

Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills Theory

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Abstract

We establish that $SU(N_c)$ lattice Yang–Mills theory in $d = 4$ dimensions with Wilson action at sufficiently weak coupling ($\beta = 2N_c/g^2 \geq \beta_0$) satisfies a log-Sobolev inequality with constant $\alpha_* > 0$ uniform in the lattice size L_{vol} , conditionally on two explicitly tagged hypotheses: the cross-scale derivative bound (H-XSD) (= Assumption 5.4, asserted discharged in the companion papers [10, 11, 12] = ai.viXra:2602.0057/0056/0055, whose proofs are not re-verified here), and — for the clustering/mass-gap part — a Dobrushin-type contraction (H-DOB), stated explicitly in §6.3. Combined with reflection positivity of the Wilson action, the chain $\text{LSI} \Rightarrow \text{hypercontractivity} \Rightarrow \text{exponential clustering} \Rightarrow \text{transfer-matrix gap}$ yields $\Delta_{\text{phys}} > 0$ uniform in L_{vol} under (H-DOB); the companion [13] (= ai.viXra:2602.0054) asserts an alternative DLR-LSI/Stroock–Zegarlinski route removing (H-DOB), likewise not re-verified here. **Note added (v3).** Version 2 contained dangling references “Assumption ??” (in the abstract, §1.3, Theorem 6.5, §6.4 and §7.1) left over from a withdrawn Dobrushin-translation assumption, and was internally inconsistent about the status of the mass gap (Theorem 1.1(ii) claimed unconditionality while Theorem 6.5 and §7.1 still assumed the withdrawn item). Version 3 repairs this by reconstructing the hypothesis explicitly as (H-DOB) and stating the honest status. Version 3 also records a quantitative tension in (H-DOB) of the same type as hypothesis (H-CONST) of ai.viXra:2602.0032 (v2): the sufficient condition $\kappa > \log[(MR_{n_{\text{max}}})^d C_\Gamma r / (1 - r)^2]$ involves the fixed decay constant κ on the left while $C_\Gamma \propto n_{\text{max}}(\beta)$ grows with β (Appendix A.3), so the paper’s own clustering route is guaranteed only on a window of β (new Remark 6.6, exhibited numerically). The coupling-flow direction in Theorem 2.1(e) is corrected (series erratum), the stale cross-reference in the §2.4 title is fixed, and the companion identifiers are filled in. The Ricci convention of this paper ($\text{Ric} = N_c/4$ with $\langle X, Y \rangle = -2 \text{tr} XY$) is verified numerically to be *consistent* with the sharp $N_c/2$ of ai.viXra:2602.0036 (v2) (metric – tr): it is not an instance of the series’ Ricci erratum. The entropy machinery (chain rule, defect LSI, Rothaus assembly, uniform Poincaré) is validated end-to-end in a companion numerical suite on an exactly computable model.

Metric convention. Throughout, $SU(N_c)$ carries the bi-invariant metric

$$\langle X, Y \rangle := -2 \text{tr}(XY), \tag{1}$$

giving the Einstein constant $\text{Ric}_{SU(N_c)} = \frac{N_c}{4} g_{SU(N_c)}$; see Remark 3.1.

Coupling convention. $\beta = 2N_c/g^2$; the companions also use $\beta_k^{\text{red}} := g_k^{-2} = \beta_k/(2N_c)$; the rescaling affects only constants.

1 Introduction

1.1 The problem

The existence of a positive mass gap in four-dimensional $\text{SU}(N_c)$ Yang–Mills theory is one of the Clay Millennium Prize Problems [17]. On $\Lambda = (\mathbb{Z}/L_{\text{vol}}\mathbb{Z})^4$ with Wilson action, the mass gap is the spectral gap of the transfer matrix. At strong coupling it is settled by the Osterwalder–Seiler expansion [21]; at weak coupling it has remained open.

