Uniform Log-Sobolev Inequality and Mass Gap for Lattice Yang–Mills Theory

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Abstract

We establish that $SU(N_c)$ lattice Yang–Mills theory in $d = 4$ dimensions with Wilson action at sufficiently weak coupling ($\beta = 2N_c/g^2 \geq \beta_0$) satisfies a log-Sobolev inequality with constant $\alpha_* > 0$ uniform in the lattice size L_{vol}. Combined with reflection positivity of the Wilson action and the DLR-LSI extension plus Stroock–Zegarlinski mixing route developed in [13], this yields a mass gap $\Delta_{\text{phys}} > 0$ uniform in L_{vol} without additional assumptions.

Note added (v2). In v1 of this paper, the uniform LSI (Theorem 1.1(i)) was conditional on Assumption 5.4. This assumption has since been verified in the companion papers [10, 11, 12]: the integrated cross-scale derivative bounds are established in [10] (using Hypotheses 3.2 and 4.2 therein), Hypothesis 4.2 is proved in [11] via an interface with Balaban's renormalization group, and Hypothesis 3.2 is verified in [12] using polymer locality, Cauchy estimates on analytic domains, and lattice-animal counting bounds. The uniform LSI is therefore unconditional in the present version. Moreover, the mass gap is now unconditional via the DLR-LSI boundary extension and Stroock–Zegarlinski equivalence approach proved in [13], which bypasses Assumption **??**.

The proof combines three ingredients: (i) Balaban's constructive renormalization group, which produces controlled effective actions at all scales; (ii) the orbit space Ricci curvature bound $\text{Ric}_{\mathcal{B}} \geq N_c/4$, which gives a uniform log-Sobolev constant for conditional measures of fast modes at each RG scale via the Bakry–Émery criterion; and (iii) a multiscale entropy decomposition with sweeping-out bounds, where the geometric scaling factor $\|Q^*_{(k)}\|^2 = 2^{-(d-1)k}$ of transversal block averaging ensures summability of cross-scale errors.

A key technical point is that Balaban's RG "absorption" mechanism isolates the dependence on scale-k fast variables into polymer residuals with exponential decay. Under k-uniform analyticity/derivative inputs from Balaban's scheme (made explicit as Assumption 5.4), Cauchy-type estimates yield the derivative bounds needed for the sweeping-out inequality. We provide a detailed audit trail from Balaban's published work (Comm. Math. Phys. **98**, **116**, **122**) supporting Assumption 5.4; see Appendix A.

Contents

Metric convention. Throughout this paper, $SU(N_c)$ is equipped with the bi-invariant metric

$$\langle X, Y \rangle := -2 \operatorname{tr}(XY), \tag{1}$$

where tr is the trace in the fundamental representation and $X, Y \in \mathfrak{su}(N_c)$ are anti-Hermitian. This gives the Einstein constant $\mathrm{Ric}_{SU(N_c)} = \frac{N_c}{4} g_{SU(N_c)}$, scaling linearly with N_c. See Remark 3.1 for the relation to other conventions.

Coupling convention. Throughout, $\beta = 2N_c/g^2$ denotes the coefficient of the Wilson action. The running coupling β_k in Theorem 2.1(e) uses this same normalization: $\beta_0 = \beta$. The companion papers [10, 11, 12] also employ the *reduced* inverse coupling $\beta_k^{\text{red}} := g_k^{-2} = \beta_k/(2N_c)$. Since this is a fixed rescaling, it affects only numerical constants and never the volume-uniformity of any bound.

1 Introduction

1.1 The problem

The existence of a positive mass gap in four-dimensional $SU(N_c)$ Yang–Mills theory is one of the Clay Millennium Prize Problems [17]. On the lattice $\Lambda = (\mathbb{Z}/L_{\text{vol}}\mathbb{Z})^d$ with Wilson action

$S_W[U] = \beta \sum_P (1 - \frac{1}{N_c} \mathrm{Re\ tr}\, U_P)$, $\beta = 2N_c/g^2$, the mass gap is the spectral gap of the transfer matrix.

At strong coupling ($\beta < \beta_0$), the mass gap is established by the Osterwalder–Seiler cluster expansion [21]. At weak coupling ($\beta \geq \beta_0$), the problem has remained open for 40 years.

1.2 Main result

Theorem 1.1 (Main Theorem)**.** *For $SU(N_c)$ lattice Yang–Mills theory on $(\mathbb{Z}/L_{\mathrm{vol}}\mathbb{Z})^4$ with Wilson action at $\beta = 2N_c/g^2 \geq \beta_0$ (sufficiently large):*

(i) ***(Uniform LSI; see Remark 1.2 for provenance)*** *There exists $\alpha_* > 0$, depending on N_c and β_0 but not on L_{vol}, such that for all gauge-invariant $f \in L^2(\mu_\beta)$:*

$$\mathrm{Ent}_{\mu_\beta}(f^2) \leq \frac{2}{\alpha_*} \int |\nabla f|^2 \, d\mu_\beta. \tag{2}$$

(ii) ***(Mass gap; unconditional)*** *The transfer matrix T has spectral gap*

$$\Delta_{\mathrm{phys}} \geq c(N_c, \beta_0) > 0,$$

uniformly in L_{vol}. This is proved unconditionally in [13] via the chain periodic uniform LSI $\Rightarrow$ DLR-LSI $\Rightarrow$ Stroock–Zegarlinski mixing/clustering $\Rightarrow$ reflection positivity $\Rightarrow$ gap.

The proof uses Balaban's constructive renormalization group [1, 2, 3, 4, 5, 6], the orbit space Ricci curvature bound $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$*, and the multiscale entropy decomposition with sweeping-out bounds. Appendix A provides an audit trail supporting Assumption 5.4.*

Remark 1.2 (Status of Assumption 5.4). Assumption 5.4 has been verified in subsequent work. The companion paper [10] replaces Assumption 5.4 by two weaker inputs: a pointwise gradient bound (Hypothesis 3.2 therein) and a conditional large-field suppression (Hypothesis 4.2 therein), and proves that these suffice for the integrated cross-scale derivative bounds via a small-field/large-field decomposition. Hypothesis 4.2 is verified in [11] using Balaban's T-operation and uniformity estimates. Hypothesis 3.2 is verified in [12] using polymer locality. As a consequence, Theorem 1.1(i) is unconditional. Theorem 1.1(ii) is also unconditional: see [13].

1.3 Logical structure

The proof follows the chain:

$$\boxed{\text{Balaban RG} + \mathrm{Ric}_{\mathcal{B}} \geq \tfrac{N_c}{4} \xrightarrow{\text{entropy decomposition}} \alpha_*^{\mathrm{LSI}} > 0 \xrightarrow{\text{RP + locality}} \Delta_{\mathrm{phys}} > 0}$$

The first arrow is unconditional: Assumption 5.4 has been verified in [10, 11, 12] (see Remark 1.2). The second arrow is also unconditional: it is superseded by the DLR-LSI plus Stroock–Zegarlinski route of [13], which yields exponential clustering and a transfer-matrix gap without Assumption **??**.

Each arrow is established by combining published results with the new ingredients of this paper:

- **Arrow 1** (§4–§5): Terminal LSI via Holley–Stroock, conditional fast LSI via Bakry–Émery, sweeping-out inequality from Balaban's polymer bounds, assembly into uniform LSI.
- **Arrow 2** (§6): LSI $\Rightarrow$ hypercontractivity $\Rightarrow$ exponential clustering $\Rightarrow$ mass gap via reflection positivity.

1.4 Relation to the Clay Millennium Problem

This result establishes the mass gap on the lattice at weak coupling, uniform in the volume. The Clay Millennium Problem additionally requires the construction of a continuum QFT satisfying the Osterwalder–Schrader axioms with a mass gap persisting as the lattice spacing $a \to 0$. Our result is a necessary ingredient for, but does not complete, this program.

2 Inputs from Balaban's constructive RG

2.1 Effective actions with polymer bounds

We use the following result of Balaban.

Theorem 2.1 (Balaban [6, Theorem 1])**.** *For $SU(N_c)$ lattice gauge theory on $\Lambda = (\mathbb{Z}/L_{\mathrm{vol}}\mathbb{Z})^4$ with $\beta = 2N_c/g^2 \geq \beta_0$: there exist effective actions S_k on coarsened lattices $\Lambda_k = (\mathbb{Z}/L_{\mathrm{vol},k}\mathbb{Z})^4$, $0 \leq k \leq n_{\max}$, where $L_{\mathrm{vol},k} := L_{\mathrm{vol}}/L_{\mathrm{RG}}^k$ denotes the linear volume size at scale k. Assume (for simplicity) that L_{vol} is divisible by $L_{\mathrm{RG}}^{n_{\max}}$, so that $L_{\mathrm{vol},k} \in \mathbb{Z}$ for all $0 \leq k \leq n_{\max}$. Then:*

(a) $Z(\Lambda, S_0) = Z(\Lambda_k, S_k)$ for all k.

(b) The effective action has the structure ([6, eq. (1.101), p. 388])*:*

$$S_k = \beta_k S_W + \sum_X \mathbf{R}^{(k)}(X) + \sum_X \mathbf{B}^{(k)}(X), \tag{3}$$

where $\mathbf{R}^{(k)}(X)$ are irrelevant polymer activities and $\mathbf{B}^{(k)}(X)$ are boundary terms near large-field regions.

(c) The polymer activities satisfy the exponential decay bound ([6, eq. (1.100), p. 388])*:*

$$|\mathbf{R}^{(k)}(X, (\mathbf{U}, \mathbf{J}))| \leq e^{-p_0(g_k)} e^{-\kappa\, d_k(X)}, \tag{4}$$

where $d_k(X) := |X|$ denotes the number of scale-k blocks in the polymer X, and $\kappa > 0$ is a universal decay constant.

(d) The polymer activities are analytic functions of the field variables on a complex domain $\widetilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, eq. (1.65), p. 383])*.*

(e) The coupling evolution satisfies $\beta_k = \beta + 2b_0 k \ln 2 + O(1/\beta)$ with $b_0 = 11N_c/(48\pi^2)$.

2.2 The absorption property

The central structural property of Balaban's RG is the *absorption of cross-scale coupling*. At each RG step, the Wilson action term coupling fine and coarse variables is decomposed and reabsorbed into the next-scale effective Wilson action. The key identity is [6, eq. (1.37), p. 367]:

$$\begin{aligned} &-\frac{1}{g_k^2} A(\zeta_0, U_{k,Z}(V_k' V_\Lambda)) - A\left(\frac{1}{(g_k''(\cdot))^2}, U_k''\right) + \frac{1}{g_k^2} A(\zeta_0, U_0) \\ &\qquad = -A\left(\frac{1}{(g_k(\cdot))^2}, U_k\right) + O(1), \end{aligned} \tag{5}$$

where the "$O(1)$" terms are explicitly bounded by absolute constants (independent of g_k, L_{vol}, and k) in the text following eq. (1.37) of [6]. After this absorption, the dependence on scale-k variables enters only through the polymer residuals $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$.

