Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills Theory: A Conditional Reduction

Retitling and scope correction (version 4)

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Publication corrected.

This note corrects the title and foregrounds the logical status of ai.viXra:2602.0041. The eleven pages of public version 3 follow unchanged.

Conditional result only.

The uniform log-Sobolev conclusion of Theorem 1.1(i) requires the cross-scale derivative hypothesis (H-XSD). Version 3 points to companion papers as a purported discharge but does not re-verify them. The mass-gap conclusion of Theorem 1.1(ii) additionally requires the Dobrushin-type hypothesis (H-DOB), or an independently valid alternative DLR-LSI/mixing route. Neither input is proved by this manuscript alone.

No unconditional weak-coupling mass gap.

The replacement does not assert an unconditional volume-uniform LSI, an unconditional lattice mass gap, a continuum construction, Osterwalder–Schrader reconstruction, or a solution of the Clay problem. The quantitative tension in (H-DOB) recorded by version 3 remains part of the scope: the paper's route is guaranteed only on the stated parameter window.

Evidence class.

The entropy identities and finite numerical model checks reported in version 3 do not prove the missing Yang–Mills hypotheses. They remain supporting, author-reported checks. Any downstream use must carry (H-XSD), (H-DOB), the volume/parameter quantifiers and the precise operator and measure.

Provenance.

Pages 2–12 reproduce public version 3 with identical rendered content and unchanged page-content streams. Where the old title or any summary suggests unconditionality, this scope correction and the explicit hypothesis boxes of version 3 control.

Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills Theory

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Abstract

We establish that $\mathrm{SU}(N_c)$ lattice Yang–Mills theory in $d = 4$ dimensions with Wilson action at sufficiently weak coupling ($\beta = 2N_c/g^2 \geq \beta_0$) satisfies a log-Sobolev inequality with constant $\alpha_* > 0$ uniform in the lattice size L_{vol}, conditionally on two explicitly tagged hypotheses: the cross-scale derivative bound (H-XSD) (= Assumption 5.4, asserted discharged in the companion papers [10, 11, 12] = ai.viXra:2602.0057/0056/0055, whose proofs are not re-verified here), and — for the clustering/mass-gap part — a Dobrushin-type contraction (H-DOB), stated explicitly in §6.3. Combined with reflection positivity of the Wilson action, the chain LSI ⇒ hypercontractivity ⇒ exponential clustering ⇒ transfer-matrix gap yields $\Delta_{\mathrm{phys}} > 0$ uniform in L_{vol} under (H-DOB); the companion [13] (= ai.viXra:2602.0054) asserts an alternative DLR-LSI/Stroock–Zegarlinski route removing (H-DOB), likewise not re-verified here. **Note added (v3).** Version 2 contained dangling references "Assumption ??" (in the abstract, §1.3, Theorem 6.5, §6.4 and §7.1) left over from a withdrawn Dobrushin-translation assumption, and was internally inconsistent about the status of the mass gap (Theorem 1.1(ii) claimed unconditionality while Theorem 6.5 and §7.1 still assumed the withdrawn item). Version 3 repairs this by reconstructing the hypothesis explicitly as (H-DOB) and stating the honest status. Version 3 also records a quantitative tension in (H-DOB) of the same type as hypothesis (H-CONST) of ai.viXra:2602.0032 (v2): the sufficient condition $\kappa > \log[(MR_{n_{\max}})^d C_\Gamma\, r/(1-r)^2]$ involves the fixed decay constant κ on the left while $C_\Gamma \propto n_{\max}(\beta)$ grows with β (Appendix A.3), so the paper's own clustering route is guaranteed only on a window of β (new Remark 6.6, exhibited numerically). The coupling-flow direction in Theorem 2.1(e) is corrected (series erratum), the stale cross-reference in the §2.4 title is fixed, and the companion identifiers are filled in. The Ricci convention of this paper ($\mathrm{Ric} = N_c/4$ with $\langle X, Y\rangle = -2\,\mathrm{tr}\,XY$) is verified numerically to be *consistent* with the sharp $N_c/2$ of ai.viXra:2602.0036 (v2) (metric $-\,\mathrm{tr}$): it is not an instance of the series' Ricci erratum. The entropy machinery (chain rule, defect LSI, Rothaus assembly, uniform Poincaré) is validated end-to-end in a companion numerical suite on an exactly computable model.

Metric convention. Throughout, $\mathrm{SU}(N_c)$ carries the bi-invariant metric

$$\langle X, Y\rangle := -2\,\mathrm{tr}(XY), \tag{1}$$

giving the Einstein constant $\mathrm{Ric}_{\mathrm{SU}(N_c)} = \frac{N_c}{4} g_{\mathrm{SU}(N_c)}$; see Remark 3.1.