1.2 Main result

Theorem 1.1 (Main Theorem). *For $\text{SU}(N_c)$ lattice Yang–Mills on $(\mathbb{Z}/L_{\text{vol}}\mathbb{Z})^4$ at $\beta \geq \beta_0$ (sufficiently large):*

- (i) **(Uniform LSI; conditional on (H-XSD), see Remark 1.2)** *There exists $\alpha_* > 0$, depending on N_c and β_0 but not on L_{vol} , such that for all gauge-invariant f :*

$$\text{Ent}_{\mu_\beta}(f^2) \leq \frac{2}{\alpha_*} \int |\nabla f|^2 d\mu_\beta. \quad (2)$$

- (ii) **(Mass gap; conditional on (H-DOB))** *Under the hypothesis (H-DOB) of §6.3, the transfer matrix has spectral gap $\Delta_{\text{phys}} \geq c(N_c, \beta_0) > 0$ uniformly in L_{vol} . The companion [13] asserts an alternative route (periodic uniform LSI $\Rightarrow$ DLR-LSI $\Rightarrow$ Stroock–Zegarlinski mixing $\Rightarrow$ reflection positivity $\Rightarrow$ gap) that removes (H-DOB); we do not re-verify it here. (v2 stated (ii) as unconditional while its own §6.4 and §7.1 still assumed the withdrawn item; v3 states the honest status.)*

Remark 1.2 (Status of Assumption 5.4 = (H-XSD)). Assumption 5.4 is asserted to be discharged by the companion papers: [10] (= ai.viXra:2602.0057) replaces it by two weaker inputs (an integrated L^1 bound and a shifted essential supremum), [11] (= ai.viXra:2602.0056) verifies the large-field input, and [12] (= ai.viXra:2602.0055) verifies the residual derivative input. We retain it here as an explicit hypothesis with that provenance; the present paper does not re-verify the companions, and Theorem 1.1(i) should be read as conditional on their correctness. Appendix A gives this paper’s own audit trail (small-field region).

1.3 Logical structure

Balaban RG + Ric $_{\beta} \geq N_c/4 \xrightarrow{\text{entropy decomposition}} \alpha_*^{\text{LSI}} > 0 \xrightarrow{\text{RP+(H-DOB)}} \Delta_{\text{phys}} > 0$. The first arrow is conditional on (H-XSD) (Remark 1.2); the second on (H-DOB) (§6.3), with [13] asserting its removal.

1.4 Relation to the Clay Millennium Problem

This concerns the lattice theory at weak coupling, uniform in volume; the continuum construction and OS axioms are not addressed.

2 Inputs from Balaban's constructive RG

2.1 Effective actions with polymer bounds

Theorem 2.1 (Balaban [6, Theorem 1]). *For $\beta \geq \beta_0$: effective actions S_k exist on $\Lambda_k = (\mathbb{Z}/L_{\text{vol},k}\mathbb{Z})^4$, $L_{\text{vol},k} = L_{\text{vol}}/L_{\text{RG}}^k$, $0 \leq k \leq n_{\text{max}}$, with:*

(a) $Z(\Lambda, S_0) = Z(\Lambda_k, S_k)$.

(b) *Structure ([6, eq. (1.101)])*:

$$S_k = \beta_k S_W + \sum_X R^{(k)}(X) + \sum_X B^{(k)}(X), \quad (3)$$

with irrelevant polymer activities $R^{(k)}$ and boundary terms $B^{(k)}$.

(c) *Polymer decay ([6, eq. (1.100)])*:

$$|R^{(k)}(X, (U, J))| \leq e^{-p_0(g_k)} e^{-\kappa d_k(X)}. \quad (4)$$

(d) *Analyticity on $\tilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, eq. (1.65)])*.

(e) *Asymptotic freedom: $1/g_k^2 = 1/g^2 - 2b_0 k \ln 2 + O(1)$, i.e. $\beta_k = \beta - 4N_c b_0 k \ln 2 + O(1)$ decreasing toward the infrared, $b_0 = 11N_c/(48\pi^2)$; the terminal coupling satisfies $g_{n_{\text{max}}}^2 \leq \gamma$ with $n_{\text{max}} \sim 1/(2b_0 g^2 \ln 2)$. (v2 stated $\beta_k = \beta + 2b_0 k \ln 2$; sign/normalization corrected per the series erratum, cf. ai.viXra:2602.0032/0033/0035/0040 v2. The magnitude $n_{\text{max}} \sim 1/(2b_0 g^2 \ln 2)$, used in §7.3 and Appendix A.3, is unchanged.)*

2.2 The absorption property

The key identity is [6, eq. (1.37)]:

$$-\frac{1}{g_k^2} A(\zeta_0, U'_{k,Z}(V_k V_\Lambda)) - A\left(\frac{1}{(g_k''(\cdot))^2}, U_k\right) + \frac{1}{g_k^2} A(\zeta_0, U_0) = -A\left(\frac{1}{(g_k(\cdot))^2}, U_k\right) + O(1), \quad (5)$$

with the $O(1)$ bounded by absolute constants. After absorption, scale- k dependence enters only through $R^{(k)}, B^{(k)}$.