2.3 Block-averaging and geometric scaling

The block-averaging operation is defined in [1, eq. (15), p. 21]:

$$\bar{U}_c = \exp\left[i \sum_{x \in B(c_-)} L_{\mathrm{RG}}^{-d} \frac{1}{i} \log U(\Gamma_{c,x}) U(c)^{-1} \right] U(c). \tag{6}$$

Its linearisation is given by [1, eq. (125), p. 40]:

$$(Q_0 A)_c = \sum_{x \in B(c_-)} L_{\mathrm{RG}}^{-(d+1)} (R_{0,c_-} A)([x, x']). \tag{7}$$

This averages over L_{RG}^{d-1} fine links in a transversal section. The adjoint distributes uniformly:

Lemma 2.2 (Adjoint scaling)**.** *Let $L_{\mathrm{RG}} > 1$ denote the fixed RG block size (Balaban's parameter L). The linearised block-averaging operator Q_0 defined in* (7) *satisfies $\|Q_0^*\|^2 = L_{\mathrm{RG}}^{-(d-1)}$ as an operator on $\mathfrak{su}(N_c)$-valued fields. Iterating k steps: $\|Q_{(k)}^*\|^2 = L_{\mathrm{RG}}^{-(d-1)k}$. In our dyadic RG, $L_{\mathrm{RG}} = 2$, $d = 4$, hence $\|Q_{(k)}^*\|^2 = 2^{-3k}$.*

Proof. The non-linear averaging (6) involves the group exponential and logarithm, but its linearisation Q_0 ([1, eq. (125)]) is a linear operator on $\mathfrak{su}(N_c)$-valued bond fields. For a fixed coarse bond c in direction μ, the operator $(Q_0 A)_c$ averages over the L_{RG}^{d-1} fine links in the transversal $(d-1)$-dimensional block $B(c_-)$, each contributing with weight $L_{\mathrm{RG}}^{-(d-1)}$ (the factor $L_{\mathrm{RG}}^{-(d+1)}$ in (7) accounts for $L_{\mathrm{RG}}^{-(d-1)}$ from transversal averaging and L_{RG}^{-2} from the path length). The adjoint Q_0^* distributes a coarse-bond perturbation uniformly among the $n = L_{\mathrm{RG}}^{d-1}$ transversal fine links, giving $\|Q_0^* t\|^2 = n|t/n|^2 = |t|^2/n$.

The non-linear corrections to this scaling are controlled by [1, Proposition 4, eq. (133)]: the full non-linear k-fold averaging $Q_k(U_0, \eta A)$ satisfies $|Q_k(U_0, \eta A) - Q_k(U_0) A| < C_2 |A|^2$, where $Q_k(U_0)$ is the composition of linearised operators. The non-linear remainder is quadratic in A and does not affect the leading scaling. Thus $\|Q_{(k)}^*\|^2 = L_{\mathrm{RG}}^{-(d-1)k}$ for the linearised operator, which governs the scaling of the sweeping-out coefficients. □

The uniform bounds on functional derivatives of the k-fold averaging are given by [1, Proposition 5, eqs. (156)–(157), p. 43]:

$$\left| \frac{\delta}{\delta A_b} Q_k(U_0, \eta A, c) \right| \leq 1 + 2C_1' \alpha_0 + C_3 \alpha_1, \tag{8}$$

with constants independent of k.

2.4 Polymer derivative bounds (Proposition 2.6)

We now record the key single-scale derivative bound needed for the sweeping-out estimate. Its derivation uses Balaban's published polymer analyticity framework together with a k-uniform analyticity-radius input, which we isolate as part of Assumption 5.4. The resulting cross-scale composition is recorded in Assumption 5.4 and is supported by the audit trail in Appendix A.

Proposition 2.3 (Single-scale polymer derivative bound on the small-field region)**.** *Assume, in addition to the hypotheses of Theorem 2.1, that the polymer analyticity domain $\widetilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ in [6, eq. (1.65), p. 383] can be chosen with $\hat{\alpha}_1 = \hat{\alpha}_1(\gamma) > 0$ uniformly in k throughout Balaban's inductive scheme (cf. Remark A.1).*

Under the hypotheses of Theorem 2.1, the effective potential $V_{<k}$ at scale k satisfies: after Balaban's RG absorption, the dependence on scale-k fast variables enters only through polymer residuals. On the small-field region Ω_k^{sf}:

$$\sup_{\Omega_k^{\mathrm{sf}}} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \le C_{\mathrm{poly}}^2\, e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k}. \tag{9}$$

The extension to the full essential supremum (including the large-field region) is the content of Assumption 5.4:

$$\operatorname*{ess\,sup}_{\mathcal{G}_k} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \le C_{\mathrm{poly}}^2\, e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k}. \tag{10}$$

Here L_{RG} denotes the fixed RG block size (Balaban's parameter $L > 1$); in our dyadic RG one has $L_{\mathrm{RG}} = 2$, hence $L_{\mathrm{RG}}^{-(d-1)k} = 2^{-(d-1)k}$. The constants $C_{\mathrm{poly}}, \kappa > 0$ are the universal constants from Balaban's construction.

Proof. The proof combines three ingredients from Balaban's papers.

Step 1: Structure. By the absorption identity (5) ([6, eq. (1.37)]), the Wilson action cross-coupling is absorbed into $\beta_{k+1} S_W$, and the residual dependence on scale-k variables is through the polymer terms $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$ in (3).

Step 2: Uniform analytic radius and derivative bounds. Balaban's Theorem 1 [6] is proved under the hypothesis that all effective couplings satisfy $g_k^2 \le \gamma$ for a fixed small $\gamma > 0$. Within this regime, the polymer activities $\mathbf{R}^{(k)}(X)$ are analytic functions of the gauge field variables on the complex domain $\widetilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, eq. (1.65), p. 383]), where the analytic radius $\hat{\alpha}_1 = \hat{\alpha}_1(\gamma) > 0$ depends only on γ and is *uniform in* k. This uniformity rests on two explicit results in Balaban's work:

(i) *Uniform averaging maps.* The k-fold averaging map Q_k admits an analytic expansion with constants independent of k ([1, Proposition 4]), its functional derivatives satisfy bounds with constants independent of k ([1, Proposition 5]; cf. our (8)), and these bounds persist uniformly under small complex perturbations of the background field ([1, Proposition 7]).

(ii) *Uniform inductive scheme.* Provided the effective couplings remain in $(0, \gamma]$, the effective densities p_k satisfy *uniformly in* k all inductive analyticity conditions and bounds of the effective-action framework (Theorem 1 in [6], referring to the conditions described in Sect. 2 of [4]). In particular, the polymer activities appearing after absorption are analytic on a complex neighbourhood whose radius depends only on γ and fixed geometric parameters of the RG step, not on k.

By the Cauchy integral formula applied within this uniform analytic domain, combined with the polymer bound (4):

$$\left|\frac{\delta \mathbf{R}^{(k)}(X)}{\delta A'(b)}\right| \le \frac{1}{\hat{\alpha}_1(\gamma)} \sup_{|A'|<\hat{\alpha}_1} |\mathbf{R}^{(k)}(X)| \le \frac{C(\gamma)}{\hat{\alpha}_1(\gamma)}\, e^{-\kappa\, d_k(X)}, \tag{11}$$

where $C(\gamma) = e^{-p_0(g_k)}$ from (4). Since $\hat{\alpha}_1(\gamma)$ is a fixed positive constant (not depending on k), the derivative bound has the same exponential decay as the polymer bound itself.

Step 2b: Boundary terms $\mathbf{B}^{(k)}(X)$ (small-field contribution only). The boundary terms in (3) are localised near the large-field region Z_k and satisfy the bound ([6, eq. (1.69), p. 377]):

$$|\mathbf{B}^{(k)}(X)| \le O(1) \sum_{j=1}^{k} |\Gamma_j^0 \cap X|$$

when X is a component of $Z^{\sim}$ (the enlarged large-field region).

On the *small-field* region Ω_k^{sf}, the boundary terms $\mathbf{B}^{(k)}(X)$ vanish by definition (they are supported on $Z_k^{\sim}$), so the Cauchy derivative bound from Step 2 applies without modification and yields:

$$|\nabla_{E_k} V_{<k}^{\mathrm{sf}}(b)|^2 \leq C_{\mathrm{poly}}^2 \, e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k}$$

uniformly on Ω_k^{sf}.

On the *large-field* region $Z_k^{\sim}$, the boundary terms $\mathbf{B}^{(k)}(X)$ are analytic on the domain $\widetilde{U}_k^c(X, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, p. 377]), and the Cauchy estimate gives a pointwise derivative bound $|\nabla \mathbf{B}^{(k)}(X)| \leq C_B(\gamma, k)/\hat{\alpha}_1 \cdot |X|$ with $C_B(\gamma, k) = O(k)$. However, this bound grows with k and does not by itself yield the geometric decay factor $L_{\mathrm{RG}}^{-(d-1)k}$ needed in (10). The full control of the large-field contribution to the essential supremum—specifically, that the large-field derivative bound also carries the factor $L_{\mathrm{RG}}^{-(d-1)k}$—is subsumed into Assumption 5.4. See Appendix A for a discussion of the evidence from Balaban's construction.

Remark 2.4 (Essential supremum and large-field regions). The essential supremum in (10) cannot be controlled by probabilistic suppression arguments: ess sup ignores sets of measure zero but not sets of small positive measure. On the small-field region, the Cauchy estimate from Step 2 provides the required pointwise bound. On the large-field region, a separate mechanism is needed to establish the geometric decay factor; this is part of the content of Assumption 5.4.

Step 3: Geometric scaling via block-averaging adjoint. The dependence of $V_{<k}$ on scale-k fast variables arises through the polymer residuals, whose gradients are bounded by Steps 2 and 2b. When projected onto the fast subspace E_k, the gradient picks up the block-averaging adjoint factor. By the linearisation of Balaban's block-averaging ([1, eq. (125)]) and the composition bounds ([1, Proposition 4]), perturbations in the fast directions at scale k contribute to the polymer gradient with an additional factor $\|Q_{(k)}^*\| = L_{\mathrm{RG}}^{-(d-1)k/2}$ per component. Squaring:

$$\mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \leq \Big(\sum_{X \ni \text{fast link}} \frac{C(\gamma)}{\hat{\alpha}_1} \, e^{-\kappa \, d_k(X)} \Big)^2 \cdot \|Q_{(k)}^*\|^2.$$

The sum over polymers containing a given fast link is bounded by $C' e^{-\kappa}$ (Lemma 5.6). Thus, on the small-field region:

$$\sup_{\Omega_k^{\mathrm{sf}}} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \leq C_{\mathrm{poly}}^2 \, e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k},$$

with $C_{\mathrm{poly}} = C'(\gamma)/\hat{\alpha}_1(\gamma)$. The extension to $\operatorname{ess\,sup}_{\mathcal{G}_k}$ (including the large-field region) is the content of Assumption 5.4. □

3 Orbit space Ricci curvature

The configuration space $\mathcal{A} = \prod_\ell SU(N_c)$ carries the product bi-invariant metric (1) on each factor. With this normalisation, the standard generators $\tilde{T}^a$ satisfying $\mathrm{tr}(\tilde{T}^a \tilde{T}^b) = -\frac{1}{2}\delta^{ab}$ form an orthonormal basis, and the structure constants satisfy

$$f^{acd} f^{bcd} = N_c \, \delta^{ab}. \tag{12}$$

The Ricci curvature of each factor is $\mathrm{Ric}_{SU(N_c)} = \frac{N_c}{4} \, g_{SU(N_c)}$ (see [27, Proposition 7.10]).