Coupling convention. $\beta = 2N_c/g^2$; the companions also use $\beta_k^{\rm red} := g_k^{-2} = \beta_k/(2N_c)$; the rescaling affects only constants.

1 Introduction

1.1 The problem

The existence of a positive mass gap in four-dimensional $\mathrm{SU}(N_c)$ Yang–Mills theory is one of the Clay Millennium Prize Problems [17]. On $\Lambda = (\mathbb{Z}/L_{\rm vol}\mathbb{Z})^4$ with Wilson action, the mass gap is the spectral gap of the transfer matrix. At strong coupling it is settled by the Osterwalder–Seiler expansion [21]; at weak coupling it has remained open.

1.2 Main result

Theorem 1.1 (Main Theorem)**.** *For $\mathrm{SU}(N_c)$ lattice Yang–Mills on $(\mathbb{Z}/L_{\rm vol}\mathbb{Z})^4$ at $\beta \geq \beta_0$ (sufficiently large):*

(i) ***(Uniform LSI; conditional on (H-XSD), see Remark 1.2)*** *There exists $\alpha_* > 0$, depending on N_c and β_0 but not on $L_{\rm vol}$, such that for all gauge-invariant f:*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq \frac{2}{\alpha_*} \int |\nabla f|^2 \, d\mu_\beta. \tag{2}$$

(ii) ***(Mass gap; conditional on (H-DOB))*** *Under the hypothesis (H-DOB) of §6.3, the transfer matrix has spectral gap $\Delta_{\rm phys} \geq c(N_c, \beta_0) > 0$ uniformly in $L_{\rm vol}$. The companion [13] asserts an alternative route (periodic uniform LSI ⇒ DLR-LSI ⇒ Stroock–Zegarlinski mixing ⇒ reflection positivity ⇒ gap) that removes (H-DOB); we do not re-verify it here.* (*v2 stated (ii) as unconditional while its own §6.4 and §7.1 still assumed the withdrawn item; v3 states the honest status.*)

Remark 1.2 (Status of Assumption 5.4 = (H-XSD)). Assumption 5.4 is asserted to be discharged by the companion papers: [10] (= ai.viXra:2602.0057) replaces it by two weaker inputs (an integrated L^1 bound and a shifted essential supremum), [11] (= ai.viXra:2602.0056) verifies the large-field input, and [12] (= ai.viXra:2602.0055) verifies the residual derivative input. We retain it here as an explicit hypothesis with that provenance; the present paper does not re-verify the companions, and Theorem 1.1(i) should be read as conditional on their correctness. Appendix A gives this paper's own audit trail (small-field region).

1.3 Logical structure

Balaban RG + $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4 \xrightarrow{\text{entropy decomposition}} \alpha_*^{\rm LSI} > 0 \xrightarrow{\text{RP+(H-DOB)}} \Delta_{\rm phys} > 0$. The first arrow is conditional on (H-XSD) (Remark 1.2); the second on (H-DOB) (§6.3), with [13] asserting its removal.

1.4 Relation to the Clay Millennium Problem

This concerns the lattice theory at weak coupling, uniform in volume; the continuum construction and OS axioms are not addressed.

2 Inputs from Balaban's constructive RG

2.1 Effective actions with polymer bounds

Theorem 2.1 (Balaban [6, Theorem 1]). *For $\beta \geq \beta_0$: effective actions S_k exist on $\Lambda_k = (\mathbb{Z}/L_{\mathrm{vol},k}\mathbb{Z})^4$, $L_{\mathrm{vol},k} = L_{\mathrm{vol}}/L_{\mathrm{RG}}^k$, $0 \leq k \leq n_{\max}$, with:*

(a) $Z(\Lambda, S_0) = Z(\Lambda_k, S_k)$.

(b) Structure ([6, eq. (1.101)]):

$$S_k = \beta_k S_W + \sum_X R^{(k)}(X) + \sum_X B^{(k)}(X), \tag{3}$$

with irrelevant polymer activities $R^{(k)}$ and boundary terms $B^{(k)}$.

(c) Polymer decay ([6, eq. (1.100)]):

$$|R^{(k)}(X,(U,J))| \leq e^{-p_0(g_k)} e^{-\kappa\, d_k(X)}. \tag{4}$$

(d) Analyticity on $\widetilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, eq. (1.65)]).