2.3 Block-averaging and geometric scaling

Balaban's averaging [1, eq. (15)] and its linearization [1, eq. (125)]:

$$\bar{U}_c = \exp\left(i \sum_{x \in B(c_-)} L_{\text{RG}}^{-d} \log U(\Gamma_{c,x}) U(c)^{-1}\right) U(c), \quad (6)$$

$$(Q_0 A)_c = \sum_{x \in B(c_-)} L_{\text{RG}}^{-(d+1)} (R_{0,c_-} A)([x, x']). \quad (7)$$

Lemma 2.2 (Adjoint scaling). $\|Q_0^*\|^2 = L_{\text{RG}}^{-(d-1)}$; iterating, $\|Q_{(k)}^*\|^2 = L_{\text{RG}}^{-(d-1)k}$ ($= 2^{-3k}$ for $L_{\text{RG}} = 2$, $d = 4$). Nonlinear corrections are quadratic ([1, Prop. 4, eq. (133)]) and do not affect the leading scaling; functional derivatives of Q_k are bounded uniformly in k ([1, Prop. 5]):

$$\left| \frac{\delta}{\delta A_b} Q_k(U_0, \eta A, c) \right| \leq 1 + 2C'_1 \alpha_0 + C_3 \alpha_1. \quad (8)$$

2.4 Polymer derivative bounds (Proposition 2.3)

(v3: the v2 title of this subsection referred to a stale “Proposition 2.6”; the item is Proposition 2.3.)

Proposition 2.3 (Single-scale polymer derivative bound, small-field region). *Assume, beyond Theorem 2.1, that the analyticity radius $\hat{\alpha}_1 = \hat{\alpha}_1(\gamma) > 0$ can be chosen uniformly in k (cf. Remark A.1). Then on the small-field region Ω_k^{sf} :*

$$\sup_{\Omega_k^{\text{sf}}} \mathbb{E}[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k] \leq C_{\text{poly}}^2 e^{-2\kappa} L_{\text{RG}}^{-(d-1)k}, \quad (9)$$

via the absorption identity (5), Cauchy estimates on the analyticity domain,

$$\left| \frac{\delta R^{(k)}(X)}{\delta A'(b)} \right| \leq \frac{1}{\hat{\alpha}_1(\gamma)} \sup_{|A'| < \hat{\alpha}_1} |R^{(k)}(X)| \leq \frac{C(\gamma)}{\hat{\alpha}_1(\gamma)} e^{-\kappa d_k(X)}, \quad (10)$$

polymer counting (Lemma 5.6), and the adjoint scaling (Lemma 2.2). The extension to the full essential supremum,

$$\text{ess sup}_{\mathcal{G}_k} \mathbb{E}[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k] \leq C_{\text{poly}}^2 e^{-2\kappa} L_{\text{RG}}^{-(d-1)k}, \quad (11)$$

is the content of Assumption 5.4 (the boundary terms $B^{(k)}$ vanish on Ω_k^{sf} but on the large-field region their Cauchy bound grows like $O(k)$ and does not by itself carry the geometric factor).

Remark 2.4 (Essential supremum and large-field regions). The ess sup cannot be controlled by probabilistic suppression (small measure is not measure zero); on the large-field region a separate mechanism is needed for the geometric factor. This is part of (H-XSD).

3 Orbit space Ricci curvature

With the metric (1), the generators $\tilde{T}^a$ with $\text{tr}(\tilde{T}^a \tilde{T}^b) = -\frac{1}{2} \delta^{ab}$ are orthonormal and

$$f^{acd} f^{bcd} = N_c \delta^{ab}, \quad (12)$$

and $\text{Ric}_{\text{SU}(N_c)} = \frac{N_c}{4} g$ [27, Prop. 7.10].

Remark 3.1 (Metric convention; extended in v3). With $\langle X, Y \rangle = -N_c \text{tr}(XY)$ one gets $\text{Ric} = \frac{1}{2} g$; with $\langle X, Y \rangle = -\text{tr}(XY)$ (the convention of ai.viXra:2602.0036 v2) one gets $\text{Ric} = \frac{N_c}{2} g$. All three describe the same tensor; the companion suite verifies the three values numerically for $\text{SU}(2)$ and $\text{SU}(3)$, together with (12). In particular the $N_c/4$ used here is *not* an instance of the series’ Ricci erratum (which concerned the $-\text{tr}$ convention): the conventions are mutually consistent.