Remark 3.1 (Metric convention). Some references use $\langle X, Y \rangle = -N_c \, \mathrm{tr}(XY)$. With that convention, the orthonormal generators are $T^a = \tilde{T}^a / \sqrt{N_c/2}$, the rescaled structure constants satisfy $\hat{f}^{acd} \hat{f}^{bcd} = 2\,\delta^{ab}$, and $\mathrm{Ric} = \frac{1}{2}\, g$. The two conventions yield different numerical constants but are physically equivalent. We use (1) throughout because it gives the Einstein constant $N_c/4$ that scales linearly with N_c.

Theorem 3.2 ([9, Theorem 1.1])**.** *With the metric convention* (1)*, the orbit space* $\mathcal{B} = \mathcal{A}/\mathcal{G}$ *satisfies* $\mathrm{Ric}_{\mathcal{B}} \geq \frac{N_c}{4}$.

Proof. The argument proceeds in three steps: Ricci curvature of $\mathcal{A}$, passage to the quotient via the synthetic Riemannian theory, and treatment of the singular stratum.

Step 1: Ricci curvature of $\mathcal{A}$. With the metric (1), the Killing form of $\mathfrak{su}(N_c)$ is $B(X,Y) = 2N_c\ \mathrm{tr}(XY)$ (for anti-Hermitian matrices in the fundamental representation). For the bi-invariant metric $g = -2\ \mathrm{tr}(XY)$, the standard formula $\mathrm{Ric} = -\frac{1}{4}B$ gives:

$$\mathrm{Ric}(X,Y) = -\frac{1}{4} \cdot 2N_c\ \mathrm{tr}(XY) = \frac{N_c}{4} \cdot \big(-2\ \mathrm{tr}(XY)\big) = \frac{N_c}{4}\, g(X,Y).$$

The product metric gives $\mathrm{Ric}_{\mathcal{A}} = \bigoplus_\ell \frac{N_c}{4}\, g_\ell$, hence $\mathrm{Ric}_{\mathcal{A}} \geq \frac{N_c}{4}\, g_{\mathcal{A}}$.

Step 2: Passage to the quotient via RCD.* The gauge group $\mathcal{G} = \prod_{x\in\Lambda} SU(N_c)$ acts on $\mathcal{A}$ by isometries (since $|uXu^{-1}| = |X|$ for the bi-invariant norm). The Riemannian volume measure $\mathrm{vol}_{\mathcal{A}}$ is invariant under this action (as bi-invariant metrics are preserved by conjugation). Since $(\mathcal{A}, g_{\mathcal{A}}, \mathrm{vol}_{\mathcal{A}})$ is a smooth compact Riemannian manifold with $\mathrm{Ric}_{\mathcal{A}} \geq \frac{N_c}{4}$, it satisfies $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ (the smooth Riemannian condition implies the synthetic one; see [30, Theorem 4.24]).

By the Galaz-García–Kell–Mondino–Sosa quotient theorem [29, Theorem 1.2]: if a compact Lie group G acts by measure-preserving isometries on a metric-measure space $(X, d, \mathfrak{m})$ satisfying $\mathrm{RCD}^*(K,N)$, then the quotient $(X/G, d_{X/G}, \mathfrak{m}_{X/G})$ satisfies $\mathrm{RCD}^*(K,N)$ with the same curvature parameter K. Applying this with $G = \mathcal{G}$, $X = \mathcal{A}$, $K = N_c/4$:

$$(\mathcal{B}, d_{\mathcal{B}}, \mathrm{vol}_{\mathcal{B}}) = (\mathcal{A}/\mathcal{G}, d_{\mathcal{A}/\mathcal{G}}, (\mathrm{vol}_{\mathcal{A}})_{\mathcal{G}}) \quad \text{satisfies} \quad \mathrm{RCD}^*(N_c/4,\ \dim \mathcal{A}).$$

This holds on the *full* orbit space $\mathcal{B}$, including the singular stratum $\mathcal{B}_{\mathrm{sing}}$ (reducible connections), without requiring a separate analysis of singularities.

Step 3: Bakry–Émery from RCD.* The $\mathrm{RCD}^*(K,N)$ condition with $K > 0$ implies the log-Sobolev inequality with constant $\alpha \geq K$ for the canonical measure on $\mathcal{B}$ ([25, Corollary 5.7.2]). This is the input used in Lemma 5.2 and Theorem 4.1. □

Remark 3.3. The RCD* quotient theorem [29] handles the singular stratum $\mathcal{B}_{\mathrm{sing}}$ (reducible connections) automatically: the curvature bound $\mathrm{RCD}^*(N_c/4, \dim \mathcal{A})$ holds on the full orbit space without requiring separate treatment of singularities. This is one advantage of the synthetic approach over O'Neill-type arguments.

4 Terminal scale LSI

Theorem 4.1 (Terminal LSI)**.** *At Balaban's terminal scale* $n_{\max}$*, the effective measure* $\mu_{n_{\max}}$ *satisfies an LSI with constant* $\alpha_{\mathrm{term}} > 0$ *depending only on* N_c*,* γ*, and* κ*.*

Proof. The terminal orbit space $\mathcal{B}_{n_{\max}}$ has fixed finite dimension $n = 3L^3_{\mathrm{vol},n_{\max}}(N_c^2 - 1) = O(1)$, Ricci curvature $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ (Theorem 3.2), and is compact.

The uniform (Haar-product) measure ν on $\mathcal{B}_{n_{\max}}$ satisfies, by the Bakry–Émery criterion [7] under the convention $\mathrm{Ent}_\nu(f^2) \leq \frac{2}{\alpha}\int |\nabla f|^2\, d\nu$ (Remark 4.2):

$$\alpha_{\mathrm{LSI}}(\nu) \geq \frac{N_c}{4}. \tag{13}$$

The effective measure is $\mu_{n_{\max}} = e^{-f}\nu/Z$ with $f = S_{n_{\max}}/(g^2_{n_{\max}}/2)$. On the terminal lattice ($O(1)$ sites), the number of plaquettes is $O(1)$ and the polymer sum $\sum_X |\epsilon_{n_{\max}}(X)|$ is finite (bounded number of terms, each $\leq e^{-\kappa}$), giving $\mathrm{osc}(f) \leq K/\gamma =: M_{\mathrm{term}}$.

By Holley–Stroock [16]: $\alpha_{\mathrm{term}} \geq \frac{N_c}{4}\, e^{-M_{\mathrm{term}}} > 0$. □

Remark 4.2 (LSI convention). We use the convention $\mathrm{Ent}_\mu(f^2) \le \frac{2}{\alpha}\mathcal{E}(f,f)$ throughout. Under this convention, the Bakry–Émery criterion with $\mathrm{CD}(K,\infty)$ gives $\alpha = K$ (not $2K$). Some references define LSI as $\mathrm{Ent}(f^2) \le \frac{1}{\alpha'}\mathcal{E}(f,f)$, in which case $\alpha' = K/2$. The factor of 2 in our convention is absorbed into the definition of α.

5 Multiscale entropy decomposition

5.1 Setup and filtration

We work with the decreasing filtration $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{n_{\max}}$, where $\mathcal{G}_k = \sigma(\bar{A}^{(k)})$ is generated by the block-averaged field at scale k. The orthogonal scale decomposition gives $T_A\mathcal{A} = \bigoplus_{k=0}^{n_{\max}} E_k \oplus V_{n_{\max}}$ with $\sum_k |\nabla_{E_k} f|^2 + |\nabla_{V_{n_{\max}}} f|^2 = |\nabla f|^2$.

5.2 Entropy chain rule

Lemma 5.1 (Entropy telescoping)**.** *For any $f \in L^2(\mu)$ with $f > 0$:*

$$\mathrm{Ent}_\mu(f) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\,\mathrm{Ent}(\mathbb{E}[f \mid \mathcal{G}_k] \mid \mathcal{G}_{k+1})\big] + \mathrm{Ent}_\mu(\mathbb{E}[f \mid \mathcal{G}_{n_{\max}}]). \tag{14}$$

Proof. Standard entropy chain rule; see Guionnet–Zegarlinski [15, Proposition 4.1]. □

5.3 Conditional fast LSI

Lemma 5.2. *There exists $\alpha_0 > 0$ (depending on N_c and κ but not on k or L_{vol}) such that for each $0 \le k \le n_{\max} - 1$:*

$$\mathrm{Ent}(h^2 \mid \mathcal{G}_{k+1}) \le \frac{2}{\alpha_0}\,\mathbb{E}\big[|\nabla_{E_k} h|^2 \mid \mathcal{G}_{k+1}\big] \tag{15}$$

for all smooth gauge-invariant h, μ-a.e. in $\mathcal{G}_{k+1}$.

Proof. Step 1: Small-field conditional LSI. Fix a coarse configuration $\bar{A}^{(k+1)} \in \mathcal{G}_{k+1}$. The conditional measure on fast modes (scale-k variables given $\mathcal{G}_{k+1}$) is supported on $\prod_{\ell \in E_k} SU(N_c)$ (a product of compact Lie groups), with conditional density proportional to $e^{-W_k(\cdot|\bar{A}^{(k+1)})}$.

Balaban's small-field/large-field decomposition ([6, Theorem 1]) provides: on the small-field region Ω_k^{sf} (where $|U(\partial p) - 1| < \varepsilon_k$ for all plaquettes), the conditional potential W_k admits the expansion ([6, eq. (1.2), p. 356]):

$$W_k = \frac{1}{2}\langle H_{1,k}B', \Delta_1(\zeta_0) H_{1,k}B'\rangle + \frac{1}{g_k}\langle DH_{1,k}B', \zeta_0\eta^{-2}\mathrm{Im}\,\partial U_0\rangle + \frac{1}{g_k^2}V(\zeta_0, g_k H_{1,k}B'),$$

where the quadratic form satisfies the lower bound ([6, eq. (1.9), p. 358]):

$$\langle H_{1,k}B', \Delta_1(\zeta_0) H_{1,k}B'\rangle \ge \frac{\gamma_0}{2d(100M)^5}\|B'\|^2.$$

Combined with the fiber Ricci curvature $\mathrm{Ric}_{\mathrm{fiber}} \ge N_c/4$ (each $SU(N_c)$ factor, Theorem 3.2), the Bakry–Émery criterion (Remark 4.2) gives a conditional LSI on the small-field region with constant $\alpha_0^{\mathrm{sf}} \ge N_c/4$.

Step 2: Large-field patching via bounded perturbation. On the full fiber (including the large-field region), the conditional measure is $d\mu_k^{\mathrm{full}} = e^{-\Phi} d\mu_k^{\mathrm{sf}}/Z$, where Φ is the "large-field correction" to the potential. By Balaban's large-field bounds ([6, eq. (0.1), p. 355]):

$$\chi_k \exp\Big[-\frac{1}{g_k^2}A(U_k) - E_-|T_\eta|\Big] \le \rho_k \le \exp\big[E_+|T_\eta|\big],$$

the oscillation of the log-density is bounded by

$$\operatorname{osc}(\log \rho_k) \le \frac{1}{g_k^2} \sup A(U_k) + (E_- + E_+)|T_\eta|.$$

On the finite-dimensional *fiber* (fast modes at scale k, which are $O((100MR_k)^d)$ link variables), this gives $\operatorname{osc}(\Phi) \le C(M, d, N_c, R_k)$, a constant depending on the geometry of the RG block but *not* on L_{vol}.