(e) Asymptotic freedom: $1/g_k^2 = 1/g^2 - 2b_0 k \ln 2 + O(1)$, i.e. $\beta_k = \beta - 4N_c b_0 k \ln 2 + O(1)$ decreasing *toward the infrared, $b_0 = 11N_c/(48\pi^2)$; the terminal coupling satisfies $g_{n_{\max}}^2 \leq \gamma$ with $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$. (v2 stated $\beta_k = \beta + 2b_0 k \ln 2$; sign/normalization corrected per the series erratum, cf. ai.viXra:2602.0032/0033/0035/0040 v2. The magnitude $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$, used in §7.3 and Appendix A.3, is unchanged.)*

2.2 The absorption property

The key identity is [6, eq. (1.37)]:

$$-\frac{1}{g_k^2} A(\zeta_0, U'_{k,Z}(V_k V_\Lambda)) - A\Big(\frac{1}{(g''_k(\cdot))^2}, U_k\Big) + \frac{1}{g_k^2} A(\zeta_0, U_0) = -A\Big(\frac{1}{(g_k(\cdot))^2}, U_k\Big) + O(1), \tag{5}$$

with the $O(1)$ bounded by absolute constants. After absorption, scale-k dependence enters only through $R^{(k)}, B^{(k)}$.

2.3 Block-averaging and geometric scaling

Balaban's averaging [1, eq. (15)] and its linearization [1, eq. (125)]:

$$\bar{U}_c = \exp\Big(i \sum_{x \in B(c_-)} L_{\mathrm{RG}}^{-d} \log U(\Gamma_{c,x}) U(c)^{-1}\Big) U(c), \tag{6}$$

$$(Q_0 A)_c = \sum_{x \in B(c_-)} L_{\mathrm{RG}}^{-(d+1)} (R_{0,c_-} A)([x, x']). \tag{7}$$

Lemma 2.2 (Adjoint scaling). *$\|Q_0^*\|^2 = L_{\mathrm{RG}}^{-(d-1)}$; iterating, $\|Q_{(k)}^*\|^2 = L_{\mathrm{RG}}^{-(d-1)k}$ ($= 2^{-3k}$ for $L_{\mathrm{RG}} = 2$, $d = 4$). Nonlinear corrections are quadratic ([1, Prop. 4, eq. (133)]) and do not affect the leading scaling; functional derivatives of Q_k are bounded uniformly in k ([1, Prop. 5]):*

$$\Big|\frac{\delta}{\delta A_b} Q_k(U_0, \eta A, c)\Big| \leq 1 + 2C'_1 \alpha_0 + C_3 \alpha_1. \tag{8}$$

2.4 Polymer derivative bounds (Proposition 2.3)

(v3: the v2 title of this subsection referred to a stale "Proposition 2.6"; the item is Proposition 2.3.)

Proposition 2.3 (Single-scale polymer derivative bound, small-field region)**.** *Assume, beyond Theorem 2.1, that the analyticity radius $\hat{\alpha}_1 = \hat{\alpha}_1(\gamma) > 0$ can be chosen uniformly in k (cf. Remark A.1). Then on the small-field region Ω_k^{sf}:*

$$\sup_{\Omega_k^{\mathrm{sf}}} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \le C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}, \tag{9}$$

via the absorption identity (5), Cauchy estimates on the analyticity domain,

$$\Big|\frac{\delta R^{(k)}(X)}{\delta A'(b)}\Big| \le \frac{1}{\hat{\alpha}_1(\gamma)} \sup_{|A'|<\hat{\alpha}_1} |R^{(k)}(X)| \le \frac{C(\gamma)}{\hat{\alpha}_1(\gamma)} e^{-\kappa d_k(X)}, \tag{10}$$

polymer counting (Lemma 5.6), and the adjoint scaling (Lemma 2.2). The extension to the full essential supremum,

$$\operatorname*{ess\,sup}_{\mathcal{G}_k} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \le C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}, \tag{11}$$

is the content of Assumption 5.4 (the boundary terms $B^{(k)}$ vanish on Ω_k^{sf} but on the large-field region their Cauchy bound grows like $O(k)$ and does not by itself carry the geometric factor).

Remark 2.4 (Essential supremum and large-field regions)*.* The ess sup cannot be controlled by probabilistic suppression (small measure is not measure zero); on the large-field region a separate mechanism is needed for the geometric factor. This is part of (H-XSD).

3 Orbit space Ricci curvature

With the metric (1), the generators $\tilde{T}^a$ with $\mathrm{tr}(\tilde{T}^a\tilde{T}^b) = -\frac{1}{2}\delta^{ab}$ are orthonormal and

$$f^{acd} f^{bcd} = N_c\, \delta^{ab}, \tag{12}$$

and $\mathrm{Ric}_{\mathrm{SU}(N_c)} = \frac{N_c}{4} g$ [27, Prop. 7.10].

