 Main results

Theorem 1.1 (Synthetic Ricci bound for the orbit space)**.** *With G, $\mathcal{A}$, $\mathcal{G}$, $d_{\mathcal{B}}$, $\nu = \pi_\#(\mathrm{Vol}_{\mathcal{A}})$ as above, the metric-measure space $(\mathcal{B}, d_{\mathcal{B}}, \nu)$ satisfies* $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$*; in particular* $CD(N_c/4, \infty)$ *[7, 8].*

Theorem 1.2 (Conditional log-Sobolev inequality)**.** *Let $d\mu = e^{-\Phi} d\nu/Z$ with $\mathrm{osc}(\Phi) < \infty$. Then μ satisfies* $\mathrm{LSI}(\alpha)$ *(Definition 4.1) with*

$$\alpha = \frac{N_c}{4} e^{-\operatorname{osc}(\Phi)}, \tag{5}$$

that is, for all Lipschitz f:

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{\alpha} \int_{\mathcal{B}} |\nabla f|^2 \, d\mu. \tag{6}$$

1.4 Relation to prior work (corrected in v3)

A Ricci curvature bound for the orbit space with a proof sketch based on the O'Neill submersion formula was stated as Theorem 3.1 of the companion ai.viXra:2602.0036 (Paper II of the geometric series); its version 2 sharpened the constant in the $-\operatorname{tr}$ convention to $N_c/2$, which is *exactly consistent* with the $N_c/4$ of the present $-2\operatorname{tr}$ convention (the Einstein constant scales as λ/c under $g \mapsto cg$; Remark 2.3, verified numerically in the suite). The uniform-LSI program that consumes the present result is ai.viXra:2602.0041 (v3), whose conditional fast LSI (Lemma 5.2 therein) is the intended application of Theorem 1.2. (v2 of this paper attributed both items to a single reference [3]; the attributions above are the corrected ones.) Our approach: avoids O'Neill computations via quotient stability [13]; handles the singular stratum automatically; and provides explicit constants.

1.5 Organization

§2: geometric setup and the Einstein property. §3: proof of Theorem 1.1. §4: Theorem 1.2. §5: explicit computations. §6: role in the program.

1.6 Notation and conventions

Orthonormal basis $\{T^a\}$ of $\mathfrak{su}(N_c)$ with $-2\,\mathrm{tr}(T^aT^b) = \delta^{ab}$; real totally antisymmetric structure constants

$$[T^a, T^b] = f^{abc}T^c, \tag{7}$$

$$f^{acd}f^{bcd} = N_c\,\delta^{ab}, \tag{8}$$

verified in §5 and numerically in the suite. No factors of i appear in (7).

2 Riemannian geometry of the orbit space

2.1 The configuration space as a Riemannian manifold

$$T_U\mathcal{A} = \bigoplus_{b\in B_1} T_{U_b}G \cong \bigoplus_{b\in B_1} \mathfrak{g}, \tag{9}$$

$$\langle X, Y\rangle_{\mathcal{A}} = \sum_b \langle X_b, Y_b\rangle_G = -2\sum_b \mathrm{tr}(X_bY_b). \tag{10}$$

2.2 Vertical and horizontal subspaces

The infinitesimal action of $\xi \in \mathfrak{g}^{|\Lambda|}$ generates

$$V_\xi(U)_b = \xi_{b_+} - \mathrm{Ad}_{U_b}(\xi_{b_-}), \tag{11}$$

$$\mathbb{V}_U = \{V_\xi(U) : \xi \in \mathfrak{g}^{|\Lambda|}\} \subset T_U\mathcal{A}, \tag{12}$$

$$\mathbb{H}_U = \mathbb{V}_U^\perp. \tag{13}$$

Lemma 2.1 (Horizontal characterization)**.** *$X \in T_U\mathcal{A}$ is horizontal iff for every $x \in \Lambda$:*

$$\sum_{b\colon b_+=x} X_b - \sum_{b\colon b_-=x} \mathrm{Ad}_{U_b^{-1}}(X_b) = 0. \tag{14}$$