Theorem 3.2 ([9, Theorem 1.1]). *With (1), the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ satisfies $\text{Ric}_{\mathcal{B}} \geq \frac{N_c}{4}$.*

Proof. As in v2: (1) $\text{Ric}_{\mathcal{A}} \geq N_c/4$ from $\text{Ric} = -\frac{1}{4} B$ with $B(X, Y) = 2N_c \text{tr}(XY)$; (2) the Galaz-García-Kell-Mondino-Sosa quotient theorem [29] for RCD^* spaces passes the bound to $\mathcal{B} = \mathcal{A}/\mathcal{G}$ including the singular stratum; (3) $\text{RCD}^*(K, N)$ with $K > 0$ gives LSI with constant $\geq K$ [25, Cor. 5.7.2]. $\square$

Remark 3.3. The synthetic route handles $\mathcal{B}_{\text{sing}}$ automatically — an advantage over O’Neill-type arguments.

4 Terminal scale LSI

Theorem 4.1 (Terminal LSI). *At the terminal scale, $\mu_{n_{\max}}$ satisfies an LSI with $\alpha_{\text{term}} > 0$ depending only on N_c, γ, κ : the terminal orbit space has $O(1)$ dimension, $\text{Ric} \geq N_c/4$, the uniform measure satisfies*

$$\alpha_{\text{LSI}}(\nu) \geq \frac{N_c}{4}, \quad (13)$$

and $\text{osc}(f) \leq K/\gamma =: M_{\text{term}}$ for the terminal density, so Holley–Stroock [16] gives $\alpha_{\text{term}} \geq \frac{N_c}{4} e^{-M_{\text{term}}}$.

Remark 4.2 (LSI convention). We use $\text{Ent}_\mu(f^2) \leq \frac{2}{\alpha} \mathcal{E}(f, f)$; Bakry–Émery with $CD(K, \infty)$ gives $\alpha = K$ in this convention.

5 Multiscale entropy decomposition

5.1 Setup and filtration

Decreasing filtration $\mathcal{G}_0 \supset \dots \supset \mathcal{G}_{n_{\max}}$, $\mathcal{G}_k = \sigma(\bar{A}^{(k)})$; orthogonal decomposition $T_A \mathcal{A} = \bigoplus_k E_k \oplus V_{n_{\max}}$ with $\sum_k |\nabla_{E_k} f|^2 + |\nabla_{V_{n_{\max}}} f|^2 = |\nabla f|^2$.

5.2 Entropy chain rule

Lemma 5.1 (Entropy telescoping). *For $f > 0$:*

$$\text{Ent}_\mu(f) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}[\text{Ent}(\mathbb{E}[f \mid \mathcal{G}_k] \mid \mathcal{G}_{k+1})] + \text{Ent}_\mu(\mathbb{E}[f \mid \mathcal{G}_{n_{\max}}]). \quad (14)$$

(Guionnet–Zegarlinski [15, Prop. 4.1]; verified exactly on the suite’s model.)

5.3 Conditional fast LSI

Lemma 5.2. *There is $\alpha_0 > 0$ (independent of k, L_{vol}) with*

$$\text{Ent}(h^2 \mid \mathcal{G}_{k+1}) \leq \frac{2}{\alpha_0} \mathbb{E}[|\nabla_{E_k} h|^2 \mid \mathcal{G}_{k+1}] \quad (15)$$

for smooth gauge-invariant h , μ -a.e.: small-field conditional LSI from the fiber Ricci bound plus Balaban’s quadratic-form lower bound ([6, eqs. (1.2), (1.9)]), Bakry–Émery, then large-field patching by Holley–Stroock with oscillation bounded by the RG-block geometry (uniform in k, L_{vol}).

5.4 Influence coefficients and sweeping-out inequality

Definition 5.3 (Influence coefficients). $\Gamma_{k\ell} := \sup_\varphi \mathbb{E}[|\nabla_{E_k} \mathbb{E}[\varphi \mid \mathcal{G}_\ell]|^2]^{1/2} / \mathbb{E}[|\nabla_{E_\ell} \varphi|^2]^{1/2}$, $\Gamma_{k\ell} = 0$ for $\ell > k$.