Remark 3.1 (Metric convention; extended in v3)*.* With $\langle X, Y\rangle = -N_c\,\mathrm{tr}(XY)$ one gets $\mathrm{Ric} = \frac{1}{2}g$; with $\langle X, Y\rangle = -\mathrm{tr}(XY)$ (the convention of ai.viXra:2602.0036 v2) one gets $\mathrm{Ric} = \frac{N_c}{2}g$. All three describe the same tensor; the companion suite verifies the three values numerically for SU(2) and SU(3), together with (12). In particular the $N_c/4$ used here is *not* an instance of the series' Ricci erratum (which concerned the $-\,\mathrm{tr}$ convention): the conventions are mutually consistent.

Theorem 3.2 ([9, Theorem 1.1])**.** *With (1), the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ satisfies* $\mathrm{Ric}_{\mathcal{B}} \ge \frac{N_c}{4}$.

Proof. As in v2: (1) $\mathrm{Ric}_{\mathcal{A}} \ge N_c/4$ from $\mathrm{Ric} = -\frac{1}{4}B$ with $B(X,Y) = 2N_c\,\mathrm{tr}(XY)$; (2) the Galaz-García–Kell–Mondino–Sosa quotient theorem [29] for RCD* spaces passes the bound to $\mathcal{B} = \mathcal{A}/\mathcal{G}$ including the singular stratum; (3) $\mathrm{RCD}^*(K, N)$ with $K > 0$ gives LSI with constant $\ge K$ [25, Cor. 5.7.2]. □

Remark 3.3*.* The synthetic route handles $\mathcal{B}_{\mathrm{sing}}$ automatically — an advantage over O'Neill-type arguments.

4 Terminal scale LSI

Theorem 4.1 (Terminal LSI)**.** *At the terminal scale, $\mu_{n_{\max}}$ satisfies an LSI with $\alpha_{\text{term}} > 0$ depending only on N_c, γ, κ: the terminal orbit space has $O(1)$ dimension,* Ric $\geq N_c/4$*, the uniform measure satisfies*

$$\alpha_{\text{LSI}}(\nu) \geq \frac{N_c}{4}, \tag{13}$$

and $\operatorname{osc}(f) \leq K/\gamma =: M_{\text{term}}$ *for the terminal density, so Holley–Stroock [16] gives $\alpha_{\text{term}} \geq \frac{N_c}{4} e^{-M_{\text{term}}}$.*

Remark 4.2 (LSI convention). We use $\operatorname{Ent}_\mu(f^2) \leq \frac{2}{\alpha}\mathcal{E}(f,f)$; Bakry–Émery with $CD(K,\infty)$ gives $\alpha = K$ in this convention.

5 Multiscale entropy decomposition

5.1 Setup and filtration

Decreasing filtration $\mathcal{G}_0 \supset \cdots \supset \mathcal{G}_{n_{\max}}$, $\mathcal{G}_k = \sigma(\bar{A}^{(k)})$; orthogonal decomposition $T_A\mathcal{A} = \bigoplus_k E_k \oplus V_{n_{\max}}$ with $\sum_k |\nabla_{E_k} f|^2 + |\nabla_{V_{n_{\max}}} f|^2 = |\nabla f|^2$.

5.2 Entropy chain rule

Lemma 5.1 (Entropy telescoping)**.** *For $f > 0$:*

$$\operatorname{Ent}_\mu(f) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\operatorname{Ent}(\mathbb{E}[f \mid \mathcal{G}_k] \mid \mathcal{G}_{k+1})\big] + \operatorname{Ent}_\mu(\mathbb{E}[f \mid \mathcal{G}_{n_{\max}}]). \tag{14}$$

(Guionnet–Zegarlinski [15, Prop. 4.1]; verified exactly on the suite's model.)

5.3 Conditional fast LSI

Lemma 5.2. *There is $\alpha_0 > 0$ (independent of k, L_{vol}) with*

$$\operatorname{Ent}(h^2 \mid \mathcal{G}_{k+1}) \leq \frac{2}{\alpha_0} \mathbb{E}\big[|\nabla_{E_k} h|^2 \mid \mathcal{G}_{k+1}\big] \tag{15}$$

for smooth gauge-invariant h, μ-a.e.: small-field conditional LSI from the fiber Ricci bound plus Balaban's quadratic-form lower bound ([6, eqs. (1.2), (1.9)]), Bakry–Émery, then large-field patching by Holley–Stroock with oscillation bounded by the RG-block geometry (uniform in k, L_{vol}).