By the Holley–Stroock perturbation lemma for LSI ([16]): if $d\mu = e^{-\Phi} d\nu / Z$ and ν satisfies LSI(α), then μ satisfies LSI($\alpha \cdot e^{-\operatorname{osc}(\Phi)}$). Applying this with $\nu = \mu_k^{\mathrm{sf}}$ (satisfying LSI($N_c/4$) by Step 1) and $\mu = \mu_k^{\mathrm{full}}$:

$$\alpha_0 \ge \frac{N_c}{4} \cdot e^{-C(M,d,N_c,R_k)} > 0.$$

The constant C depends on the RG block geometry (through M, R_k) but is *uniform in k and L_{vol}* because $R_k \le R_{n_{\max}}$ is bounded by a function of β alone. □

5.4 Influence coefficients and sweeping-out inequality

Definition 5.3 (Influence coefficients across scales)**.** For $0 \le \ell \le k \le n_{\max} - 1$, define the influence coefficient

$$\Gamma_{k\ell} := \sup_{\substack{\varphi \in C^\infty(\mathcal{A}) \\ \mathbb{E}[|\nabla_{E_\ell}\varphi|^2] \neq 0}} \frac{\mathbb{E}\big[|\nabla_{E_k} \mathbb{E}[\varphi \mid \mathcal{G}_\ell]|^2\big]^{1/2}}{\mathbb{E}\big[|\nabla_{E_\ell}\varphi|^2\big]^{1/2}},$$

with $\Gamma_{k\ell} = 0$ for $\ell > k$.

Assumption 5.4 (Cross-scale derivative bound)**.** For all $0 \le \ell < k \le n_{\max} - 1$:

$$\operatorname*{ess\,sup}_{\mathcal{G}_\ell} \mathbb{E}\big[|\nabla_{E_k} V_{<\ell}|^2 \mid \mathcal{G}_\ell\big] \le C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}. \tag{16}$$

Here $V_{<\ell} = -\log(\text{conditional density of modes } <\ell \text{ given } \mathcal{G}_\ell)$ (normalised), $E_k = \ker(Q_{k+1})^\perp \cap \ker(Q_k)$ is the scale-k fluctuation subspace, and $C_{\mathrm{poly}}, \kappa$ are the universal constants from Proposition 2.3.

On the small-field region, this bound follows from Balaban's polymer analyticity and decay (Proposition 2.3) composed across intermediate scales $\ell < m < k$, combined with the block-averaging adjoint scaling (Lemma 2.2). The extension to the full essential supremum (including the large-field region, where the geometric decay factor $L_{\mathrm{RG}}^{-(d-1)k}$ does not follow from the Cauchy estimate alone; see Remark 2.4) is part of the content of this Assumption. The derivation and supporting evidence are given in Appendix A.

Status. Assumption 5.4 has been proved in [10, 11, 12]; it is retained here as an Assumption only to preserve the original logical structure of v1.

Lemma 5.5 (Geometric decay of influence)**.** *Under Proposition 2.3, the influence coefficients satisfy:*

$$\Gamma_{k\ell} \le C_\Gamma \, e^{-\kappa} \, L_{\mathrm{RG}}^{-(d-1)k/2} \tag{17}$$

for all $\ell \le k$, with C_Γ depending on N_c, γ, κ but not on k, ℓ, or L_{vol}.

Proof. Fix $\ell < k$ and a smooth function φ. The conditional expectation is

$$\mathbb{E}[\varphi \mid \mathcal{G}_\ell](\bar{A}^{(\ell)}) = \frac{\int \varphi(A)\, e^{-V_{<\ell}(A|\bar{A}^{(\ell)})}\, dA_{<\ell}}{\int e^{-V_{<\ell}(A|\bar{A}^{(\ell)})}\, dA_{<\ell}},$$

where $dA_{<\ell}$ denotes integration over scales finer than ℓ, and $V_{<\ell}$ is the effective potential from Balaban's RG.

Step 1: Differentiation. For $v \in E_k$ with $k > \ell$, the direction v acts on the integration variables $A_{<\ell}$ (which include scale k). Differentiating:

$$v\,\mathbb{E}[\varphi \mid \mathcal{G}_\ell] = \mathbb{E}[v\varphi \mid \mathcal{G}_\ell] - \mathrm{Cov}(\varphi,\, vV_{<\ell} \mid \mathcal{G}_\ell). \tag{18}$$

This is the standard identity for differentiating a ratio of integrals with respect to a parameter appearing in the integrand.

Step 2: Bounding the direct term. The first term satisfies

$$\mathbb{E}\big[|\mathbb{E}[v\varphi \mid \mathcal{G}_\ell]|^2\big] \le \mathbb{E}\big[\mathbb{E}[|v\varphi|^2 \mid \mathcal{G}_\ell]\big] = \mathbb{E}[|v\varphi|^2],$$

by Jensen's inequality. Summing over $v \in E_k$: $\mathbb{E}[|\nabla_{E_k}\mathbb{E}[\varphi \mid \mathcal{G}_\ell]|^2]_{\mathrm{direct}} \le \mathbb{E}[|\nabla_{E_k}\varphi|^2]$, which gives $\Gamma_{k\ell}^{\mathrm{direct}} \le 1$ (the "diagonal" contribution, relevant only for $k = \ell$).

Step 3: Bounding the indirect term. By Cauchy–Schwarz in the conditional expectation:

$$|\,\mathrm{Cov}(\varphi, vV_{<\ell} \mid \mathcal{G}_\ell)|^2 \le \mathrm{Var}(\varphi \mid \mathcal{G}_\ell) \cdot \mathbb{E}[|vV_{<\ell}|^2 \mid \mathcal{G}_\ell].$$

For $k > \ell$: the variable $v \in E_k$ is a fast direction at scale k, and $V_{<\ell}$ is the effective potential for scales $< \ell$. The dependence of $V_{<\ell}$ on scale-k variables arises through the multi-step RG: scale-k variables affect the conditional density at scale ℓ through $k - \ell$ intermediate scales.

By Assumption 5.4:

$$\mathbb{E}[|vV_{<\ell}|^2 \mid \mathcal{G}_\ell] \le C_{\mathrm{poly}}^2 e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k}.$$

The factor $L_{\mathrm{RG}}^{-(d-1)k}$ arises from the block-averaging adjoint at scale k (Lemma 2.2), regardless of the number of intermediate scales.

For the variance factor: by the conditional Poincaré inequality (which follows from the conditional LSI, Lemma 5.2):

$$\mathrm{Var}(\varphi \mid \mathcal{G}_\ell) \le \frac{1}{\alpha_0}\mathbb{E}[|\nabla_{<\ell}\varphi|^2 \mid \mathcal{G}_\ell].$$

Combining: the indirect contribution satisfies

$$\mathbb{E}\big[|\nabla_{E_k}\mathbb{E}[\varphi \mid \mathcal{G}_\ell]|^2\big]_{\mathrm{indirect}} \le \frac{C_{\mathrm{poly}}^2 e^{-2\kappa}}{\alpha_0} \cdot L_{\mathrm{RG}}^{-(d-1)k} \cdot \mathbb{E}[|\nabla\varphi|^2].$$

Taking the ratio with $\mathbb{E}[|\nabla_{E_\ell}\varphi|^2]$ and using $\mathbb{E}[|\nabla\varphi|^2] \ge \mathbb{E}[|\nabla_{E_\ell}\varphi|^2]$:

$$\Gamma_{k\ell} \le 1 \cdot \mathbf{1}_{k=\ell} + \frac{C_{\mathrm{poly}} e^{-\kappa}}{\sqrt{\alpha_0}} \cdot L_{\mathrm{RG}}^{-(d-1)k/2} \cdot \mathbf{1}_{k>\ell}.$$

Setting $C_\Gamma = C_{\mathrm{poly}}/\sqrt{\alpha_0}$ gives (17). □

Lemma 5.6 (Polymer counting)**.** *Let b be a fixed bond of the lattice at scale k. The number of connected polymers X (unions of M-cubes at scale k) with $d_k(X) = n$ and $X \ni b$ satisfies*

$$\#\{X : X \ni b,\, d_k(X) = n\} \le C_d^n$$

for a constant C_d depending only on d. Consequently, for $\kappa > \log C_d$:

$$\sum_{X \ni b} e^{-\kappa\, d_k(X)} \le \frac{C_d e^{-\kappa}}{1 - C_d e^{-\kappa}} < \infty. \tag{19}$$

Proof. Fix a reference bond b at scale k. A connected polymer X with $d_k(X) = |X| = n$ is a connected union of n scale-k blocks whose spanning tree has at most n vertices. Starting from the block containing b, each subsequent block in a depth-first exploration of the spanning tree is chosen from at most $(3^d - 1)$ neighbouring blocks. Thus the number of connected polymers with $|X| = n$ containing b is at most $(3^d - 1)^n =: C_d^n$.

For the sum: since $d_k(X) \geq 1$ for any polymer with $|X| \geq 2$ cubes, and the single-cube polymer ($n = 0$) contributes at most 1:

$$\sum_{X \ni b} e^{-\kappa\, d_k(X)} \leq 1 + \sum_{n=1}^{\infty} C_d^n e^{-\kappa n} = 1 + \frac{C_d e^{-\kappa}}{1 - C_d e^{-\kappa}} < \infty$$

for $\kappa > \log C_d$. □

Lemma 5.7 (Sweeping-out inequality (bounded observables))**.** *Assume Lemma 5.2 (conditional fast LSI with constant α_0) and Lemma 5.5 (geometric influence decay). Let g be a bounded, smooth, gauge-invariant function, and set $g_k := \sqrt{\mathbb{E}[g^2 \mid \mathcal{G}_k]}$. Then for each $k \leq n_{\max} - 1$:*

$$\mathbb{E}\big[|\nabla_{E_k} g_k|^2\big] \leq 2\,\mathbb{E}\big[|\nabla_{E_k} g|^2\big] + 2 \sum_{\ell < k} \Gamma_{k\ell}^2\, \mathbb{E}\big[|\nabla_{E_\ell} g|^2\big] + D_k\, \|g\|_\infty^2, \tag{20}$$

where $D_k := C_{\mathrm{poly}}^2 e^{-2\kappa}\, L_{\mathrm{RG}}^{-(d-1)k}$. Consequently:

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[|\nabla_{E_k} g_k|^2\big] \leq (2 + D_*)\, \mathbb{E}\big[|\nabla g|^2\big] + D_*\, \|g\|_\infty^2,$$

where $D_* := \sum_{k \geq 0} D_k \leq C_{\mathrm{poly}}^2 e^{-2\kappa}/(1 - L_{\mathrm{RG}}^{-(d-1)})$. For $L_{\mathrm{RG}} = 2$, $d = 4$: $D_* \leq \frac{8}{7} C_{\mathrm{poly}}^2 e^{-2\kappa}$.

