Ricci Curvature of the Orbit Space of Lattice Gauge Theory
and Single-Scale Log-Sobolev Inequalities

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Abstract

We establish that the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of $\mathrm{SU}(N_c)$ lattice gauge theory satisfies the Riemannian curvature-dimension condition $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$; in particular, it satisfies $\mathrm{CD}(N_c/4, \infty)$ in the sense of Lott–Villani–Sturm. The proof proceeds by showing that the configuration space $\mathcal{A} = \mathrm{SU}(N_c)^{|\mathcal{B}_1(\Lambda)|}$, equipped with the bi-invariant product metric $\langle X, Y \rangle = -2\,\mathrm{tr}(XY)$, is an Einstein manifold with $\mathrm{Ric}_{\mathcal{A}} = (N_c/4)\, g_{\mathcal{A}}$ (see Proposition 2.2), and then applying the stability of the RCD^* condition under quotients by compact groups of measure-preserving isometries (Galaz-García–Kell–Mondino–Sosa). This approach bypasses the need for explicit O'Neill curvature computations and handles the singular stratum (reducible connections) automatically. As a consequence, we derive a conditional log-Sobolev inequality for measures on $\mathcal{B}$ of the form $d\mu = e^{-\Phi}\, d\nu/Z$ with constant $\alpha = (N_c/4)\, e^{-\operatorname{osc}(\Phi)}$. All constants are computed explicitly for SU(2) and SU(3). This provides the geometric input in a program aiming at a volume-uniform log-Sobolev inequality for $\mathrm{SU}(N_c)$ lattice Yang–Mills theory at weak coupling; the complementary analytic input (uniform bounds on the effective potential oscillation, via Balaban's renormalization group) is the subject of ongoing work.

Contents

1 Introduction

1.1 Context and motivation

The Yang–Mills mass gap problem, as formulated by Jaffe and Witten [1], asks for the rigorous construction of a quantum Yang–Mills theory in four-dimensional Minkowski space-time satisfying the Wightman axioms, together with the proof that the mass operator has a strictly positive lower bound (mass gap). This is one of the seven Millennium Prize Problems of the Clay Mathematics Institute.

A natural strategy, pursued in various forms since the pioneering work of Wilson [2], is to:

(i) establish a spectral gap (mass gap) for the lattice-regularized theory, uniformly in the lattice volume;

(ii) take the continuum limit and verify that the gap persists;

(iii) reconstruct the continuum theory in Wightman axioms.

The present paper addresses the geometric foundations needed for step (i). Specifically, we provide a complete, self-contained proof that the orbit space of lattice gauge theory has positive Ricci curvature in a precise synthetic sense, and we derive the resulting log-Sobolev inequality for fiber-conditioned measures.

1.2 The orbit space and its geometry

Let $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ be a finite periodic lattice with $d \geq 2$, and let $G = \mathrm{SU}(N_c)$ with $N_c \geq 2$. The configuration space of lattice gauge theory is

$$\mathcal{A} = \prod_{b \in \mathcal{B}_1(\Lambda)} G \,, \tag{1}$$

where $\mathcal{B}_1(\Lambda)$ denotes the set of oriented bonds (edges) of Λ. An element $U = (U_b)_{b \in \mathcal{B}_1} \in \mathcal{A}$ assigns a group element $U_b \in G$ to each bond b.

The gauge group acts on $\mathcal{A}$ by

$$(\mathcal{G} \ni g) : U_b \longmapsto g_{b_+} U_b \, g_{b_-}^{-1} , \tag{2}$$

where $b_\pm$ denote the endpoints of the bond b. The orbit space is the quotient

$$\mathcal{B} = \mathcal{A}/\mathcal{G} . \tag{3}$$

We equip G with the bi-invariant Riemannian metric

$$\langle X, Y \rangle_G = -2 \, \mathrm{tr}(XY) , \qquad X, Y \in \mathfrak{g} = \mathfrak{su}(N_c) , \tag{4}$$

where tr denotes the trace in the fundamental representation and X, Y are anti-Hermitian. This is the standard normalization in which the Killing form satisfies $B(X,Y) = 2N_c \, \mathrm{tr}(XY)$ and the Einstein constant is $N_c/4$ (see Proposition 2.2). The metric (4) induces the product metric on $\mathcal{A}$ and, via the quotient map $\pi : \mathcal{A} \to \mathcal{B}$, a distance on $\mathcal{B}$.