Proof. $X \perp V_\xi$ for all ξ; expanding,

$$\langle X, V_\xi\rangle_{\mathcal{A}} = -2\sum_b \mathrm{tr}\big[X_b(\xi_{b_+} - U_b\xi_{b_-}U_b^{-1})\big] \tag{15}$$

$$= -2\sum_x \mathrm{tr}\Big[\xi_x\Big(\sum_{b_+=x} X_b - \sum_{b_-=x} U_b^{-1}X_bU_b\Big)\Big], \tag{16}$$

which vanishes for all ξ_x iff (14) holds. (v3: the equivalence orthogonality $\Leftrightarrow$ (14) is verified at machine precision on a 2×2 lattice with random U in the suite.) □

2.3 The Einstein property of the configuration space

Proposition 2.2 ($\mathcal{A}$ is Einstein)**.**

$$\mathrm{Ric}_{\mathcal{A}} = \frac{N_c}{4} g_{\mathcal{A}}. \tag{17}$$

Proof. Each factor is compact simple with bi-invariant metric, so

$$\mathrm{Ric}_G(X,Y) = -\frac{1}{4}B(X,Y), \tag{18}$$

and in the orthonormal basis

$$\mathrm{Ric}_G(T^a,T^b) = \frac{1}{4}\sum_c \langle [T^a,T^c],[T^b,T^c]\rangle_G = \frac{1}{4} f^{acd} f^{bcd} = \frac{N_c}{4}\delta^{ab}, \tag{19}$$

by (8). For the product, mixed components vanish and

$$\mathrm{Ric}_{\mathcal{A}}(X,X) = \sum_b \mathrm{Ric}_G(X_b,X_b) = \frac{N_c}{4}|X|^2_{\mathcal{A}}. \tag{20}$$

(v3: the *full* Ricci tensor $\mathrm{Ric}(T^a,T^b) = (N_c/4)\delta^{ab}$ is verified numerically for $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$, together with (8).) □

Remark 2.3 (Metric convention and scaling). Under $g' = cg$ the Ricci $(0,2)$-tensor is unchanged and the Einstein constant scales as $\lambda' = \lambda/c$. With $\langle\cdot,\cdot\rangle' = -N_c \,\mathrm{tr}$ $(= \frac{N_c}{2}\langle\cdot,\cdot\rangle_G)$ the constant is $\frac{1}{2}$; with $-\mathrm{tr}$ $(= \frac{1}{2}\langle\cdot,\cdot\rangle_G)$ it is $\frac{N_c}{2}$ — the value used in ai.viXra:2602.0036 (v2). We adopt $-2\,\mathrm{tr}$ because $N_c/4$ scales linearly with N_c. (v3: the three conventions and the scaling law are verified numerically; this closes the series' Ricci bookkeeping — the $N_c/4$ here is *not* an instance of the erratum corrected in 2602.0036 v2, which concerned the $-\mathrm{tr}$ convention.)

Proposition 2.4. *$\mathcal{G}$ acts on $(\mathcal{A}, g_{\mathcal{A}}, \mathrm{Vol}_{\mathcal{A}})$ by measure-preserving isometries (left and right translations on each bond; bi-invariance).*

3 Proof of Theorem 1.1: synthetic Ricci bound via quotient stability

Theorem 3.1 (Galaz-García–Kell–Mondino–Sosa [13])**.** *If (X,d,m) satisfies $\mathrm{RCD}^*(K,N)$ and a compact group H acts by m-preserving isometries, then $(X/H, d_{X/H}, m_{X/H})$ satisfies $\mathrm{RCD}^*(K,N)$; in particular $CD(K,\infty)$.*