Assumption 5.4 (Cross-scale derivative bound; hypothesis (H-XSD)). *For $0 \leq \ell < k \leq n_{\max} - 1$:*

$$\text{ess sup}_{\mathcal{G}_\ell} \mathbb{E}[|\nabla_{E_k} V_{<\ell}|^2 \mid \mathcal{G}_\ell] \leq C_{\text{poly}}^2 e^{-2\kappa} L_{\text{RG}}^{-(d-1)k}. \quad (16)$$

On the small-field region this follows from Proposition 2.3 composed across scales (Appendix A); the large-field extension is the assumption's content. **Status (v3):** asserted discharged in [10, 11, 12] (= ai.viXra:2602.0057/0056/0055); retained as an explicit hypothesis with that provenance (Remark 1.2).

Lemma 5.5 (Geometric decay of influence). *Under Assumption 5.4:*

$$\Gamma_{k\ell} \leq C_\Gamma e^{-\kappa} L_{\text{RG}}^{-(d-1)k/2}, \quad C_\Gamma = C_{\text{poly}}/\sqrt{\alpha_0}, \quad (17)$$

via the differentiation identity

$$v \mathbb{E}[\varphi \mid \mathcal{G}_\ell] = \mathbb{E}[v\varphi \mid \mathcal{G}_\ell] - \text{Cov}(\varphi, vV_{<\ell} \mid \mathcal{G}_\ell), \quad (18)$$

Jensen for the direct term, and Cauchy–Schwarz plus the conditional Poincaré for the indirect term.

Lemma 5.6 (Polymer counting). $\#\{X : X \ni b, d_k(X) = n\} \leq C_d^n$ with $C_d = 3^d - 1$; hence for $\kappa > \log C_d$:

$$\sum_{X \ni b} e^{-\kappa d_k(X)} \leq 1 + \frac{C_d e^{-\kappa}}{1 - C_d e^{-\kappa}} < \infty. \quad (19)$$

(The counting bound is verified by brute force for small polymers in the suite.)

Lemma 5.7 (Sweeping-out, bounded observables). With $g_k := \sqrt{\mathbb{E}[g^2 \mid \mathcal{G}_k]}$, for bounded smooth gauge-invariant g :

$$\mathbb{E}[|\nabla_{E_k} g_k|^2] \leq 2 \mathbb{E}[|\nabla_{E_k} g|^2] + 2 \sum_{\ell < k} \Gamma_{k\ell}^2 \mathbb{E}[|\nabla_{E_\ell} g|^2] + D_k \|g\|_\infty^2, \quad (20)$$

$D_k := C_{\text{poly}}^2 e^{-2\kappa} L_{\text{RG}}^{-(d-1)k}$, whence $\sum_k \mathbb{E}[|\nabla_{E_k} g_k|^2] \leq (2 + D_*) \mathcal{E}(g, g) + D_* \|g\|_\infty^2$ with $D_* \leq \frac{8}{7} C_{\text{poly}}^2 e^{-2\kappa}$ for $L_{\text{RG}} = 2, d = 4$.

Remark 5.8 (No entropy circularity; extension by truncation). The error is controlled by energy plus a pointwise bound, not by the entropy being estimated; extension to $\text{Dom}(\mathcal{E})$ by truncation and lower semicontinuity [25, Prop. 5.2.5].

5.5 Assembly: uniform LSI

Lemma 5.9 (Variance telescoping). With $m_k := \mathbb{E}[g \mid \mathcal{G}_k]$:

$$\text{Var}_\mu(g) = \sum_{k=0}^{n_{\text{max}}-1} \mathbb{E}[\text{Var}(m_k \mid \mathcal{G}_{k+1})] + \text{Var}_\mu(m_{n_{\text{max}}}). \quad (21)$$

Lemma 5.10 (Energy-only sweeping-out). For bounded smooth gauge-invariant g :

$$\sum_{k=0}^{n_{\text{max}}-1} \mathbb{E}[|\nabla_{E_k} m_k|^2] \leq C \mathcal{E}(g, g), \quad C = 2 + \frac{2}{\alpha_0} C_{\text{poly}}^2 e^{-2\kappa} (1 - L_{\text{RG}}^{-(d-1)})^{-1}, \quad (22)$$

using the conditional Poincaré (from Lemma 5.2) inside the covariance bound; no $\|g\|_\infty$ appears since the covariance involves g linearly.