1.3 Main results

Our main results are the following.

Theorem 1.1 (Synthetic Ricci bound for the orbit space). *Let $G = \mathrm{SU}(N_c)$ with the metric (4), $\mathcal{A} = G^{|\mathcal{B}_1(\Lambda)|}$ the configuration space with product metric, $\mathcal{G} = G^{|\Lambda|}$ the gauge group acting by (2), $d_{\mathcal{B}}$ the quotient distance on $\mathcal{B} = \mathcal{A}/\mathcal{G}$, and $\nu = \pi_\#(\mathrm{Vol}_{\mathcal{A}})$ the pushforward of Riemannian volume. Then the metric-measure space $(\mathcal{B}, d_{\mathcal{B}}, \nu)$ satisfies the Riemannian curvature-dimension condition $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ (in the sense of the RCD theory; see [20] and the refinement to RCD^* in [21]); in particular, it satisfies $\mathrm{CD}(N_c/4, \infty)$ in the sense of Lott–Villani [6] and Sturm [7].*

Theorem 1.2 (Conditional log-Sobolev inequality). *Let ν be as in Theorem 1.1 and let $d\mu = e^{-\Phi} \, d\nu / Z$ with $\Phi : \mathcal{B} \to \mathbb{R}$ measurable and $\mathrm{osc}(\Phi) := \sup \Phi - \inf \Phi < \infty$. Then μ satisfies the log-Sobolev inequality $\mathrm{LSI}(\alpha)$ (Definition 4.1) with*

$$\alpha = \frac{N_c}{4} \, e^{-\,\mathrm{osc}(\Phi)} . \tag{5}$$

That is, for all Lipschitz $f : \mathcal{B} \to \mathbb{R}$:

$$\mathrm{Ent}_\mu(f^2) \le \frac{2}{\alpha} \int_{\mathcal{B}} |\nabla f|^2 \, d\mu . \tag{6}$$

1.4 Relation to prior work

A Ricci curvature bound $\mathrm{Ric}_{\mathcal{B}} \ge N_c/4$ was stated as Theorem 3.1 in Eriksson [3], with a proof sketch based on the O'Neill submersion formula. The present paper establishes a stronger result — the full synthetic $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ condition on the orbit space as a metric-measure space — using a different and more robust method. Our approach:

- Avoids explicit O'Neill curvature computations entirely, instead using the stability of synthetic Ricci bounds under quotients by compact groups of measure-preserving isometries [16].

- Handles the singular stratum $\mathcal{B}_{\text{sing}}$ (reducible connections) automatically, since the quotient theorem of [16] applies to arbitrary isometric group actions, including non-free ones.

- Provides explicit curvature and LSI constants for SU(2) and SU(3).

We note that the O'Neill approach sketched in [3] requires careful treatment of the Lie bracket of horizontal extensions (which is controlled by the curvature of the mechanical connection, not by the componentwise Lie bracket) and of the non-totally-geodesic nature of the gauge orbits. The synthetic approach adopted here sidesteps both issues.

Theorem 1.2 follows from Theorem 1.1 via the Bakry–Émery criterion combined with the Holley–Stroock perturbation lemma. This conditional LSI is Lemma 5.2 of [3], here proven unconditionally with explicit constants.

1.5 Organization

Section 2 establishes the geometric setup: the configuration space, gauge action, vertical/horizontal decomposition, and the Einstein property of $\mathcal{A}$. Section 3 proves Theorem 1.1 using the stability of synthetic Ricci bounds under isometric quotients. Remark 3.2 provides supplementary geometric information on the singular stratum (not required for the main theorem, but useful for context). Section 4 derives Theorem 1.2. Section 5 contains the explicit computations for SU(2) and SU(3). Section 6 discusses the role of these results in the broader program toward the Yang–Mills mass gap.