Proof of Theorem 1.1. Step 1: $(\mathcal{A}, d_{\mathcal{A}}, \mathrm{Vol}_{\mathcal{A}})$ is smooth compact with $\mathrm{Ric}_{\mathcal{A}} = (N_c/4)g_{\mathcal{A}}$ (Proposition 2.2), hence $\mathrm{RCD}^*(N_c/4, \dim\mathcal{A})$ [17, 18]. Step 2: $\mathcal{G}$ compact acting by measure-preserving isometries (Proposition 2.4). Step 3: $(\mathcal{B}, d_{\mathcal{B}})$ is a compact geodesic space with $\nu = \pi_\# \mathrm{Vol}_{\mathcal{A}}$; Theorem 3.1 applies. □

Remark 3.2 (The singular stratum). $\mathcal{B}$ is generally not a manifold (reducible connections). The smallest non-central stabilizer type $S(U(1)\times U(N_c-1))$ suggests codimension $\geq 2(N_c-1) \geq 2$; the precise codimension requires an orbit-type analysis. None of this is needed: Theorem 3.1 handles arbitrary compact actions.

Remark 3.3 (Comparison with the O'Neill approach)*.* O'Neill requires the regular stratum, the integrability tensor (the curvature of the mechanical connection — a subtle point), and separate treatment of singularities; the synthetic route avoids all three. Pointwise sharp bounds on $\mathcal{B}_{\mathrm{reg}}$ are not attempted here.

4 The conditional log-Sobolev inequality

4.1 Convention

Definition 4.1 (LSI constant)**.** *μ satisfies* $\mathrm{LSI}(\alpha)$ *if for all Lipschitz f:*

$$\mathrm{Ent}_\mu(f^2) \le \frac{2}{\alpha} \int |\nabla f|^2 \, d\mu. \tag{21}$$

Remark 4.2 (LSI conventions)*.* With (21), Bakry–Émery gives $\alpha = K$; the convention $\mathrm{Ent}(f^2) \le \frac{1}{\rho}\mathcal{E}$ gives $\rho = K/2 = \alpha/2$. (v3: the $\alpha = K$ normalization is verified on the 1D Gaussian with $CD(K,\infty)$ using the *exact* Gross optimizers $f_t = e^{tx/2}$, whose ratio equals K identically — $\mathrm{Ent}(f_t^2) = \frac{t^2}{2K} e^{t^2/2K}$, $\int |f_t'|^2 = \frac{t^2}{4} e^{t^2/2K}$ — and no random trial function violates it; suite.)

4.2 The Bakry–Émery criterion

Theorem 4.3 (Bakry–Émery [9]; synthetic version [15, Thm 5.7.4])**.** *If (X, d, m) satisfies* $\mathrm{RCD}^*(K, N)$ *with $K > 0$ and m a probability measure, then m satisfies* $\mathrm{LSI}(K)$*.*

Remark 4.4. RCD^* (not merely CD) excludes Finslerian pathologies; our $\mathcal{B}$ is a Riemannian quotient, so the hypothesis holds.

Corollary 4.5. *$\bar{\nu} = \nu/\nu(\mathcal{B})$ satisfies* $\mathrm{LSI}(N_c/4)$*:*

$$\mathrm{Ent}_{\bar{\nu}}(f^2) \le \frac{8}{N_c} \int_{\mathcal{B}} |\nabla f|^2 \, d\bar{\nu}. \tag{22}$$

4.3 Holley–Stroock perturbation

Lemma 4.6 (Holley–Stroock [10])**.** *If ν satisfies* $\mathrm{LSI}(\alpha_0)$ *and $d\mu = e^{-\Phi} d\nu / Z$ with* $\mathrm{osc}(\Phi) < \infty$*, then μ satisfies* $\mathrm{LSI}(\alpha_0 e^{-\mathrm{osc}(\Phi)})$*. (v3: verified in LSI form on the discretized circle with directly minimized constants; suite.)*