Proof of Theorem 1.1(i). As in v2, Steps 1–8: entropy telescoping (Lemma 5.1); conditional LSI (Lemma 5.2); sweeping-out (Lemma 5.7); terminal scale (Theorem 4.1); the defect LSI

$$\text{Ent}_\mu(g^2) \leq C_E \mathcal{E}(g, g) + C_B \|g\|_\infty^2; \quad (23)$$

the uniform Poincaré, proved independently of the global LSI via Lemmas 5.9–5.10,

$$\text{Var}_\mu(g) \leq C_P \mathcal{E}(g, g), \quad C_P = \frac{2C}{\alpha_0} + \frac{2}{\alpha_{\text{term}}}; \quad (24)$$

Rothaus’ lemma [25, Prop. 5.1.3] converting defect LSI + Poincaré into the tight LSI

$$\text{Ent}_\mu(g^2) \leq \frac{2}{\alpha_*} \mathcal{E}(g, g); \quad (25)$$

and extension to all bounded g by the homogeneity

$$\text{Ent}_\mu(\lambda f) = \lambda \text{Ent}_\mu(f) \quad (\lambda > 0, f \geq 0), \quad (26)$$

then to $\text{Dom}(\mathcal{E})$ by Corollary 5.11. (v3: the entire pipeline — (14), (23) with recipe constants, (24), the Rothaus assembly and (26) — is validated on an exactly computable compact model in the companion suite, with the assembled α_* confirmed below the model’s true LSI constant obtained by direct minimization.) $\square$

Corollary 5.11 (Extension to $\text{Dom}(\mathcal{E})$). *By truncation $f^{(m)} = (-m) \vee f \wedge m$ and lower semicontinuity [25, Prop. 5.2.5], the tight LSI extends with the same α_* .*

6 From uniform LSI to mass gap

6.1 Reversible diffusion and its Dirichlet form

$$Lf := \sum_{\ell} (\Delta_{\ell} f - \langle \nabla_{\ell} S_{\beta}, \nabla_{\ell} f \rangle), \quad (27)$$

$$\mathcal{E}(f, f) = - \int f Lf d\mu_{\beta} = \int |\nabla f|^2 d\mu_{\beta}, \quad P_t = e^{tL}. \quad (28)$$

6.2 Hypercontractivity and spectral gap

Proposition 6.1. *The uniform LSI implies (i) hypercontractivity (Gross [14]); (ii) $\lambda_1 \geq \alpha_*$ and $\|P_t f - \mathbb{E}f\|_2 \leq e^{-\lambda_1 t} \|f - \mathbb{E}f\|_2$.*

6.3 Exponential clustering

(H-DOB) Site-to-site Dobrushin contraction (reconstructed in v3). *The site-to-site influence matrix C_{xy} of Lemma 6.4 satisfies $\sup_x \sum_{y \neq x} C_{xy} < 1$; by Lemmas 6.2–6.4, a sufficient condition is*

$$\kappa > \log \left[(MR_{n_{\max}})^d C_{\Gamma} \frac{r}{(1-r)^2} \right], \quad r = L_{\text{RG}}^{-(d-1)/2}. \quad (\text{H-DOB})$$

This is the hypothesis that v2 referenced as a dangling “Assumption ??” after withdrawing its earlier Dobrushin-translation form (Remark 6.3); v3 states it explicitly. See Remark 6.6 for its quantitative status.

Lemma 6.2 (Quantified contraction condition). *With $\Gamma_{k\ell} \leq C_\Gamma e^{-\kappa} r^k$: $D_{\text{row}} := \sup_k \sum_{\ell < k} \Gamma_{k\ell} \leq C_\Gamma e^{-\kappa} \frac{r}{(1-r)^2}$ (using $kr^k \leq \sum_j jr^j$; the elementary bound is machine-verified). For $L_{\text{RG}} = 2$, $d = 4$: $r = 2^{-3/2}$, $\frac{r}{(1-r)^2} \approx 0.846$.*

Remark 6.3 (Withdrawn assumption; superseded route). The Dobrushin-translation assumption of v1 is withdrawn; [13] asserts that the DLR-LSI extension plus Stroock–Zegarlinski equivalence, with reflection positivity, yields the gap without it. (v3: within the present paper the clustering chain still requires (H-DOB) above; v2’s text claimed unconditionality here while assuming the withdrawn item in Theorem 6.5 and §6.4 — repaired.)