1.6 Notation and conventions

Throughout, $G = \mathrm{SU}(N_c)$, $\mathfrak{g} = \mathfrak{su}(N_c)$, $\dim G = N_c^2 - 1$. We use the inner product (4) on $\mathfrak{g}$. We work with an orthonormal basis $\{T^a\}_{a=1}^{N_c^2-1}$ of $\mathfrak{g} = \mathfrak{su}(N_c)$ consisting of *anti-Hermitian* matrices, satisfying $-2\,\mathrm{tr}(T^aT^b) = \delta^{ab}$. The (real, totally antisymmetric) structure constants f^{abc} are defined by

$$[T^a, T^b] = f^{abc}T^c\,. \tag{7}$$

The quadratic Casimir in the adjoint representation satisfies

$$f^{acd}f^{bcd} = N_c\,\delta^{ab} \tag{8}$$

with our normalization (4); this is verified in §5.1 and §5.2. For SU(2): $f^{abc} = \varepsilon^{abc}$. No factors of i appear in (7) since T^a are anti-Hermitian.

2 Riemannian geometry of the orbit space

2.1 The configuration space as a Riemannian manifold

The configuration space $\mathcal{A} = G^{|\mathcal{B}_1|}$ inherits the product Riemannian structure from $(G, \langle\cdot,\cdot\rangle_G)$. At a point $U = (U_b) \in \mathcal{A}$, the tangent space is

$$T_U\mathcal{A} = \bigoplus_{b\in\mathcal{B}_1} T_{U_b}G \cong \bigoplus_{b\in\mathcal{B}_1} \mathfrak{g}\,, \tag{9}$$

where we use left-trivialization: $T_{U_b}G \ni \dot{U}_b = X_bU_b$ with $X_b \in \mathfrak{g}$. The metric on $\mathcal{A}$ is

$$\langle X, Y\rangle_{\mathcal{A}} = \sum_{b\in\mathcal{B}_1} \langle X_b, Y_b\rangle_G = -2\sum_{b\in\mathcal{B}_1} \mathrm{tr}(X_bY_b)\,. \tag{10}$$

2.2 Vertical and horizontal subspaces

The gauge group $\mathcal{G} = G^{|\Lambda|}$ acts on $\mathcal{A}$ by (2). The infinitesimal action of $\xi = (\xi_x)_{x\in\Lambda} \in \mathrm{Lie}(\mathcal{G}) = \mathfrak{g}^{|\Lambda|}$ generates the vertical vector field

$$V_\xi(U)_b = \xi_{b_+} - \mathrm{Ad}_{U_b}(\xi_{b_-})\,, \tag{11}$$

where $\mathrm{Ad}_{U_b}(\eta) = U_b \eta U_b^{-1}$ is the adjoint action. The vertical subspace at U is

$$_U = \{V_\xi(U) : \xi \in \mathfrak{g}^{|\Lambda|}\} \subset T_U\mathcal{A}\,. \tag{12}$$

The horizontal subspace is the orthogonal complement with respect to (10):

$$\mathcal{H}_U = {}^{\perp}_{U}\ . \tag{13}$$

Lemma 2.1 (Horizontal characterization)**.** *A vector $X = (X_b)_{b\in\mathcal{B}_1} \in T_U\mathcal{A}$ is horizontal if and only if it satisfies the lattice Coulomb condition: for every $x \in \Lambda$,*

$$\sum_{b:\, b_+=x} X_b - \sum_{b:\, b_-=x} \mathrm{Ad}_{U_b^{-1}}(X_b) = 0\,. \tag{14}$$

Proof. The vector X is horizontal iff $\langle X, V_\xi\rangle_{\mathcal{A}} = 0$ for all $\xi \in \mathfrak{g}^{|\Lambda|}$. Using (11) and (10):

$$\langle X, V_\xi\rangle_{\mathcal{A}} = -2\sum_b \mathrm{tr}\big(X_b(\xi_{b_+} - U_b\xi_{b_-}U_b^{-1})\big) \tag{15}$$

$$= -2\sum_x \mathrm{tr}\Big(\xi_x\Big[\sum_{b:\, b_+=x} X_b - \sum_{b:\, b_-=x} U_b^{-1}X_bU_b\Big]\Big)\,. \tag{16}$$