Proof. Write $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot \mid \mathcal{G}_k]$. For $v \in E_k$, differentiating the conditional expectation gives

$$v\, \mathbb{E}_k[g^2] = \mathbb{E}_k[v(g^2)] - \mathrm{Cov}(g^2, vV_{<k} \mid \mathcal{G}_k).$$

Since $g_k = \sqrt{\mathbb{E}_k[g^2]}$:

$$|\nabla_{E_k} g_k|^2 = \frac{|\nabla_{E_k} \mathbb{E}_k[g^2]|^2}{4\, \mathbb{E}_k[g^2]}.$$

Direct term. Using $v(g^2) = 2g \cdot vg$ and Cauchy–Schwarz in $\mathbb{E}_k$:

$$\frac{|\mathbb{E}_k[v(g^2)]|^2}{4\mathbb{E}_k[g^2]} \leq \mathbb{E}_k[|vg|^2].$$

Covariance term. By Cauchy–Schwarz and $\mathrm{Var}_k(g^2) \leq \mathbb{E}_k[g^4]$:

$$\frac{|\,\mathrm{Cov}(g^2, vV_{<k} \mid \mathcal{G}_k)|^2}{4\mathbb{E}_k[g^2]} \leq \frac{\mathbb{E}_k[g^4]}{4\mathbb{E}_k[g^2]} \cdot \mathbb{E}_k[|vV_{<k}|^2] \leq \|g\|_\infty^2 \cdot \mathbb{E}_k[|vV_{<k}|^2],$$

where we used $\mathbb{E}_k[g^4]/\mathbb{E}_k[g^2] \leq \|g\|_\infty^2$. By Assumption 5.4: $\mathbb{E}_k[|vV_{<k}|^2] \leq C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}$.

Summing over an orthonormal basis of E_k, taking expectations, and using the definition of $\Gamma_{k\ell}$ for the cross-scale transfer of the direct term gives (20). □

Remark 5.8 (No entropy circularity; extension by truncation). Lemma 5.7 bounds the multi-scale commutator error by the Dirichlet form plus (for bounded g) a term $D_* \|g\|_\infty^2$. This creates no entropy circularity: the error is controlled by energy and a pointwise bound, not by the entropy being estimated. The LSI is first established for bounded smooth gauge-invariant functions and then extended to all $g \in \mathrm{Dom}(\mathcal{E})$ by truncation and lower semicontinuity; see [25, Proposition 5.2.5].

5.5 Assembly: Uniform LSI

Lemma 5.9 (Variance telescoping)**.** *For all $g \in L^2(\mu)$, setting $m_k := \mathbb{E}[g \mid \mathcal{G}_k]$:*

$$\mathrm{Var}_\mu(g) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\,\mathrm{Var}(m_k \mid \mathcal{G}_{k+1})\big] + \mathrm{Var}_\mu(m_{n_{\max}}). \tag{21}$$

Proof. By the law of total variance applied iteratively: $\mathrm{Var}_\mu(m_k) = \mathbb{E}[\mathrm{Var}(m_k \mid \mathcal{G}_{k+1})] + \mathrm{Var}_\mu(m_{k+1})$. Summing over $k = 0, \dots, n_{\max} - 1$ and using $m_0 = g$ gives (21). □

Lemma 5.10 (Energy-only sweeping-out for conditional expectations)**.** *Assume Lemma 5.2 (conditional Poincaré with constant $2/\alpha_0$) and Proposition 2.3 (polymer derivative bounds). For all bounded smooth gauge-invariant g, setting $m_k := \mathbb{E}[g \mid \mathcal{G}_k]$:*

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[|\nabla_{E_k} m_k|^2\big] \le \mathfrak{C}\,\mathcal{E}(g,g), \tag{22}$$

where $\mathfrak{C} = 2 + D_*^{(\mathrm{mart})}$ with $D_*^{(\mathrm{mart})} = (2/\alpha_0) C_{\mathrm{poly}}^2 e^{-2\kappa}/(1 - L_{\mathrm{RG}}^{-(d-1)}) < \infty$, independent of L_{vol}.

Proof. For $v \in E_k$, differentiating the conditional expectation:

$$v\, m_k = v\,\mathbb{E}[g \mid \mathcal{G}_k] = \mathbb{E}[vg \mid \mathcal{G}_k] - \mathrm{Cov}(g, vV_{<k} \mid \mathcal{G}_k).$$

By Cauchy–Schwarz in the conditional expectation:

$$|\,\mathrm{Cov}(g, vV_{<k} \mid \mathcal{G}_k)|^2 \le \mathrm{Var}(g \mid \mathcal{G}_k) \cdot \mathbb{E}\big[|vV_{<k}|^2 \mid \mathcal{G}_k\big].$$

The key difference from Lemma 5.7 is that no ratio $\mathbb{E}[g^4 \mid \mathcal{G}_k]/\mathbb{E}[g^2 \mid \mathcal{G}_k]$ appears: the covariance involves g linearly, not g^2. Since $\mathcal{G}_k \supset \mathcal{G}_{k+1}$ (more information decreases conditional variance):

$$\mathrm{Var}(g \mid \mathcal{G}_k) \le \mathrm{Var}(g \mid \mathcal{G}_{k+1}).$$

By the conditional Poincaré inequality on the scale-k fiber (Lemma 5.2):

$$\mathrm{Var}(g \mid \mathcal{G}_{k+1}) \le \frac{2}{\alpha_0}\,\mathbb{E}\big[|\nabla_{E_k} g|^2 \mid \mathcal{G}_{k+1}\big].$$

Since $\mathcal{G}_{k+1} \subset \mathcal{G}_k$, the right-hand side is $\mathcal{G}_{k+1}$-measurable hence $\mathcal{G}_k$-measurable. By Proposition 2.3 at scale k: $\mathbb{E}\big[|vV_{<k}|^2 \mid \mathcal{G}_k\big] \le C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}$. Combining:

$$|\,\mathrm{Cov}(g, vV_{<k} \mid \mathcal{G}_k)|^2 \le \frac{2}{\alpha_0}\,\mathbb{E}\big[|\nabla_{E_k} g|^2 \mid \mathcal{G}_{k+1}\big] \cdot C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}.$$

Since $v\, m_k = \mathbb{E}[vg \mid \mathcal{G}_k] - \mathrm{Cov}(g, vV_{<k} \mid \mathcal{G}_k)$ and $|\mathbb{E}[vg \mid \mathcal{G}_k]|^2 \le \mathbb{E}[|vg|^2 \mid \mathcal{G}_k]$ by Jensen:

$$|v\, m_k|^2 \le 2\,\mathbb{E}\big[|vg|^2 \mid \mathcal{G}_k\big] + \frac{2}{\alpha_0}\,\mathbb{E}\big[|\nabla_{E_k} g|^2 \mid \mathcal{G}_{k+1}\big] \cdot C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k}.$$

Summing over an orthonormal basis of E_k, taking full expectations (using $\mathbb{E}[\mathbb{E}[\cdot \mid \mathcal{G}_{k+1}]] = \mathbb{E}[\cdot]$), and summing over k with orthogonality $\sum_k \mathbb{E}[|\nabla_{E_k} g|^2] \le \mathcal{E}(g,g)$:

$$\sum_k \mathbb{E}[|\nabla_{E_k} m_k|^2] \le 2\,\mathcal{E}(g,g) + \frac{2}{\alpha_0}\,\mathcal{E}(g,g) \cdot \sum_k C_{\mathrm{poly}}^2 e^{-2\kappa} L_{\mathrm{RG}}^{-(d-1)k} = (2 + D_*^{(\mathrm{mart})})\,\mathcal{E}(g,g). \quad \square$$

Proof of Theorem 1.1(i). Step 1: Entropy telescoping. By Lemma 5.1 with $f = g^2$, writing $g_k = \sqrt{\mathbb{E}[g^2 \mid \mathcal{G}_k]}$:

$$\mathrm{Ent}_\mu(g^2) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}[\mathrm{Ent}(g_k^2 \mid \mathcal{G}_{k+1})] + \mathrm{Ent}_\mu(g_{n_{\max}}^2).$$

Step 2: Conditional LSI. By Lemma 5.2: $\mathbb{E}[\mathrm{Ent}(g_k^2 \mid \mathcal{G}_{k+1})] \le (2/\alpha_0)\,\mathbb{E}[|\nabla_{E_k} g_k|^2]$.

Step 3: Sweeping-out. By Lemma 5.7 (for bounded g), summing over k and using orthogonality $\sum_k |\nabla_{E_k} g|^2 \le |\nabla g|^2$:

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}[|\nabla_{E_k} g_k|^2] \le (2 + D_*)\,\mathcal{E}(g,g) + D_*\,\|g\|_\infty^2.$$

Step 4: Terminal scale. By Theorem 4.1: $\mathrm{Ent}_\mu(g_{n_{\max}}^2) \le (2/\alpha_{\mathrm{term}})\,\mathbb{E}[|\nabla_{V_{n_{\max}}} g_{n_{\max}}|^2]$. The terminal sweeping-out (the same argument as Lemma 5.7 at the terminal scale) gives $\mathbb{E}[|\nabla_{V_{n_{\max}}} g_{n_{\max}}|^2] \le 2\,\mathbb{E}[|\nabla_{V_{n_{\max}}} g|^2] + D_{\mathrm{term}}\,\|g\|_\infty^2$ with D_{term} independent of L_{vol}.

Step 5: Defect LSI. Combining Steps 1–4 and using orthogonality:

$$\mathrm{Ent}_\mu(g^2) \le C_E\,\mathcal{E}(g,g) + C_B\,\|g\|_\infty^2 \tag{23}$$

for all bounded smooth gauge-invariant g, where $C_E, C_B > 0$ depend on $\alpha_0, \alpha_{\mathrm{term}}, D_*, D_{\mathrm{term}}$ but *not on* L_{vol}.

Step 6: Uniform Poincaré inequality. Set $m_k := \mathbb{E}[g \mid \mathcal{G}_k]$. By Lemma 5.9:

$$\mathrm{Var}_\mu(g) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\,\mathrm{Var}(m_k \mid \mathcal{G}_{k+1})\big] + \mathrm{Var}_\mu(m_{n_{\max}}).$$

By the conditional Poincaré inequality (from Lemma 5.2):

$$\mathbb{E}\big[\,\mathrm{Var}(m_k \mid \mathcal{G}_{k+1})\big] \le \frac{2}{\alpha_0}\,\mathbb{E}\big[|\nabla_{E_k} m_k|^2\big].$$

At the terminal scale, $m_{n_{\max}} = \mathbb{E}[g \mid \mathcal{G}_{n_{\max}}]$ satisfies $\nabla_{V_{n_{\max}}} m_{n_{\max}} = \mathbb{E}[\nabla_{V_{n_{\max}}} g \mid \mathcal{G}_{n_{\max}}]$, so by Jensen: $\mathbb{E}[|\nabla_{V_{n_{\max}}} m_{n_{\max}}|^2] \le \mathbb{E}[|\nabla_{V_{n_{\max}}} g|^2] \le \mathcal{E}(g,g)$. By the terminal Poincaré (Theorem 4.1): $\mathrm{Var}_\mu(m_{n_{\max}}) \le (2/\alpha_{\mathrm{term}})\,\mathbb{E}[|\nabla_{V_{n_{\max}}} g|^2]$. Therefore:

$$\mathrm{Var}_\mu(g) \le \frac{2}{\alpha_0} \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[|\nabla_{E_k} m_k|^2\big] + \frac{2}{\alpha_{\mathrm{term}}}\,\mathcal{E}(g,g).$$

Applying Lemma 5.10 to control the sum:

$$\mathrm{Var}_\mu(g) \le C_P\,\mathcal{E}(g,g), \qquad C_P := \frac{2\mathfrak{C}}{\alpha_0} + \frac{2}{\alpha_{\mathrm{term}}}, \tag{24}$$

with C_P independent of L_{vol}. This Poincaré inequality holds for all $g \in L^2(\mu)$ with no defect term. It is proved without using the global LSI.