Lemma 6.4 (From multiscale influence to a site-to-site Dobrushin matrix). *With*

$$C_{xy} := \sum_{k=0}^{n_{\max}} \sum_{\ell=0}^{k-1} \Gamma_{k\ell} \mathbf{1}_{x \in \text{Block}_k} \mathbf{1}_{y \in \text{Block}_\ell}, \quad (29)$$

one has $\sup_x \sum_y C_{xy} \leq (\sup_k |\text{Block}_k|) D_{\text{row}} \leq (MR_{n_{\max}})^d D_{\text{row}}$; hence (H-DOB) holds if κ exceeds the threshold displayed above.

Theorem 6.5 (Clustering; conditional on (H-DOB)). *Assume the uniform LSI and (H-DOB). Then for gauge-invariant O_A, O_B at distance r : $|\langle O_A O_B \rangle_c| \leq C \|O_A\|_\infty \|O_B\|_\infty e^{-r/\xi}$, $\xi = O(1/\sqrt{\alpha_*})$, uniformly in L_{vol} : finite-range structure plus polymer tails (Lemma 5.6), Dobrushin–Shlosman complete analyticity [23, 24] under the site-to-site contraction, and gauge-invariant conditioning.*

Remark 6.6 (New in v3: the (H-DOB) window). (H-DOB) has the same structure as hypothesis (H-CONST) of ai.viXra:2602.0032 (v2): the left side κ is a *fixed* output of Balaban’s expansion, while the right side grows with β — both through $(MR_{n_{\max}})^d$ and through $C_\Gamma \propto C_{\text{poly}} \propto n_{\max} \sim 1/(2b_0 g^2 \ln 2)$ (Appendix A.3). Numerically (suite): already for $(MR)^d = 16^4$ the threshold is $\kappa \gtrsim 12$ –14 for $g^2 \in [0.05, 0.5]$, growing logarithmically as $g^2 \rightarrow 0$. Hence the present paper’s clustering route is guaranteed only on a window $\beta_0 \leq \beta \leq \beta_1(\kappa)$, unless the constants of Balaban’s construction are favorable; outside the window one must rely on the asserted route of [13]. This does not affect Theorem 1.1(i) (the LSI itself needs no (H-DOB)), nor uniformity in L_{vol} at fixed admissible β .

6.4 Mass gap via reflection positivity

Proof of Theorem 1.1(ii). Assume (H-DOB). Reflection positivity of the Wilson action [21] gives the spectral representation $\langle O(t) O(0) \rangle_c = \sum_{n \geq 1} |\langle \Omega | \hat{O} | n \rangle|^2 e^{-E_n t}$; by Theorem 6.5 the correlator decays at rate $M = 1/\xi > 0$, forcing $E_1 \geq M$; taking $O = O_P = \frac{1}{N_c} \text{Re tr } U_P$ (which has positive variance) yields $\Delta_{\text{phys}} = E_1 \geq M > 0$. $\square$

7 Discussion

7.1 What is proved

Uniform LSI (Theorem 1.1(i)), conditional on (H-XSD) with the provenance of Remark 1.2; mass gap $\Delta_{\text{phys}} > 0$ uniform in L_{vol} , conditional on (H-DOB) (Theorem 1.1(ii)); the companion [13] asserts removal of (H-DOB).

7.2 What is not proved

Continuum limit and OS axioms; persistence of the gap as $a \rightarrow 0$; the strong-coupling regime (covered by [21]).

7.3 On the expected scaling $m_{\text{gap}} \sim e^{-C/g^2}$

We do not establish the expected scaling: $\alpha_{\text{term}} \geq \frac{N_c}{4} e^{-M_{\text{term}}}$ with $M_{\text{term}} = O(1/\gamma)$, and tracking α_* through $n_{\text{max}} \sim 1/(2b_0 g^2 \ln 2)$ steps is beyond this paper.

7.4 Relation to the Millennium Problem

Infrared component only; the ultraviolet construction is beyond scope.

7.5 Conditional aspects and the logical structure

(1) Balaban’s RG (published); (2) (H-XSD) with the audit trail of Appendix A and the asserted discharges [10, 11, 12]; (3) (H-DOB) with the explicit threshold of Lemma 6.4 and the window caveat of Remark 6.6; (4) $\text{Ric}_{\mathcal{B}} \geq N_c/4$ via RCD* quotient [29]; (5) the sweeping-out/assembly steps, non-circular, validated numerically. All conditions require γ small; how small is determined by Balaban’s constants and the thresholds here.

7.6 The role of each input

As in v2’s table, with identifiers: cross-scale bounds [10] (=2602.0057); large-field suppression [11] (=2602.0056); residual derivatives [12] (=2602.0055); LSI→gap route [13] (=2602.0054); Ricci source [9] (=2602.0046).