Since this must vanish for all ξ_x, the bracketed expression must vanish for each x, giving (14) after noting that $U_b^{-1}X_bU_b = \mathrm{Ad}_{U_b^{-1}}(X_b)$. □

2.3 The Einstein property of the configuration space

Proposition 2.2 ($\mathcal{A}$ is Einstein)**.** *The configuration space $(\mathcal{A}, g_{\mathcal{A}})$ is an Einstein manifold:*

$$\mathrm{Ric}_{\mathcal{A}} = \frac{N_c}{4}\, g_{\mathcal{A}}\,. \tag{17}$$

Proof. Each factor $(G, \langle\cdot,\cdot\rangle_G)$ is a compact simple Lie group with bi-invariant metric. For such groups, the Ricci tensor is (see e.g. [5, Proposition 7.4] or [17, Proposition 9.14]):

$$\mathrm{Ric}_G(X,Y) = -\frac{1}{4}B(X,Y)\,, \tag{18}$$

where B is the Killing form. With our normalization $\langle X,Y\rangle_G = -2\ \mathrm{tr}(XY)$ (4) and an orthonormal basis $\{T^a\}$ of $\mathfrak{g}$ satisfying $-2\ \mathrm{tr}(T^aT^b) = \delta^{ab}$:

$$\mathrm{Ric}_G(T^a,T^b) = \frac{1}{4}\sum_c \langle [T^a,T^c],\, [T^b,T^c]\rangle_G = \frac{1}{4} f^{acd}f^{bcd} = \frac{N_c}{4}\delta^{ab}\,, \tag{19}$$

where the first equality is the standard formula for the Ricci tensor of a compact Lie group with bi-invariant metric (see e.g. [5, Proposition 7.4]; in particular, for any fixed bi-invariant metric, if $\{T^a\}$ is orthonormal with respect to that metric then $\mathrm{Ric}_G(T^a,T^b) =$

$\frac{1}{4}f^{acd}f^{bcd}$), the second uses the expansion in structure constants with respect to the $\{T^a\}$ orthonormal basis, and the third applies the quadratic Casimir identity (8). Thus $\mathrm{Ric}_G = \frac{N_c}{4}\, g_G$.

For the product $\mathcal{A} = G^{|\mathcal{B}_1|}$, the Ricci tensor is the direct sum of factor Ricci tensors (mixed components vanish identically). Since all factors have the same Einstein constant:

$$\mathrm{Ric}_{\mathcal{A}}(X,X) = \sum_{b\in\mathcal{B}_1} \mathrm{Ric}_G(X_b, X_b) = \frac{N_c}{4}\sum_b |X_b|_G^2 = \frac{N_c}{4}|X|_{\mathcal{A}}^2\,. \tag{20}$$

□

Remark 2.3 (Metric convention and scaling)**.** Some references use the metric $\langle X,Y\rangle' = -N_c\ \mathrm{tr}(XY) = \frac{N_c}{2}\,\langle X,Y\rangle_G$. Under rescaling $g' = c\,g$ with $c > 0$, the Ricci tensor (as a $(0,2)$-tensor) is unchanged: $\mathrm{Ric}_{g'} = \mathrm{Ric}_g$. However, the Einstein equation $\mathrm{Ric}_g = \lambda\, g$ transforms to $\mathrm{Ric}_{g'} = (\lambda/c)\, g'$, so the Einstein constant scales as $\lambda' = \lambda/c$. With the convention $\langle\cdot,\cdot\rangle' = (N_c/2)\,\langle\cdot,\cdot\rangle_G$, the Einstein constant becomes $(N_c/4)/(N_c/2) = 1/2$, independent of N_c. We adopt $\langle X,Y\rangle_G = -2\ \mathrm{tr}(XY)$ throughout because the resulting constant $N_c/4$ scales linearly with N_c, which is favorable for the large-N_c limit and matches the convention used in [3].