Step 7: Tight LSI via Rothaus' lemma. From Step 5 we have the defect inequality (23):

$$\mathrm{Ent}_\mu(g^2) \le C_E\,\mathcal{E}(g,g) + C_B\,\|g\|_\infty^2$$

for all bounded smooth gauge-invariant g. In particular, restricting to $\|g\|_\infty \le 1$ yields a defect LSI with additive constant:

$$\mathrm{Ent}_\mu(g^2) \le C_E\,\mathcal{E}(g,g) + C_B \qquad \text{for all bounded smooth gauge-invariant } g \text{ with } \|g\|_\infty \le 1.$$

Together with the uniform Poincaré (24), Rothaus' lemma ([25, Proposition 5.1.3]) yields a tight LSI

$$\mathrm{Ent}_\mu(g^2) \le \frac{2}{\alpha_*}\,\mathcal{E}(g,g) \tag{25}$$

for all bounded smooth gauge-invariant g with $\|g\|_\infty \le 1$, where $\alpha_* = \alpha_*(C_E, C_B, C_P) > 0$ is independent of L_{vol}.

Step 8: Extension to all bounded g by homogeneity. The tight LSI of Step 7 holds for $\|g\|_\infty \le 1$. For arbitrary bounded g with $\|g\|_\infty = M > 0$, set $h = g/M$ so that $\|h\|_\infty = 1$. By the degree-2 homogeneity of entropy (see, e.g., [25, §5.1]): for any $\lambda > 0$ and $f \ge 0$,

$$\mathrm{Ent}_\mu(\lambda f) = \lambda\,\mathrm{Ent}_\mu(f). \tag{26}$$

Applying with $\lambda = M^2$ and $f = h^2$:

$$\mathrm{Ent}_\mu(g^2) = \mathrm{Ent}_\mu(M^2 h^2) = M^2\,\mathrm{Ent}_\mu(h^2) \le M^2 \cdot \frac{2}{\alpha_*}\,\mathcal{E}(h,h) = \frac{2}{\alpha_*}\,\mathcal{E}(g,g).$$

Hence the tight LSI holds for *all* bounded smooth gauge-invariant g with the same constant α_*. The extension to $\mathrm{Dom}(\mathcal{E})$ follows by Corollary 5.11. □

Corollary 5.11 (Extension to $\mathrm{Dom}(\mathcal{E})$)**.** *Steps 7–8 establish $\mathrm{Ent}_\mu(f^2) \le (2/\alpha_*)\,\mathcal{E}(f,f)$ for all bounded smooth gauge-invariant f (with no restriction on $\|f\|_\infty$). For unbounded $f \in \mathrm{Dom}(\mathcal{E})$, define truncations $f^{(m)} := (-m) \vee f \wedge m$. Each $f^{(m)}$ is bounded, so the tight LSI applies:*

$$\mathrm{Ent}_\mu((f^{(m)})^2) \le \frac{2}{\alpha_*}\,\mathcal{E}(f^{(m)}, f^{(m)}).$$

By [25, Proposition 5.2.5]: both $f \mapsto \mathrm{Ent}_\mu(f^2)$ and $f \mapsto \mathcal{E}(f,f)$ are lower semicontinuous in $L^2(\mu)$, and $f^{(m)} \to f$ in $L^2(\mu)$ with $\mathcal{E}(f^{(m)}, f^{(m)}) \le \mathcal{E}(f,f)$ (since truncation is a contraction for the Dirichlet form on compact Riemannian manifolds). Passing to the limit $m \to \infty$:

$$\mathrm{Ent}_\mu(f^2) \le \liminf_{m\to\infty} \mathrm{Ent}_\mu((f^{(m)})^2) \le \frac{2}{\alpha_*}\,\liminf_{m\to\infty} \mathcal{E}(f^{(m)}, f^{(m)}) \le \frac{2}{\alpha_*}\,\mathcal{E}(f,f).$$

The constant α_* is preserved in the limit.

6 From uniform LSI to mass gap

6.1 Reversible diffusion and its Dirichlet form

Let $\mathcal{A} = \prod_\ell SU(N_c)$ with the product bi-invariant Riemannian metric. The Yang–Mills measure is $\mu_\beta(dU) = Z_\beta^{-1} e^{-S_\beta(U)} dU$, where dU is the Haar-product measure.

Define the symmetric diffusion generator:

$$\mathcal{L} f := \sum_\ell \Big(\Delta_\ell f - \langle \nabla_\ell S_\beta, \nabla_\ell f \rangle \Big), \tag{27}$$

where Δ_ℓ is the Laplace–Beltrami operator on the ℓ-th $SU(N_c)$ factor and ∇_ℓ is the corresponding Riemannian gradient. Then $\mathcal{L}$ is reversible with respect to μ_β, and its Dirichlet form is

$$\mathcal{E}(f,f) := -\int f\,\mathcal{L} f\,d\mu_\beta = \sum_\ell \int \|\nabla_\ell f\|^2\,d\mu_\beta = \int |\nabla f|^2\,d\mu_\beta. \tag{28}$$

This is precisely the gradient-squared integral appearing in the LSI (2). Let $P_t := e^{t\mathcal{L}}$ be the associated Markov semigroup.

6.2 Hypercontractivity and spectral gap

Proposition 6.1. *The uniform LSI (Theorem 1.1(i)) implies:*

(i) ***(Gross [14])*** *P_t is hypercontractive: $\|P_t\|_{L^2(\mu_\beta)\to L^{p(t)}(\mu_\beta)} \le 1$ with $p(t) = 1 + e^{2\alpha_* t}$.*

(ii) ***(Poincaré)*** $\mathrm{Var}_{\mu_\beta}(f) \le \frac{1}{\lambda_1}\mathcal{E}(f,f)$ *with* $\lambda_1 \ge \alpha_*$. $\|P_t f - \mathbb{E}_{\mu_\beta}[f]\|_{L^2} \le e^{-\lambda_1 t}\|f - \mathbb{E}_{\mu_\beta}[f]\|_{L^2}$.

6.3 Exponential clustering

Lemma 6.2 (Quantified contraction condition)**.** *Assume Lemma 5.5:*

$$\Gamma_{k\ell} \le C_\Gamma e^{-\kappa} L_{\mathrm{RG}}^{-(d-1)k/2} \qquad \text{for all } \ell \le k.$$

Define the row-sum influence parameter

$$\mathfrak{D}_{\mathrm{row}} := \sup_k \sum_{\ell<k} \Gamma_{k\ell}.$$

Then $\mathfrak{D}_{\mathrm{row}} < \infty$ and

$$\mathfrak{D}_{\mathrm{row}} \le C_\Gamma e^{-\kappa} \frac{L_{\mathrm{RG}}^{-(d-1)/2}}{\left(1 - L_{\mathrm{RG}}^{-(d-1)/2}\right)^2}.$$

In particular, $\mathfrak{D}_{\mathrm{row}} < 1$ holds whenever

$$\kappa > \log C_\Gamma + \log\left(\frac{L_{\mathrm{RG}}^{-(d-1)/2}}{\left(1 - L_{\mathrm{RG}}^{-(d-1)/2}\right)^2} \right).$$

For $L_{\mathrm{RG}} = 2$ and $d = 4$, one has $L_{\mathrm{RG}}^{-(d-1)/2} = 2^{-3/2}$ and

$$\frac{2^{-3/2}}{(1-2^{-3/2})^2} \approx 0.846, \qquad \log\left(\frac{2^{-3/2}}{(1-2^{-3/2})^2} \right) \approx -0.17,$$

so it suffices that $\kappa > \log C_\Gamma - 0.17$.

Proof. Set $r := L_{\mathrm{RG}}^{-(d-1)/2} \in (0,1)$. Then

$$\Gamma_{k\ell} \le C_\Gamma e^{-\kappa} r^k.$$

For each k,

$$\sum_{\ell<k} \Gamma_{k\ell} \le k \cdot C_\Gamma e^{-\kappa} r^k \le C_\Gamma e^{-\kappa} \sum_{j\ge 1} j r^j = C_\Gamma e^{-\kappa} \frac{r}{(1-r)^2}.$$

Taking $\sup_k$ gives the claimed bound. The displayed threshold is the sufficient condition $C_\Gamma e^{-\kappa} \frac{r}{(1-r)^2} < 1$. □

Remark 6.3 (Withdrawn assumption: superseded by [13]). The Dobrushin-translation assumption used in earlier versions is no longer needed. The mass gap is obtained unconditionally in [13] via the DLR-LSI extension and the Stroock–Zegarlinski equivalence, combined with reflection positivity.

Lemma 6.4 (From multiscale influence to a site-to-site Dobrushin matrix)**.** *Let $\Gamma_{k\ell}$ be the multiscale influence coefficients of Definition 5.3. Define the site-to-site matrix C_{xy} on Λ by*

$$C_{xy} := \sum_{k=0}^{n_{\max}} \sum_{\ell=0}^{k-1} \Gamma_{k\ell}\, \mathbf{1}_{x\in \mathrm{Block}_k}\, \mathbf{1}_{y\in \mathrm{Block}_\ell}, \tag{29}$$

i.e. we sum only over $\ell < k$ (off-diagonal scales). Then

$$\sup_x \sum_y C_{xy} \le \big(\sup_k |\mathrm{Block}_k|\big)\, \mathfrak{D}_{\mathrm{row}},$$

where

$$\mathfrak{D}_{\mathrm{row}} := \sup_k \sum_{\ell<k} \Gamma_{k\ell} \qquad \textit{(as in Lemma 6.2).}$$

In particular, since $\sup_k |\mathrm{Block}_k| \le (MR_{n_{\max}})^d$,

$$\sup_x \sum_y C_{xy} \le (MR_{n_{\max}})^d\, \mathfrak{D}_{\mathrm{row}}.$$

Consequently, if $(MR_{n_{\max}})^d\, \mathfrak{D}_{\mathrm{row}} < 1$, *then the site-to-site Dobrushin condition* $\sup_x \sum_{y\neq x} C_{xy} < 1$ *holds. Using Lemma 6.2, a sufficient condition for* $(MR_{n_{\max}})^d\, \mathfrak{D}_{\mathrm{row}} < 1$ *is*

$$\kappa > \log\big((MR_{n_{\max}})^d C_\Gamma\big) + \log\left(\frac{L_{\mathrm{RG}}^{-(d-1)/2}}{\big(1 - L_{\mathrm{RG}}^{-(d-1)/2}\big)^2}\right).$$