Proof of Theorem 1.2. Apply Lemma 4.6 to Corollary 4.5:

$$\alpha = \frac{N_c}{4} e^{-\mathrm{osc}(\Phi)}. \tag{23}$$

□

5 Explicit computations

5.1 SU(2)

$G = \mathrm{SU}(2) \cong S^3$, $\dim G = 3$. *Orthonormal basis (sign corrected in v3):* $T^a = -\frac{i}{2}\sigma^a$ satisfies $\langle T^a, T^b\rangle_G = \delta^{ab}$ and $[T^a, T^b] = +\varepsilon^{abc}T^c$, confirming (8) for $N_c = 2$: $\varepsilon^{acd}\varepsilon^{bcd} = 2\delta^{ab}$. (v2 wrote $T^a = +\frac{i}{2}\sigma^a$, for which $[T^a, T^b] = -\varepsilon^{abc}T^c$; the sign is irrelevant for f^2 and for every theorem, but the stated basis now matches (7); verified numerically.)

Curvature. $K_G(T^a, T^b) = \frac{1}{4}\|[T^a, T^b]\|^2 = \frac{1}{4}$ constant (round S^3 of radius 2; constancy verified over 200 random planes to 10^{-10}); $\mathrm{Ric}_G = (3-1)\cdot\frac{1}{4} = \frac{1}{2} = N_c/4$; radius 2; diameter 2π.

LSI constants. $\alpha_0 = \frac{1}{2}$; $\alpha = \frac{1}{2}e^{-\operatorname{osc}(\Phi)}$.

5.2 SU(3)

$\dim G = 8$. With the Gell-Mann matrices, the orthonormal basis is

$$T^a = -\frac{i}{2}\lambda^a \in \mathfrak{su}(3) \tag{24}$$

(sign as in §5.1). SU(3) is not a space form: sectional curvature vanishes on Cartan planes and is maximal on $\mathfrak{su}(2)$-subalgebra planes. *Quantitative range (new in v3, measured):* $K = 0$ exactly on the Cartan plane (λ_3, λ_8); $K = \frac{1}{4}$ on root $\mathfrak{su}(2)$ planes; sampled range over 400 random planes $[0.02, 0.21] \subset [0, \frac{1}{4}]$. Ricci: by (8) with $N_c = 3$, $\mathrm{Ric}_G = \frac{3}{4}g_G$ (Einstein, verified as a full tensor numerically). *LSI constants.* $\alpha_0 = \frac{3}{4}$; $\alpha = \frac{3}{4}e^{-\operatorname{osc}(\Phi)}$.

5.3 General SU(Nc): summary table

N_c	$\dim G$	$\mathrm{Ric}_{\mathcal{A}}$ (Einstein)	Synthetic bound on $\mathcal{B}$	LSI α_0
2	3	1/2	$\mathrm{RCD}^*(1/2, 3\lvert B_1\rvert)$	1/2
3	8	3/4	$\mathrm{RCD}^*(3/4, 8\lvert B_1\rvert)$	3/4
N_c	N_c^2-1	$N_c/4$	$\mathrm{RCD}^*(N_c/4, (N_c^2-1)\lvert B_1\rvert)$	$N_c/4$

All constants grow linearly with N_c.

6 Discussion and role in the mass gap program

6.1 How this result is used (identifiers added in v3)

Theorem 1.2 provides the single-scale LSI for one fiber of the multiscale decomposition used in Balaban's RG [11, 12]. The program requires: (1) the geometric input (this paper); (2) the analytic input — volume-uniform bounds on $\operatorname{osc}(\Phi)$ for the effective potentials. (v3) The analytic input and the assembly are developed in ai.viXra:2602.0041 (v3) and its companions ai.viXra:2602.0057/0056/0055/0054; per the conventions of this series, those papers' discharges are *asserted* there and are not re-verified here; 2602.0041 v3 states its remaining hypotheses explicitly ((H-XSD), (H-DOB)). If the analytic input is established, Theorem 1.2 yields a volume-uniform LSI at weak coupling; the passage to a mass gap requires the further steps (hypercontractivity, clustering, reflection positivity) discussed there.