Note added in v3

Prepared with the companion suite `verify_2602.0041.py` (repository `THE-ERIKSSON-PROGRAMME, verification/2602-0041/`; 11 checks, all passing, ~ 7 s, NumPy/SciPy). Changes relative to v2: (1) the four dangling “Assumption ??” references (abstract, §1.3, Theorem 6.5, §6.4, §7.1) are repaired by reconstructing the hypothesis explicitly as (H-DOB) (§6.3), and the internal inconsistency about the status of Theorem 1.1(ii) (claimed unconditional while §6.4/§7.1 assumed the withdrawn item) is resolved: within this paper the gap is conditional on (H-DOB), with [13] asserting its removal; (2) new Remark 6.6: the (H-DOB) threshold has the (H-CONST) pattern of ai.viXra:2602.0032 (v2) — fixed κ against a right-hand side growing with β through $(MR_{n_{\text{max}}})^d$ and $C_{\Gamma} \propto n_{\text{max}}$ — so the paper’s own clustering route holds on a β -window (numerically exhibited); (3) Theorem 2.1(e): coupling-flow sign/normalization corrected (series erratum; n_{max} magnitude unchanged); (4) Assumption 5.4 tagged (H-XSD) with honest provenance (“asserted discharged in the companions, not re-verified here”) and companion identifiers filled in ([9]–[13] = ai.viXra:2602.0046/0057/0056/0055/0054); (5) the §2.4 title’s stale “Proposition 2.6” corrected; (6) Remark 3.1 extended: the three metric conventions verified numerically to be mutually consistent (this paper’s $N_c/4$ is correct in its convention; coherent with the sharp $N_c/2$ of ai.viXra:2602.0036 v2); (7) the entropy pipeline (chain rule, defect LSI with recipe constants, independent uniform Poincaré, Rothaus assembly, homogeneity) validated end-to-end on an exactly computable compact model, with the assembled

α_* confirmed below the model’s true LSI constant; Lemma 5.6’s counting bound verified by brute force; Lemma 6.2’s elementary bound machine-checked. All v2 numbered statements and equations keep their numbers; (H-DOB) is deliberately an unnumbered tagged hypothesis to avoid renumbering.

A Audit trail: derivation of the cross-scale derivative bound

As in v2 (Steps 1–3, small-field region): single-scale bound (9); propagation across intermediate scales via the k -fold averaging bounds ([1, Props. 4, 5, 7], eq. (146)); composition with the factor $(k - \ell + 1) \leq n_{\max}$ absorbed into $C_{\text{poly}} = n_{\max}(\beta_0)C'/\hat{\alpha}_1(\gamma)$ — which is the source of the β -growth of C_Γ recorded in Remark 6.6. The large-field extension is (H-XSD).

Remark A.1. Balaban proves k -uniform analyticity and derivative bounds for the averaging maps and states k -uniform inductive conditions for effective densities; our audit assumes the analyticity parameter $\hat{\alpha}_1(\gamma)$ can be chosen uniformly in k . Extracting a standalone lemma with all parameters is deferred (and asserted in [12]).

Remark A.2. The factor $n_{\max}$ in C_{poly} grows polynomially in $\log(1/g^2)$; it does not affect positivity of α_* at fixed β_0 , only its quantitative value — and, via C_Γ , the (H-DOB) threshold (Remark 6.6).

B Numerical verification (new in v3)

`verify_2602.0041.py`: (1) Ricci convention coherence: $N_c/4$ (-2 tr), $N_c/2$ ($-\text{tr}$), $\frac{1}{2}$ ($-N_c \text{ tr}$), and (12), verified for $\text{SU}(2)$, $\text{SU}(3)$; (2) entropy pipeline on a four-rotor model (5^4 states): chain rule (14) exact; defect LSI (23) with recipe constants over 300 random observables; uniform Poincaré (24) valid against the true gap; assembled α_* below the true LSI constant (direct minimization of $2\mathcal{E}/\text{Ent}$ with analytic gradients); homogeneity (26); (3) Lemma 6.2 exact and the (H-DOB) window exhibit ($\kappa_{\text{needed}} \approx 12\text{--}14$, growing as $g^2 \rightarrow 0$); (4) Lemma 5.6 by brute-force polyomino counting; (5) corrected coupling flow and the geometric sums for D_* . Run: `python3 verify_2602.0041.py` (exit 0 iff all pass).

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