Proof. Fix x. Using $\sum_y \mathbf{1}_{y\in\mathrm{Block}_\ell} = |\mathrm{Block}_\ell|$:

$$\begin{aligned}
\sum_y C_{xy} &= \sum_{k=0}^{n_{\max}} \sum_{\ell=0}^{k-1} \Gamma_{k\ell}\, \mathbf{1}_{x\in\mathrm{Block}_k} \sum_y \mathbf{1}_{y\in\mathrm{Block}_\ell} \\
&= \sum_{k=0}^{n_{\max}} \sum_{\ell=0}^{k-1} \Gamma_{k\ell}\, \mathbf{1}_{x\in\mathrm{Block}_k}\, |\mathrm{Block}_\ell| \\
&\le \big(\sup_\ell |\mathrm{Block}_\ell|\big) \sum_{k=0}^{n_{\max}} \mathbf{1}_{x\in\mathrm{Block}_k} \sum_{\ell<k} \Gamma_{k\ell}.
\end{aligned}$$

Since for each scale k the site x belongs to at most one block, the indicator $\mathbf{1}_{x\in\mathrm{Block}_k}$ selects at most one term per k, hence

$$\sum_y C_{xy} \le \big(\sup_\ell |\mathrm{Block}_\ell|\big) \sup_k \sum_{\ell<k} \Gamma_{k\ell} = \big(\sup_k |\mathrm{Block}_k|\big)\, \mathfrak{D}_{\mathrm{row}}.$$

Taking $\sup_x$ yields the claim. The bound for $\sum_{y\neq x} C_{xy}$ is immediate since $C_{xx} \ge 0$. □

Theorem 6.5 (Clustering (conditional))**.** *Assume the uniform LSI (Theorem 1.1(i)) and Assumption* **??**. *Then for gauge-invariant observables $\mathcal{O}_A, \mathcal{O}_B$ supported on regions A, B with* $\mathrm{dist}(A, B) = r$*:*

$$|\langle \mathcal{O}_A \mathcal{O}_B \rangle_c| \le C\, \|\mathcal{O}_A\|_\infty\, \|\mathcal{O}_B\|_\infty\, e^{-r/\xi},$$

with $\xi = O(1/\sqrt{\alpha_*})$, *independent of* L_{vol}.

Proof. We use the finite propagation property of the interaction combined with the spectral gap of $\mathcal{L}$.

Step 1: Finite-range structure. The Wilson action S_W couples each link to at most $2d(d-1)$ neighbouring plaquettes. Balaban's polymer corrections $\sum_X \mathbf{R}^{(k)}(X)$ couple variables within polymer X, but with exponential decay $e^{-\kappa\, d_k(X)}$ (eq. (4)). Thus the total interaction is of the form: nearest-neighbour + exponentially decaying corrections.

Step 2: Dobrushin–Shlosman criterion. The influence coefficients $\Gamma_{k\ell}$ of Definition 5.3 satisfy the quantitative row-sum bound $\mathfrak{D}_{\mathrm{row}} < \infty$ (Lemma 6.2), and $\mathfrak{D}_{\mathrm{row}} < 1$ once κ is sufficiently large (equivalently, β_0 sufficiently large, so that γ is sufficiently small). The translation from

multiscale influence coefficients to a site-to-site matrix is provided by Lemma 6.4, yielding a sufficient site-to-site Dobrushin contraction condition of the form

$$(MR_{n_{\max}})^d \mathfrak{D}_{\mathrm{row}} < 1.$$

An explicit sufficient lower bound on κ implying this condition is recorded in Lemma 6.4.

Step 3: Complete analyticity ⇒ clustering. By the general theory of Dobrushin–Shlosman [23] (see also Martinelli [24, Theorem 3.2] for the lattice formulation): complete analyticity for a lattice system with finite-range interaction implies exponential decay of truncated correlations, uniformly in volume.

The adaptation to our setting (where the interaction includes exponentially decaying polymer corrections) follows from the standard perturbation argument: polymer corrections satisfying $\sum_X e^{-\kappa\, d_k(X)} < \infty$ (Lemma 5.6) can be absorbed into the Dobrushin matrix without violating the contraction condition, provided κ is large enough (equivalently, provided $(MR_{n_{\max}})^d \mathfrak{D}_{\mathrm{row}}$ is sufficiently small).

Step 4: Gauge invariance. For gauge-invariant observables, the conditional expectation onto gauge-invariant functions is a contraction in $L^2(\mu_\beta)$, so the clustering rate can only improve. □

6.4 Mass gap via reflection positivity

Proof of Theorem 1.1(ii). Assume Assumption **??**. The Wilson action satisfies reflection positivity with respect to temporal hyperplanes [21]. By the reconstruction theorem ([21, Theorem 6.1]), the temporal two-point function of any gauge-invariant local observable $\mathcal{O}$ with $\mathrm{Var}_{\mu_\beta}(\mathcal{O}) > 0$ admits the spectral representation

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle_c = \sum_{n\geq 1} |\langle \Omega|\hat{\mathcal{O}}|n\rangle|^2 e^{-E_n t}$$

with $E_n > 0$ and $|\langle \Omega|\hat{\mathcal{O}}|n\rangle|^2 \geq 0$. At least one weight is strictly positive (since $\sum_n |\langle \Omega|\hat{\mathcal{O}}|n\rangle|^2 = \mathrm{Var}(\mathcal{O}) > 0$).

By Theorem 6.5, $|\langle \mathcal{O}(t)\mathcal{O}(0)\rangle_c| \leq Ce^{-Mt}$ for all $t > 0$ with $M = 1/\xi > 0$. Since the spectral representation is a sum of non-negative exponentially decaying terms, the smallest energy satisfies $E_1 \geq M$: otherwise the term $|\langle \Omega|\hat{\mathcal{O}}|1\rangle|^2 e^{-E_1 t}$ would dominate at large t, contradicting the clustering bound for any $\mathcal{O}$ with $\langle \Omega|\hat{\mathcal{O}}|1\rangle \neq 0$.

To ensure such an $\mathcal{O}$ exists, take $\mathcal{O} = \mathcal{O}_P := \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\, U_P$ for a fixed plaquette P. This is gauge-invariant, local, and has $\mathrm{Var}(\mathcal{O}_P) > 0$ for $\beta < \infty$ (the distribution of U_P under μ_β is not concentrated at a single value). By completeness of the energy eigenstates, at least one overlap $\langle \Omega|\hat{\mathcal{O}}_P|n\rangle$ is non-zero.

Therefore $\Delta_{\mathrm{phys}} = E_1 \geq M > 0$. □

7 Discussion

7.1 What is proved

For $SU(N_c)$ lattice Yang–Mills on $(\mathbb{Z}/L_{\mathrm{vol}}\mathbb{Z})^4$ at $\beta \geq \beta_0$:

- Uniform log-Sobolev inequality (Theorem 1.1(i))
- Mass gap $\Delta_{\mathrm{phys}} \geq c(N_c, \beta_0) > 0$ uniform in L_{vol}, conditional on Assumption **??** (Theorem 1.1(ii))

7.2 What is not proved

1. **Continuum limit.** We do not construct a continuum QFT or verify OS axioms.

2. **Mass gap persistence as $a \to 0$.** Our bound gives $m_{\text{phys}} \geq c' \cdot \Lambda_{\text{QCD}}$ at each fixed a, but we do not prove uniform bounds as $a \to 0$.

3. **Strong coupling.** Our results require $\beta \geq \beta_0$. At strong coupling, the mass gap follows from [21].

7.3 On the expected scaling $m_{\text{gap}} \sim e^{-C/g^2}$

The mass gap bound established in Theorem 1.1(ii) is $\Delta_{\text{phys}} > 0$ but we do not establish the expected scaling $m_{\text{gap}} \geq c \cdot e^{-C(N_c)/g^2}$. This scaling would follow if one could track how the LSI constant α_* depends on β through Balaban's RG. Specifically, the terminal-scale LSI constant $\alpha_{\text{term}} \geq (N_c/4)\, e^{-M_{\text{term}}}$ (Theorem 4.1), where $M_{\text{term}} = O(1/\gamma)$, is exponentially small in $1/\gamma$, and $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$ RG steps relate the terminal scale to the original lattice. A quantitative tracking of the LSI constant through these steps—which would require controlling how α_* degrades with $n_{\max}$—is beyond the scope of this paper but is compatible with the expected dimensional transmutation $m_{\text{phys}} \sim \Lambda_{\text{QCD}}$.

7.4 Relation to the Millennium Problem

Our result establishes the infrared component of the constructive program. The ultraviolet component—construction of the continuum limit satisfying the OS axioms—requires completing Balaban's program beyond what has been published, and is beyond the scope of this work.

7.5 Conditional aspects and the logical structure

The proof of Theorem 1.1 depends on the following chain of results, each of which we classify by its logical status:

1. **Balaban's RG (published, accepted).** The construction of effective actions with polymer bounds (Theorem 2.1) relies on the published series [1, 2, 3, 4, 5, 6]. These papers are accepted as rigorous by the mathematical physics community.

2. **Cross-scale derivative bound (Assumption 5.4).** This is derived in Appendix A from Balaban's polymer analyticity, decay bounds, and the block-averaging adjoint scaling. The derivation introduces a factor $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$ into the constant C_{poly}, making it depend on β_0 (but not on L_{vol}). In the original version of this paper, this was formulated as an Assumption because the audit trail in Appendix A, while based on explicit k-uniform analyticity/derivative bounds for the averaging maps [1, Propositions 4–5,7] and the inductive framework for effective densities [6, Theorem 1], did not extract a standalone lemma with all parameters spelled out.

   Update (v2): This Assumption has been verified by the companion papers [10, 11, 12]. Specifically, [10] replaces the essential supremum over $\mathcal{G}_k$ by two weaker inputs (an integrated L^1 bound and a shifted essential supremum over $\mathcal{G}_{k+1}$), which suffice for the sweeping-out and Poincaré arguments respectively. The large-field input (Hypothesis 4.2 of [10]) is verified in [11], and the residual derivative input (Hypothesis 3.2 of [10]) is verified in [12].

3. **Dobrushin contraction (Lemma 6.2).** Lemma 6.2 gives explicit geometric-series bounds on the row-sum influence parameter

$$\mathfrak{D}_{\text{row}} := \sup_k \sum_{\ell<k} \Gamma_{k\ell}, \qquad \mathfrak{D}_{\text{row}} \le C_\Gamma e^{-\kappa} \frac{L_{\text{RG}}^{-(d-1)/2}}{\left(1 - L_{\text{RG}}^{-(d-1)/2}\right)^2}.$$

In particular, $\mathfrak{D}_{\text{row}} < 1$ is ensured by an explicit lower bound on κ (equivalently, by choosing β_0 sufficiently large). The translation from multiscale influence coefficients to a site-to-site matrix is provided by Lemma 6.4, yielding a sufficient site-to-site Dobrushin contraction condition of the form

$$(MR_{n_{\max}})^d \, \mathfrak{D}_{\text{row}} < 1.$$

4. **Orbit space Ricci curvature (Theorem 3.2).** The bound $\text{Ric}_{\mathcal{B}} \ge N_c/4$ follows from the Galaz-García–Kell–Mondino–Sosa quotient theorem for RCD* spaces [29], applied to the isometric gauge group action on $(\mathcal{A}, g_{\mathcal{A}})$.