5.4 Influence coefficients and sweeping-out inequality

Definition 5.3 (Influence coefficients)**.** $\Gamma_{k\ell} := \sup_\varphi \mathbb{E}[|\nabla_{E_k}\mathbb{E}[\varphi \mid \mathcal{G}_\ell]|^2]^{1/2} / \mathbb{E}[|\nabla_{E_\ell}\varphi|^2]^{1/2}$, $\Gamma_{k\ell} = 0$ *for $\ell > k$.*

Assumption 5.4 (Cross-scale derivative bound; hypothesis (H-XSD))**.** *For $0 \leq \ell < k \leq n_{\max} - 1$:*

$$\operatorname*{ess\,sup}_{\mathcal{G}_\ell} \mathbb{E}\big[|\nabla_{E_k} V_{<\ell}|^2 \mid \mathcal{G}_\ell\big] \leq C_{\text{poly}}^2 e^{-2\kappa} L_{\text{RG}}^{-(d-1)k}. \tag{16}$$

On the small-field region this follows from Proposition 2.3 composed across scales (Appendix A); the large-field extension is the assumption's content. ***Status (v3):*** *asserted discharged in [10, 11, 12] (= ai.viXra:2602.0057/0056/0055); retained as an explicit hypothesis with that provenance (Remark 1.2).*

Lemma 5.5 (Geometric decay of influence)**.** *Under Assumption 5.4:*

$$\Gamma_{k\ell} \le C_\Gamma e^{-\kappa} L_{\mathrm{RG}}^{-(d-1)k/2}, \qquad C_\Gamma = C_{\mathrm{poly}}/\sqrt{\alpha_0}, \tag{17}$$

via the differentiation identity

$$v\,\mathbb{E}[\varphi \mid \mathcal{G}_\ell] = \mathbb{E}[v\varphi \mid \mathcal{G}_\ell] - \mathrm{Cov}(\varphi, vV_{<\ell} \mid \mathcal{G}_\ell), \tag{18}$$

Jensen for the direct term, and Cauchy–Schwarz plus the conditional Poincaré for the indirect term.

Lemma 5.6 (Polymer counting)**.** $\#\{X : X \ni b,\ d_k(X) = n\} \le C_d^n$ *with* $C_d = 3^d - 1$*; hence for* $\kappa > \log C_d$*:*

$$\sum_{X \ni b} e^{-\kappa d_k(X)} \le 1 + \frac{C_d e^{-\kappa}}{1 - C_d e^{-\kappa}} < \infty. \tag{19}$$

(The counting bound is verified by brute force for small polymers in the suite.)

Lemma 5.7 (Sweeping-out, bounded observables)**.** *With* $g_k := \sqrt{\mathbb{E}[g^2 \mid \mathcal{G}_k]}$*, for bounded smooth gauge-invariant* g*:*

$$\mathbb{E}[|\nabla_{E_k} g_k|^2] \le 2\,\mathbb{E}[|\nabla_{E_k} g|^2] + 2\sum_{\ell<k} \Gamma_{k\ell}^2 \mathbb{E}[|\nabla_{E_\ell} g|^2] + D_k \|g\|_\infty^2, \tag{20}$$

$D_k := C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}$*, whence* $\sum_k \mathbb{E}[|\nabla_{E_k} g_k|^2] \le (2 + D_*)\mathcal{E}(g,g) + D_* \|g\|_\infty^2$ *with* $D_* \le \frac{8}{7} C_{\mathrm{poly}}^2 e^{-2\kappa}$ *for* $L_{\mathrm{RG}} = 2, d = 4$*.*

Remark 5.8 (No entropy circularity; extension by truncation)*.* The error is controlled by energy plus a pointwise bound, not by the entropy being estimated; extension to $\mathrm{Dom}(\mathcal{E})$ by truncation and lower semicontinuity [25, Prop. 5.2.5].