2.4 The gauge action is by measure-preserving isometries

Proposition 2.4. *The gauge group $\mathcal{G} = G^{|\Lambda|}$ acts on $(\mathcal{A}, g_{\mathcal{A}}, \mathrm{Vol}_{\mathcal{A}})$ by measure-preserving isometries.*

Proof. The gauge action (2) on each bond is $U_b \mapsto g_{b_+}U_b g_{b_-}^{-1}$, which is the composition of a left translation by g_{b_+} and a right translation by $g_{b_-}^{-1}$. Both are isometries of $(G, \langle\cdot,\cdot\rangle_G)$ since the metric is bi-invariant. The product of isometries is an isometry of $(\mathcal{A}, g_{\mathcal{A}})$. Since isometries of compact Riemannian manifolds preserve the Riemannian volume, $\mathcal{G}$ acts by measure-preserving isometries. □

3 Proof of Theorem 1.1: synthetic Ricci bound via quotient stability

The key tool is the following recent result on the stability of synthetic Ricci bounds under quotients.

Theorem 3.1 (Galaz-García–Kell–Mondino–Sosa [16])**.** *Let $(X, d, \mathfrak{m})$ be a metric-measure space satisfying* $\mathrm{RCD}^*(K,N)$ *for some $K \in \mathbb{R}$ and $N \in [1,\infty)$. Let H be a compact group acting on X by $\mathfrak{m}$-preserving isometries. Then the quotient space $(X/H,\ d_{X/H},\ \mathfrak{m}_{X/H})$, equipped with the quotient distance and pushforward measure, satisfies* $\mathrm{RCD}^*(K,N)$*. In particular, it satisfies* $\mathrm{CD}(K,\infty)$.

Proof of Theorem 1.1. We verify the hypotheses of Theorem 3.1.

Step 1. The configuration space $(\mathcal{A}, g_{\mathcal{A}}) = (G^{|\mathcal{B}_1|}, g_{\mathrm{prod}})$ is a smooth, compact Riemannian manifold with $\mathrm{Ric}_{\mathcal{A}} = \frac{N_c}{4}\, g_{\mathcal{A}}$ (Proposition 2.2). Since $\mathcal{A}$ is compact, $(\mathcal{A}, d_{\mathcal{A}})$ is complete and geodesic. On a smooth complete n-dimensional Riemannian manifold, the lower Ricci bound $\mathrm{Ric} \geq K\,g$ implies the $\mathrm{RCD}^*(K,n)$ condition for the metric-measure space (M, d_g, Vol_g); this is by now standard in the synthetic geometry literature, see e.g. [21, 20]. Hence $(\mathcal{A}, d_{\mathcal{A}}, \mathrm{Vol}_{\mathcal{A}})$ satisfies $\mathrm{RCD}^*(N_c/4, \dim\mathcal{A})$.

Step 2. The gauge group $\mathcal{G}$ is compact (finite product of compact groups) and acts on $\mathcal{A}$ by measure-preserving isometries (Proposition 2.4).

Step 3. Since $\mathcal{A}$ is compact and $\mathcal{G}$ acts by isometries, the quotient $(\mathcal{B}, d_{\mathcal{B}})$ endowed with the standard metric quotient distance is a compact (hence complete and proper) length space, and therefore a geodesic metric space. Moreover, $\nu = \pi_{\#} \operatorname{Vol}_{\mathcal{A}}$ is the canonical pushforward reference measure. All hypotheses of Theorem 3.1 are satisfied, so $(\mathcal{B}, d_{\mathcal{B}}, \nu)$ satisfies $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$, and hence $\mathrm{CD}(N_c/4, \infty)$. □

Remark 3.2 (The singular stratum)**.** The quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is generally not a smooth manifold due to the presence of reducible connections (those with stabilizer strictly larger than the center $Z(G)$). For orientation, we note that the smallest non-central stabilizer type in $\mathrm{SU}(N_c)$ is conjugate to $S(U(1)\times U(N_c-1))$, whose dimension is $1+(N_c-1)^2-1 = (N_c-1)^2$. Since $\dim \mathrm{SU}(N_c) = N_c^2 - 1$, this suggests a codimension of at least $(N_c^2-1)-(N_c-1)^2 = 2(N_c-1) \geq 2$ for the singular stratum; the precise codimension in the quotient of the full configuration space depends on the lattice topology and requires a more detailed orbit-type analysis. In any case, this codimension estimate is *not needed* for Theorem 1.1: the quotient theorem (Theorem 3.1) applies to arbitrary compact group actions, including non-free ones, and handles singularities automatically through the synthetic framework.