6.2 Epistemic honesty

As in v2, we record what this paper does not establish: we do not prove the mass gap; the bound $N_c/4$ is likely not sharp on $\mathcal{B}$ (the quotient only gains curvature); the dimension parameter $\dim \mathcal{A}$ is not optimal (the effective dimension is $\dim \mathcal{A} - \dim \mathcal{G}$); pure gauge theory only. (v3 adds: all statements of this paper survived the numerical audit — the suite validates rather than refutes; the only corrections are the §5.1 sign, the §1.4 attributions, and the §6.1 identifiers.)

Note added in v3

Prepared with the companion suite `verify_2602_0046.py` (repository `THE-ERIKSSON-PROGRAMME`, `verification/2602-0046/`; 10 checks, all passing, ∼7 s, NumPy/SciPy): (1) full Einstein tensor $(N_c/4)\delta^{ab}$ for SU(2/3/4) and Casimir (8); (2) convention triangle $N_c/4$ $(-2\,\mathrm{tr}) \leftrightarrow N_c/2$ $(-\,\mathrm{tr}$, ai.viXra:2602.0036 v2) $\leftrightarrow \frac{1}{2}$ $(-N_c\,\mathrm{tr})$ with the scaling law $\lambda' = \lambda/c$; (3) the §5.1 sign ($f^{123} = -1$ for $T = +\frac{i}{2}\sigma$, $+1$ for $T = -\frac{i}{2}\sigma$); (4) SU(2) round ($K \equiv \frac{1}{4}$), SU(3) sectional range $[0, \frac{1}{4}]$ with exact endpoints; (5) Lemma 2.1 at machine precision; (6) the Bakry–Émery convention via exact Gross optimizers; (7) Holley–Stroock in LSI form; (8) the exact energy/entropy correspondence for the $S^1/\mathbb{Z}_2$ quotient (the finite-dimensional shadow of Theorem 3.1). Corrections: §5.1 basis sign; §1.4 attributions (O'Neill sketch = ai.viXra:2602.0036, conditional fast LSI = ai.viXra:2602.0041 v3, Lemma 5.2); §6.1 identifiers with honest status. All v2 numbered statements and equations keep their numbers; no statement of v2 is refuted.

References

[1] A. Jaffe and E. Witten, *Quantum Yang–Mills theory*, in: The Millennium Prize Problems, CMI & AMS (2006) 129–152.

[2] K. G. Wilson, Phys. Rev. D **10** (1974) 2445.

[3] L. Eriksson, *Uniform log-Sobolev inequality and mass gap for lattice Yang–Mills theory*, ai.viXra:2602.0041 (v3), 2026.

[4] L. Eriksson, *Geodesic convexity and structural limits of curvature methods for the Yang–Mills mass gap on the lattice*, ai.viXra:2602.0036 (v2), 2026.

[5] B. O'Neill, Michigan Math. J. **13** (1966) 459–469.

[6] J. Milnor, Advances in Math. **21** (1976) 293–329.

[7] J. Lott and C. Villani, Ann. of Math. **169** (2009) 903–991.

[8] K.-T. Sturm, Acta Math. **196** (2006) 65–131, 133–177.

[9] D. Bakry and M. Émery, Sém. Prob. XIX, LNM 1123, Springer (1985) 177–206.

[10] R. Holley and D. Stroock, J. Statist. Phys. **46** (1987) 1159–1194.

[11] T. Balaban, Comm. Math. Phys. **95** (1984) 17–40.

[12] T. Balaban, Comm. Math. Phys. **122** (1989) 355–392.

[13] F. Galaz-García, M. Kell, A. Mondino, G. Sosa, J. Funct. Anal. **275** (2018) 1368–1446.

[14] A. L. Besse, *Einstein Manifolds*, Springer (1987).

[15] D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer (2014).

[16] C. Villani, *Optimal Transport: Old and New*, Springer (2009).

[17] L. Ambrosio, N. Gigli, G. Savaré, Duke Math. J. **163** (2014) 1405–1490.

[18] M. Erbar, K. Kuwada, K.-T. Sturm, Invent. Math. **201** (2015) 993–1071.