5.5 Assembly: uniform LSI

Lemma 5.9 (Variance telescoping)**.** *With* $m_k := \mathbb{E}[g \mid \mathcal{G}_k]$*:*

$$\mathrm{Var}_\mu(g) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}[\mathrm{Var}(m_k \mid \mathcal{G}_{k+1})] + \mathrm{Var}_\mu(m_{n_{\max}}). \tag{21}$$

Lemma 5.10 (Energy-only sweeping-out)**.** *For bounded smooth gauge-invariant* g*:*

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}[|\nabla_{E_k} m_k|^2] \le C\,\mathcal{E}(g,g), \qquad C = 2 + \tfrac{2}{\alpha_0} C_{\mathrm{poly}}^2 e^{-2\kappa} \big(1 - L_{\mathrm{RG}}^{-(d-1)}\big)^{-1}, \tag{22}$$

using the conditional Poincaré (from Lemma 5.2) inside the covariance bound; no $\|g\|_\infty$ *appears since the covariance involves* g *linearly.*

Proof of Theorem 1.1(i). As in v2, Steps 1–8: entropy telescoping (Lemma 5.1); conditional LSI (Lemma 5.2); sweeping-out (Lemma 5.7); terminal scale (Theorem 4.1); the defect LSI

$$\mathrm{Ent}_\mu(g^2) \le C_E\,\mathcal{E}(g,g) + C_B\|g\|_\infty^2; \tag{23}$$

the uniform Poincaré, proved independently of the global LSI via Lemmas 5.9–5.10,

$$\mathrm{Var}_\mu(g) \le C_P\,\mathcal{E}(g,g), \qquad C_P = \frac{2C}{\alpha_0} + \frac{2}{\alpha_{\mathrm{term}}}; \tag{24}$$

Rothaus' lemma [25, Prop. 5.1.3] converting defect LSI + Poincaré into the tight LSI

$$\mathrm{Ent}_\mu(g^2) \le \frac{2}{\alpha_*}\mathcal{E}(g,g); \tag{25}$$

and extension to all bounded g by the homogeneity

$$\mathrm{Ent}_\mu(\lambda f) = \lambda\ \mathrm{Ent}_\mu(f) \qquad (\lambda > 0,\ f \ge 0), \tag{26}$$

then to $\mathrm{Dom}(\mathcal{E})$ by Corollary 5.11. (v3: the entire pipeline — (14), (23) with recipe constants, (24), the Rothaus assembly and (26) — is validated on an exactly computable compact model in the companion suite, with the assembled α_* confirmed below the model's true LSI constant obtained by direct minimization.) □

Corollary 5.11 (Extension to $\mathrm{Dom}(\mathcal{E})$)**.** *By truncation $f^{(m)} = (-m) \vee f \wedge m$ and lower semicontinuity [25, Prop. 5.2.5], the tight LSI extends with the same α_*.*

6 From uniform LSI to mass gap

6.1 Reversible diffusion and its Dirichlet form

$$Lf := \sum_\ell \big(\Delta_\ell f - \langle \nabla_\ell S_\beta, \nabla_\ell f\rangle\big), \tag{27}$$

$$\mathcal{E}(f,f) = -\int f L f\, d\mu_\beta = \int |\nabla f|^2\, d\mu_\beta, \qquad P_t = e^{tL}. \tag{28}$$

6.2 Hypercontractivity and spectral gap

Proposition 6.1. *The uniform LSI implies (i) hypercontractivity (Gross [14]); (ii) $\lambda_1 \ge \alpha_*$ and $\|P_t f - \mathbb{E}f\|_2 \le e^{-\lambda_1 t}\|f - \mathbb{E}f\|_2$.*

6.3 Exponential clustering

(H-DOB) Site-to-site Dobrushin contraction (reconstructed in v3). *The site-to-site influence matrix C_{xy} of Lemma 6.4 satisfies $\sup_x \sum_{y\neq x} C_{xy} < 1$; by Lemmas 6.2–6.4, a sufficient condition is*

$$\kappa \ > \ \log\Big[(MR_{n_{\max}})^d\, C_\Gamma\, \frac{r}{(1-r)^2}\Big], \qquad r = L_{\mathrm{RG}}^{-(d-1)/2}. \tag{H-DOB}$$

This is the hypothesis that v2 referenced as a dangling "Assumption ??" after withdrawing its earlier Dobrushin-translation form (Remark 6.3); v3 states it explicitly. See Remark 6.6 for its quantitative status.

Lemma 6.2 (Quantified contraction condition). *With $\Gamma_{k\ell} \le C_\Gamma e^{-\kappa} r^k$: $D_{\rm row} := \sup_k \sum_{\ell<k} \Gamma_{k\ell} \le C_\Gamma e^{-\kappa} \frac{r}{(1-r)^2}$ (using $kr^k \le \sum_j jr^j$; the elementary bound is machine-verified). For $L_{\rm RG} = 2$, $d = 4$: $r = 2^{-3/2}$, $\frac{r}{(1-r)^2} \approx 0.846$.*

Remark 6.3 (Withdrawn assumption; superseded route). The Dobrushin-translation assumption of v1 is withdrawn; [13] asserts that the DLR-LSI extension plus Stroock–Zegarlinski equivalence, with reflection positivity, yields the gap without it. (v3: within the present paper the clustering chain still requires (H-DOB) above; v2's text claimed unconditionality here while assuming the withdrawn item in Theorem 6.5 and §6.4 — repaired.)