Remark 3.3 (Comparison with the O'Neill approach)**.** The O'Neill submersion formula [4] provides an alternative route to Ricci bounds on quotients, but requires: (a) restriction to the regular stratum, (b) computation of the integrability tensor A (involving the curvature of the mechanical connection, *not* the componentwise Lie bracket — a subtle point), and (c) separate treatment of the singular stratum via codimension estimates. The synthetic approach via Theorem 3.1 avoids all three complications.

An O'Neill-type analysis on $\mathcal{B}_{\mathrm{reg}}$ can in principle yield pointwise lower bounds on curvature quantities, but obtaining a sharp pointwise Ricci bound requires careful control of the O'Neill tensors (in particular A) and of the horizontal distribution. We do not pursue such pointwise estimates here. For the application to log-Sobolev inequalities, the synthetic RCD^* framework is natural and sufficient.

4 The conditional log-Sobolev inequality

4.1 Convention

Definition 4.1 (LSI constant)**.** A probability measure μ on a metric-measure space satisfies $\mathrm{LSI}(\alpha)$ with $\alpha > 0$ if for all Lipschitz functions f:

$$\operatorname{Ent}_{\mu}(f^2) \leq \frac{2}{\alpha} \int |\nabla f|^2 \, d\mu \,, \tag{21}$$

where $\operatorname{Ent}_{\mu}(g) = \int g \log g \, d\mu - (\int g \, d\mu) \log(\int g \, d\mu)$, and $|\nabla f|$ denotes the minimal weak upper gradient of f (which coincides with the Riemannian gradient norm on smooth strata).

Remark 4.2 (LSI conventions)**.** Let $K > 0$ be the $\mathrm{CD}(K, \infty)$ lower bound.

(i) With our convention (21), the Bakry–Émery theorem gives $\alpha = K$.

(ii) Some references define $\mathrm{LSI}(\rho)$ via $\operatorname{Ent}(f^2) \leq \frac{1}{\rho}\mathcal{E}(f, f)$, giving $\rho = K/2$.

(iii) The two conventions are related by $\rho = \alpha/2$, equivalently $\frac{1}{\rho} = \frac{2}{\alpha}$.

When interfacing with other works, the reader should verify which convention is in use.

4.2 The Bakry–Émery criterion

Theorem 4.3 (Bakry–Émery [8]; synthetic version [18, Theorem 5.7.4])**.** *If* $(X, d, \mathfrak{m})$ *is a metric-measure space satisfying* $\mathrm{RCD}^*(K, N)$ *for some* $N \in [1, \infty)$ *with* $K > 0$*, and* $\mathfrak{m}$ *is a probability measure, then* $\mathfrak{m}$ *satisfies* $\mathrm{LSI}(K)$ *in the sense of Definition 4.1.*

Remark 4.4. The hypothesis RCD^* (rather than merely CD) is needed to exclude non-Riemannian (Finslerian) spaces where the equivalence between curvature bounds and functional inequalities may fail; see [19] for background on the interplay between synthetic curvature conditions and functional inequalities. Since our orbit space arises as the quotient of a Riemannian manifold by isometries, it satisfies $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ by Theorem 3.1, and the hypothesis is fulfilled.