5. **Sweeping-out and assembly (Lemma 5.7, Lemma 5.10, Steps 5–7).** Two sweeping-out lemmas are used: Lemma 5.7 controls $\nabla_{E_k}\sqrt{\mathbb{E}[g^2|\mathcal{G}_k]}$ (with a $\|g\|_\infty^2$ defect), yielding the defect LSI (23). Lemma 5.10 controls $\nabla_{E_k}\mathbb{E}[g \mid \mathcal{G}_k]$ purely by energy (no $\|g\|_\infty$), using the conditional Poincaré inequality (Lemma 5.2) together with the single-scale polymer derivative bound (Proposition 2.3). This energy-only bound feeds into the variance chain rule (Lemma 5.9) to give a uniform Poincaré (24) independent of the global LSI. Rothaus' lemma ([25, Proposition 5.1.3]) then converts the defect LSI plus Poincaré into a tight LSI. The extension to $\text{Dom}(\mathcal{E})$ uses [25, Proposition 5.2.5]. No step is circular.

All conditions are satisfied for γ (equivalently, g^2) sufficiently small, as guaranteed by Balaban's Theorem 1 [6]. The question of *how small* γ must be is determined by the constants in Balaban's construction and the thresholds in Lemmas 6.2 and 6.4.

7.6 The role of each input

Input	Source	Role
$\text{Ric}_{\mathcal{B}} \ge N_c/4$	RCD* quotient [29]; Paper II [9, Theorem 1.1]	Conditional fast LSI
Balaban's RG	CMP 95–122	Effective actions, polymer bounds
Absorption property	CMP 122, eq. (1.37)	Cross-scale decoupling
Polymer analyticity	CMP 122, eq. (1.65)	Derivative bounds
Block-avg. scaling	CMP 98, eq. (125)	Geometric factor 2^{-3k}
Cross-scale bounds	Paper III [10]	Replaces Assumption 5.4
Large-field suppression	Paper IV [11]	Verifies Hyp. 4.2 of [10]
Residual derivatives	Paper V [12]	Verifies Hyp. 3.2 of [10]
Holley–Stroock	[16]	Terminal LSI
Bakry–Émery	[7]	LSI from Ricci
Gross theorem	[14]	LSI $\Rightarrow$ hypercontractivity
Reflection positivity	[21]	Clustering $\Rightarrow$ gap

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A Audit trail: derivation of the cross-scale derivative bound

We derive Assumption 5.4 from Balaban's published results. The argument proceeds in three steps.

Remark A.1 (On uniform analyticity radii). Balaban proves k-uniform analyticity and derivative bounds for the averaging maps ([1, Propositions 4, 5, and 7]) and states k-uniform analyticity conditions for effective densities under the small-coupling regime (Theorem 1 in [6], referring to the inductive conditions of [4, Sect. 2]). Our audit trail assumes that the specific analyticity parameter $\hat{\alpha}_1(\gamma)$ appearing in the polymer analyticity domain ([6, eq. (1.65)]) can be chosen uniformly in k within that framework. Extracting an explicit standalone lemma with all parameters spelled out from Balaban's inductive scheme is deferred.

A.1 Step 1: Single-scale polymer derivative bound

At each scale k, Proposition 2.3 provides the small-field bound:

$$\sup_{\Omega_k^{\mathrm{sf}}} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big] \leq C_{\mathrm{poly}}^2 e^{-2\kappa} \cdot L_{\mathrm{RG}}^{-(d-1)k}. \tag{30}$$

The extension to $\operatorname{ess\,sup}_{\mathcal{G}_k}$ (including the large-field region) is Assumption 5.4. As discussed in Remark 2.4, the large-field contribution cannot be controlled by probabilistic suppression alone; the geometric decay factor $L_{\mathrm{RG}}^{-(d-1)k}$ on the large-field region requires additional input from Balaban's construction that we do not extract as a standalone lemma here.

The small-field bound (30) is established in §2.4 using:

- The absorption identity ([6, eq. (1.37), p. 367]): after RG absorption, the dependence of $V_{<k}$ on E_k-variables enters only through the polymer terms $\mathbf{R}^{(k)}(X)$ and $\mathbf{B}^{(k)}(X)$.
- Analyticity of polymer activities on the complex domain $\widetilde{U}_k^c(Y, \tilde{\alpha}_0, \hat{\alpha}_1)$ ([6, eq. (1.65), p. 383]), with uniform radius $\hat{\alpha}_1(\gamma)$.
- Cauchy derivative bounds: $|\nabla \mathbf{R}^{(k)}(X)| \leq (C/\hat{\alpha}_1) e^{-\kappa d_k(X)}$.
- Block-averaging adjoint scaling: $\|Q_{(k)}^*\|^2 = L_{\mathrm{RG}}^{-(d-1)k}$ ([1, eq. (125), Proposition 4]).

A.2 Step 2: Propagation across intermediate scales (small-field region)

As in Step 1, the bounds in Steps 2–3 are established on the small-field region Ω^{sf} only. The extension to $\operatorname{ess\,sup}_{\mathcal{G}_\ell}$ *(including the large-field region) is the content of Assumption 5.4; see Remark 2.4.*

For $\ell < k$, the potential $V_{<\ell}$ depends on scale-k variables through the chain of RG transformations at scales $\ell, \ell+1, \dots, k-1$. Balaban's effective action at scale ℓ ([6, eq. (1.101)]) depends on the background field U_ℓ, which in turn depends on all finer-scale variables through the minimisation and averaging operations.

The key structural property is that the dependence of U_ℓ on scale-k variables ($k > \ell$) enters through the k-th order block-averaging $Q_{(k)}$. By Balaban's Proposition 4 ([1, eq. (133), p. 42]), the linearised k-fold averaging $Q_k(U_0)$ satisfies:

$$|Q_k(U_0, \eta A) - Q_k(U_0)A| < C_2|A|^2,$$

with constants *independent of* k. The linear part $Q_k(U_0)$ has adjoint norm $\|Q_k(U_0)^*\|^2 \leq (1 + 2C_1'\alpha_0)^2 \cdot L_{\mathrm{RG}}^{-(d-1)k}$ by [1, eq. (146)].

Therefore, on Ω^{sf}, a perturbation in the E_k-direction propagates to scale $\ell < k$ with an amplitude bounded by $\|Q_{(k)}^*\| = L_{\mathrm{RG}}^{-(d-1)k/2}$, regardless of the number of intermediate scales $k - \ell$. This is because the block-averaging adjoint at scale k already accounts for the full compression from the fine to the coarse lattice.

A.3 Step 3: Composition and the cross-scale bound (small-field region)

Combining Steps 1 and 2, and working throughout on Ω^{sf}: the gradient $\nabla_{E_k} V_{<\ell}$ receives contributions from the polymer activities $\mathbf{R}^{(m)}(X)$ at all intermediate scales $\ell \leq m \leq k$. On the small-field region $\Omega^{\mathrm{sf}}_{\ell:k}$ (defined below), the boundary terms $\mathbf{B}^{(m)}(X)$ vanish identically for every intermediate scale m, since each $\mathbf{B}^{(m)}(X)$ is supported on the enlarged large-field region $Z_m^{\sim}$ ([6, eq. (1.69), p. 377]) and $\Omega^{\mathrm{sf}}_{\ell:k} \cap Z_m^{\sim} = \varnothing$ by definition. Thus only the polymer residuals $\mathbf{R}^{(m)}(X)$ contribute. Each contribution satisfies:

$$|\nabla_{E_k} \mathbf{R}^{(m)}(X)| \leq \frac{C}{\hat{\alpha}_1} e^{-\kappa d_m(X)} \cdot \|Q_{(k)}^*\| \leq \frac{C}{\hat{\alpha}_1} e^{-\kappa d_m(X)} \cdot L_{\mathrm{RG}}^{-(d-1)k/2}.$$

The factor $L_{\mathrm{RG}}^{-(d-1)k/2}$ is the same for all intermediate scales m, because it arises from the projection onto E_k (via the adjoint of the full k-step block-averaging), not from the individual steps.

Here $\Omega^{\mathrm{sf}}_{\ell:k} := \bigcap_{m=\ell}^{k} (\mathcal{A} \setminus Z_m^{\sim})$ denotes the region where the small-field condition holds at *all* intermediate scales $\ell \leq m \leq k$; on this region, $\mathbf{B}^{(m)}(X) = 0$ for every m in the sum.

Summing over polymers X at each intermediate scale m (using Lemma 5.6) and over scales m, the supremum is taken over all coarse-field configurations $\bar{A}^{(\ell)} \in \mathcal{G}_\ell$ consistent with $\Omega^{\mathrm{sf}}_{\ell:k}$:

$$\begin{aligned}
\sup_{\Omega^{\mathrm{sf}}_{\ell:k}} \mathbb{E}\big[|\nabla_{E_k} V_{<\ell}|^2 \mid \mathcal{G}_\ell\big] &\leq \Bigg(\sum_{m=\ell}^{k} \sum_{X \ni \text{link at scale } m} \frac{C}{\hat{\alpha}_1} e^{-\kappa d_m(X)} \cdot L_{\mathrm{RG}}^{-(d-1)k/2} \Bigg)^2 \\
&\leq \Bigg(\sum_{m=\ell}^{k} \frac{C' e^{-\kappa}}{\hat{\alpha}_1} \cdot L_{\mathrm{RG}}^{-(d-1)k/2} \Bigg)^2 \\
&\leq \left(\frac{(k - \ell + 1) C' e^{-\kappa}}{\hat{\alpha}_1} \right)^2 \cdot L_{\mathrm{RG}}^{-(d-1)k}.
\end{aligned}$$

This establishes the cross-scale gradient bound on the small-field region. The full $\operatorname{ess\,sup}_{\mathcal{G}_\ell}$ bound (16) requires, in addition, control of the large-field contribution; this is precisely the content of Assumption 5.4.

The factor $(k - \ell + 1) \leq n_{\max}$ is polynomial in $\log(1/g^2)$ (since $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$). This factor is *not* absorbed by $e^{-\kappa}$ (which is a fixed constant from Balaban's construction), but is instead incorporated into the constant $C_{\text{poly}} = C_{\text{poly}}(N_c, \beta_0) = n_{\max}(\beta_0) \cdot C'/\hat{\alpha}_1(\gamma)$. Since β_0 is fixed in Theorem 1.1, this dependence does not affect the uniformity of α_* in L_{vol}. It does affect the quantitative value of α_* (and hence of Δ_{phys}), which decreases as β_0 increases—consistent with the expected physical scaling of the mass gap.

Remark A.2. The factor $n_{\max}$ in C_{poly} means that the polymer derivative constant grows with β (since $n_{\max} \sim 1/(2b_0 g^2 \ln 2)$). This growth is polynomial in $\log(1/g^2)$ and does not affect the positivity of α_*, only its quantitative value. The mass gap bound $\Delta_{\text{phys}} \geq c(N_c, \beta_0) > 0$ remains uniform in L_{vol}.