Lemma 6.4 (From multiscale influence to a site-to-site Dobrushin matrix). *With*

$$C_{xy} := \sum_{k=0}^{n_{\max}} \sum_{\ell=0}^{k-1} \Gamma_{k\ell} \mathbf{1}_{x\in \mathrm{Block}_k} \mathbf{1}_{y\in \mathrm{Block}_\ell}, \tag{29}$$

one has $\sup_x \sum_y C_{xy} \le (\sup_k |\mathrm{Block}_k|) D_{\rm row} \le (MR_{n_{\max}})^d D_{\rm row}$; hence (H-DOB) holds if κ exceeds the threshold displayed above.

Theorem 6.5 (Clustering; conditional on (H-DOB)). *Assume the uniform LSI and* (H-DOB). *Then for gauge-invariant O_A, O_B at distance r: $|\langle O_A O_B\rangle_c| \le C\|O_A\|_\infty \|O_B\|_\infty e^{-r/\xi}$, $\xi = O(1/\sqrt{\alpha_*})$, uniformly in $L_{\rm vol}$: finite-range structure plus polymer tails (Lemma 5.6), Dobrushin–Shlosman complete analyticity [23, 24] under the site-to-site contraction, and gauge-invariant conditioning.*

Remark 6.6 (New in v3: the (H-DOB) window). (H-DOB) has the same structure as hypothesis (H-CONST) of ai.viXra:2602.0032 (v2): the left side κ is a *fixed* output of Balaban's expansion, while the right side grows with β — both through $(MR_{n_{\max}})^d$ and through $C_\Gamma \propto C_{\rm poly} \propto n_{\max} \sim 1/(2b_0 g^2 \ln 2)$ (Appendix A.3). Numerically (suite): already for $(MR)^d = 16^4$ the threshold is $\kappa \gtrsim$ 12–14 for $g^2 \in [0.05, 0.5]$, growing logarithmically as $g^2 \to 0$. Hence the present paper's clustering route is guaranteed only on a window $\beta_0 \le \beta \le \beta_1(\kappa)$, unless the constants of Balaban's construction are favorable; outside the window one must rely on the asserted route of [13]. This does not affect Theorem 1.1(i) (the LSI itself needs no (H-DOB)), nor uniformity in $L_{\rm vol}$ at fixed admissible β.

6.4 Mass gap via reflection positivity

Proof of Theorem 1.1(ii). Assume (H-DOB). Reflection positivity of the Wilson action [21] gives the spectral representation $\langle O(t)O(0)\rangle_c = \sum_{n\ge1} |\langle \Omega|\hat{O}|n\rangle|^2 e^{-E_n t}$; by Theorem 6.5 the correlator decays at rate $M = 1/\xi > 0$, forcing $E_1 \ge M$; taking $O = O_P = \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\, U_P$ (which has positive variance) yields $\Delta_{\rm phys} = E_1 \ge M > 0$. □

7 Discussion

7.1 What is proved

Uniform LSI (Theorem 1.1(i)), conditional on (H-XSD) with the provenance of Remark 1.2; mass gap $\Delta_{\rm phys} > 0$ uniform in $L_{\rm vol}$, conditional on (H-DOB) (Theorem 1.1(ii)); the companion [13] asserts removal of (H-DOB).

7.2 What is not proved

Continuum limit and OS axioms; persistence of the gap as $a \to 0$; the strong-coupling regime (covered by [21]).

7.3 On the expected scaling $m_{\text{gap}} \sim e^{-C/g^2}$

We do not establish the expected scaling: $\alpha_{\text{term}} \geq \frac{N_c}{4} e^{-M_{\text{term}}}$ with $M_{\text{term}} = O(1/\gamma)$, and tracking α_* through $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$ steps is beyond this paper.

7.4 Relation to the Millennium Problem

Infrared component only; the ultraviolet construction is beyond scope.

7.5 Conditional aspects and the logical structure

(1) Balaban's RG (published); (2) (H-XSD) with the audit trail of Appendix A and the asserted discharges [10, 11, 12]; (3) (H-DOB) with the explicit threshold of Lemma 6.4 and the window caveat of Remark 6.6; (4) $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ via RCD* quotient [29]; (5) the sweeping-out/assembly steps, non-circular, validated numerically. All conditions require γ small; how small is determined by Balaban's constants and the thresholds here.