Corollary 4.5. *Let* $\bar{\nu} := \nu/\nu(\mathcal{B})$ *be the normalization of* ν *to a probability measure. Then* $\bar{\nu}$ *satisfies* $\mathrm{LSI}(N_c/4)$*:*

$$\mathrm{Ent}_{\bar{\nu}}(f^2) \leq \frac{8}{N_c} \int_{\mathcal{B}} |\nabla f|^2 \, d\bar{\nu} \,. \tag{22}$$

Proof. The space $(\mathcal{B}, d_{\mathcal{B}}, \nu)$ satisfies $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ by Theorem 1.1. Multiplying the reference measure by a positive constant does not affect the RCD^* condition (the defining entropy convexity is scale-invariant), so $(\mathcal{B}, d_{\mathcal{B}}, \bar{\nu})$ satisfies $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ as well. Apply Theorem 4.3 with $K = N_c/4$. □

4.3 Holley–Stroock perturbation

Lemma 4.6 (Holley–Stroock [9])**.** *If* ν *satisfies* $\mathrm{LSI}(\alpha_0)$ *and* $d\mu = e^{-\Phi}\, d\nu/Z$ *with* $\mathrm{osc}(\Phi) = \sup \Phi - \inf \Phi < \infty$*, then* μ *satisfies* $\mathrm{LSI}(\alpha)$ *with* $\alpha = \alpha_0\, e^{-\,\mathrm{osc}(\Phi)}$.

Proof of Theorem 1.2. Note that replacing ν by its normalization $\bar{\nu}$ only changes the normalizing constant Z in $d\mu = e^{-\Phi}\, d\nu/Z$; the resulting probability measure μ is the same. Apply Lemma 4.6 to Corollary 4.5 with $\bar{\nu}$ in the role of ν:

$$\alpha = \frac{N_c}{4}\, e^{-\,\mathrm{osc}(\Phi)} \,. \tag{23}$$

□

5 Explicit computations

5.1 SU(2)

For $N_c = 2$: $G = \mathrm{SU}(2) \cong S^3$, $\dim G = 3$, $\mathfrak{g} = \mathfrak{su}(2)$.

Orthonormal basis. Let σ^a $(a = 1, 2, 3)$ be the Pauli matrices. Then $T^a = \frac{i}{2}\sigma^a \in \mathfrak{su}(2)$ satisfies $\langle T^a, T^b \rangle_G = -2\,\mathrm{tr}(T^a T^b) = \delta^{ab}$ (since $\mathrm{tr}(\frac{i}{2}\sigma^a \cdot \frac{i}{2}\sigma^b) = -\frac{1}{4}\,\mathrm{tr}(\sigma^a\sigma^b) = -\frac{1}{2}\delta^{ab}$) and $[T^a, T^b] = \varepsilon^{abc} T^c$, confirming (8) for $N_c = 2$: $f^{acd} f^{bcd} = \varepsilon^{acd}\varepsilon^{bcd} = 2\,\delta^{ab}$.

Curvature. Since SU(2) is a round 3-sphere:

- Sectional curvature: for orthonormal T^a, T^b, $K_G(T^a, T^b) = \frac{1}{4}|[T^a, T^b]|^2 = 1/4$ (constant, since SU(2) with this metric is isometric to the round S^3 of radius 2).
- Ricci curvature: $\mathrm{Ric}_G = (3 - 1) \times \frac{1}{4} = \frac{1}{2}$. Agrees with $N_c/4 = 1/2$.
- Radius: $K = 1/r^2 = 1/4$ gives $r = 2$.
- Diameter: $\mathrm{diam} = \pi r = 2\pi$.

LSI constants. $\alpha_0 = N_c/4 = 1/2$; $\quad \alpha = \frac{1}{2} e^{-\,\mathrm{osc}(\Phi)}$.

5.2 SU(3)

For $N_c = 3$: $\dim G = 8$, $\mathfrak{g} = \mathfrak{su}(3)$.

Orthonormal basis. Let λ^a ($a = 1, \dots, 8$) be the Gell-Mann matrices with $\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. Then

$$T^a = \frac{i}{2}\,\lambda^a \in \mathfrak{su}(3) \tag{24}$$

satisfies $\langle T^a, T^b \rangle_G = -2\ \mathrm{tr}(T^a T^b) = -2 \cdot (-\frac{1}{4})\,\mathrm{tr}(\lambda^a \lambda^b) = -2 \cdot (-\frac{1}{4}) \cdot 2\delta^{ab} = \delta^{ab}$.

Curvature. SU(3) is *not* a space form; sectional curvature varies (it vanishes for commuting elements and is maximal for subalgebras isomorphic to $\mathfrak{su}(2)$). However:

- Ricci curvature: By the Casimir identity (8) with $N_c = 3$: $f^{acd} f^{bcd} = 3\,\delta^{ab}$, so $\mathrm{Ric}_G = \frac{3}{4}\,g_G$.
- SU(3) is Einstein with constant 3/4.
- *Verification:* the Casimir $f^{acd} f^{bcd}$ for SU(3) with the Gell-Mann basis $T^a = (i/2)\lambda^a$ can be checked entry-by-entry; the nonzero structure constants yield $\sum_{c,d}(f^{acd})^2 = 3$ for each a.

LSI constants. $\alpha_0 = N_c/4 = 3/4$; $\quad \alpha = \frac{3}{4} e^{-\,\mathrm{osc}(\Phi)}$.

5.3 General SU(N_c): summary table

N_c	$\dim G$	$\mathrm{Ric}_{\mathcal{A}}$ (Einstein)	Synthetic bound on $\mathcal{B}$	LSI α_0
2	3	1/2	$\mathrm{RCD}^*(1/2,\, 3\lvert\mathcal{B}_1\rvert)$	1/2
3	8	3/4	$\mathrm{RCD}^*(3/4,\, 8\lvert\mathcal{B}_1\rvert)$	3/4
N_c	N_c^2-1	$N_c/4$	$\mathrm{RCD}^*(N_c/4,\, (N_c^2-1)\lvert\mathcal{B}_1\rvert)$	$N_c/4$

All constants grow linearly with N_c, which is favorable for the large-N_c limit.

6 Discussion and role in the mass gap program

6.1 How this result is used

Theorem 1.2 provides a *single-scale* log-Sobolev inequality for the gauge-invariant measure on one fiber of the multiscale decomposition of the Wilson action used in Balaban's constructive renormalization group [10, 11, 12, 13, 14, 15].

The intended program toward the mass gap requires two independent inputs:

(1) *Geometric input* (this paper): the synthetic Ricci bound $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ on $\mathcal{B}$, yielding the conditional LSI of Theorem 1.2.

(2) *Analytic input* (to appear): uniform bounds on $\mathrm{osc}(\Phi)$ for the effective potential arising from Balaban's constructive renormalization group [10, 11, 12, 13, 14, 15], independent of the lattice volume L_{vol}.

If the analytic input is established, the conditional LSI of Theorem 1.2 becomes an unconditional, volume-uniform log-Sobolev inequality for the full Wilson measure at weak coupling. Such a uniform LSI is expected to imply a spectral gap (mass gap) via standard mechanisms (hypercontractivity and reflection positivity), provided the remaining functional-analytic steps can be carried out.

We emphasize that the present paper establishes only the geometric input (1). The analytic input (2) and the subsequent assembly are the subject of ongoing work and are not addressed here.

6.2 Epistemic honesty

We record what this paper does *not* establish:

- We do not prove the mass gap. This paper provides one of two geometric/analytic inputs needed for the LSI.
- The Ricci bound $N_c/4$ is likely not sharp. We make no claim of optimality.
- The $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ condition on $\mathcal{B}$ is established via the quotient stability theorem of Galaz-García–Kell–Mondino–Sosa [16], which handles singularities (reducible connections) automatically. A direct O'Neill-type analysis on $\mathcal{B}_{\mathrm{reg}}$ may yield pointwise curvature information, but obtaining a sharp pointwise Ricci lower bound requires careful control of the O'Neill tensors and is not attempted here.
- The dimension parameter $N = \dim \mathcal{A}$ in the $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ bound is inherited from Theorem 3.1 and is not optimal: the effective dimension of $\mathcal{B}$ is $\dim \mathcal{A} - \dim \mathcal{G}$. Improving N to $\dim \mathcal{A} - \dim \mathcal{G}$ would require techniques beyond the scope of the present quotient stability approach.
- Our results are for the *pure gauge theory*. Extension to gauge theories coupled to fermions requires additional work.

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