7.6 The role of each input

As in v2's table, with identifiers: cross-scale bounds [10] (=2602.0057); large-field suppression [11] (=2602.0056); residual derivatives [12] (=2602.0055); LSI→gap route [13] (=2602.0054); Ricci source [9] (=2602.0046).

Note added in v3

Prepared with the companion suite `verify_2602_0041.py` (repository `THE-ERIKSSON-PROGRAMME`, `verification/2602-0041/`; 11 checks, all passing, ∼7 s, NumPy/SciPy). Changes relative to v2: (1) the four dangling "Assumption ??" references (abstract, §1.3, Theorem 6.5, §6.4, §7.1) are repaired by reconstructing the hypothesis explicitly as (H-DOB) (§6.3), and the internal inconsistency about the status of Theorem 1.1(ii) (claimed unconditional while §6.4/§7.1 assumed the withdrawn item) is resolved: within this paper the gap is conditional on (H-DOB), with [13] asserting its removal; (2) new Remark 6.6: the (H-DOB) threshold has the (H-CONST) pattern of ai.viXra:2602.0032 (v2) — fixed κ against a right-hand side growing with β through $(MR_{n_{\max}})^d$ and $C_\Gamma \propto n_{\max}$ — so the paper's own clustering route holds on a β-window (numerically exhibited); (3) Theorem 2.1(e): coupling-flow sign/normalization corrected (series erratum; $n_{\max}$ magnitude unchanged); (4) Assumption 5.4 tagged (H-XSD) with honest provenance ("asserted discharged in the companions, not re-verified here") and companion identifiers filled in ([9]–[13] = ai.viXra:2602.0046/0057/0056/0055/0054); (5) the §2.4 title's stale "Proposition 2.6" corrected; (6) Remark 3.1 extended: the three metric conventions verified numerically to be mutually consistent (this paper's $N_c/4$ is correct in its convention; coherent with the sharp $N_c/2$ of ai.viXra:2602.0036 v2); (7) the entropy pipeline (chain rule, defect LSI with recipe constants, independent uniform Poincaré, Rothaus assembly, homogeneity) validated end-to-end on an exactly computable compact model, with the assembled

α_* confirmed below the model's true LSI constant; Lemma 5.6's counting bound verified by brute force; Lemma 6.2's elementary bound machine-checked. All v2 numbered statements and equations keep their numbers; (H-DOB) is deliberately an unnumbered tagged hypothesis to avoid renumbering.

A Audit trail: derivation of the cross-scale derivative bound

As in v2 (Steps 1–3, small-field region): single-scale bound (9); propagation across intermediate scales via the k-fold averaging bounds ([1, Props. 4, 5, 7], eq. (146)); composition with the factor $(k - \ell + 1) \le n_{\max}$ absorbed into $C_{\text{poly}} = n_{\max}(\beta_0)C'/\hat{\alpha}_1(\gamma)$ — which is the source of the β-growth of C_Γ recorded in Remark 6.6. The large-field extension is (H-XSD).

Remark A.1. Balaban proves k-uniform analyticity and derivative bounds for the averaging maps and states k-uniform inductive conditions for effective densities; our audit assumes the analyticity parameter $\hat{\alpha}_1(\gamma)$ can be chosen uniformly in k. Extracting a standalone lemma with all parameters is deferred (and asserted in [12]).

Remark A.2. The factor $n_{\max}$ in C_{poly} grows polynomially in $\log(1/g^2)$; it does not affect positivity of α_* at fixed β_0, only its quantitative value — and, via C_Γ, the (H-DOB) threshold (Remark 6.6).

B Numerical verification (new in v3)

`verify_2602_0041.py`: (1) Ricci convention coherence: $N_c/4\ (-2\,\mathrm{tr})$, $N_c/2\ (-\,\mathrm{tr})$, $\frac{1}{2}\ (-N_c\,\mathrm{tr})$, and (12), verified for $\mathrm{SU}(2), \mathrm{SU}(3)$; (2) entropy pipeline on a four-rotor model (5^4 states): chain rule (14) exact; defect LSI (23) with recipe constants over 300 random observables; uniform Poincaré (24) valid against the true gap; assembled α_* below the true LSI constant (direct minimization of $2\mathcal{E}/\,\mathrm{Ent}$ with analytic gradients); homogeneity (26); (3) Lemma 6.2 exact and the (H-DOB) window exhibit ($\kappa_{\text{needed}} \approx 12$–$14$, growing as $g^2 \to 0$); (4) Lemma 5.6 by brute-force polyomino counting; (5) corrected coupling flow and the geometric sums for D_*. Run: `python3 verify_2602_0041.py` (exit 0 iff all pass).

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