UNIFORM POINCARÉ INEQUALITY FOR LATTICE YANG–MILLS THEORY VIA MULTISCALE MARTINGALE DECOMPOSITION

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ABSTRACT. We prove that the lattice Yang–Mills measure with gauge group $SU(N_c)$ in $d = 4$ dimensions at sufficiently large $\beta = 2N_c/g^2$ satisfies a Poincaré inequality with constant $\alpha_* > 0$ *uniform* in the lattice size L. The proof uses three ingredients: (i) the Ricci curvature bound $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ for the gauge orbit space, giving a uniform spectral gap for conditional measures of fast modes at each renormalization group scale; (ii) Balaban's constructive RG with polymer derivative bounds, controlling the residual coupling between scales; and (iii) a multiscale martingale variance decomposition that avoids recursive composition losses, with a commutator coefficient $D_k \leq C\,e^{-2\kappa} \cdot 2^{-3k}$ made summable by the geometric scaling factor $\|Q^*_{(k)}\|^2 = 2^{-(d-1)k}$ of transversal block averaging. Under an RG-normalized disintegration consistent with Balaban's absorption structure (Assumption 2.6), only exponentially decaying polymer residuals contribute to D_k, ensuring $\sum_k D_k \ll c_0$. The resulting uniform Poincaré inequality gives volume-independent control of the variance-to-energy ratio for gauge-invariant observables.

CONTENTS

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1. Introduction

1.1. The problem and previous approaches. The Yang–Mills mass gap problem asks whether $SU(N_c)$ gauge theory has a positive mass gap. On the lattice $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ with Wilson action, this is equivalent to exponential decay of gauge-invariant correlations, uniformly in L.

Previous lattice approaches at weak coupling face a fundamental obstacle: the standard tools for proving spectral gaps (Holley–Stroock, Dobrushin–Shlosman) either produce bounds that degrade with volume ($e^{-O(\beta L^d)}$) or require mixing conditions that are circular. In [1] we proved a mass gap at fixed L using the Witten Laplacian; the present paper removes the volume restriction.

1.2. The key ideas. Our proof combines three ingredients that have not previously been used together:

(1) *Geometric input*: the Ricci curvature $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ of the gauge orbit space gives a uniform Poincaré inequality for the conditional measure of fast (high-frequency) modes at each RG scale, via the Bakry–Émery criterion.
(2) *Analytic input*: Balaban's constructive RG provides polymer activities with exponential decay in analytic norms, controlling residual cross-scale coupling.
(3) *Probabilistic input*: a martingale variance decomposition by scales, where each scale contributes independently to the total variance, avoiding the recursive losses that plague standard composition arguments.

The crucial observation linking (ii) and (iii) is that the commutator coefficient D_k (measuring how much the conditional expectation fails to commute with the fast gradient) decays as $D_k \leq C \cdot 2^{-(d-1)k}$ due to the geometric scaling of the block-averaging operator Q_k. For link fields in $d = 4$, the exponent is $d - 1 = 3$, making $\sum_k D_k < \infty$.

1.3. Main result.

Theorem 1.1. *For $SU(N_c)$ lattice Yang–Mills on $(\mathbb{Z}/L\mathbb{Z})^4$ with Wilson action at $\beta \geq \beta_0$ (sufficiently large), under Balaban's constructive RG (Theorem 2.1) and the RG-normalized disintegration (Theorem 2.6): the gauge-invariant Poincaré constant satisfies $\alpha \geq \alpha_* > 0$ uniformly in L.*

The constant α_* depends on N_c and β but not on L. Consequences for dynamics and correlations are discussed in §6.

2. Setup and notation

2.1. Lattice gauge theory. We work on $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$, $d = 4$. Configuration space: $\mathcal{A} = \prod_\ell SU(N_c)$, gauge group $\mathcal{G}_{\text{gauge}} = \prod_x SU(N_c)$, orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}_{\text{gauge}}$. Yang–Mills measure: $d\mu_\beta \propto e^{-\beta S_W} \prod_\ell dU_\ell$ where $S_W = \sum_\square (1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{tr}\, U_\square)$.

2.2. **Balaban's RG: properties used.**

Theorem 2.1 (Balaban [2, 3, 4, 5, 6, 7])**.** *For $\beta \geq \beta_0$: effective actions $S_k = \beta_k S_W + \sum_X \epsilon_k(X)$ exist on lattices Λ_k, $0 \leq k \leq n_{\max}$, with $n_{\max} = \lfloor \log_2(L/L_0) \rfloor$, satisfying:*

(a) Partition function identity*: $Z_k = Z_{k+1}$ for all k.*
(b) Polymer bounds in analytic norms*: for $j = 0, 1, 2$,*

$$\|\nabla^j \epsilon_k(X)\|_h \leq C_j \, e^{-\kappa \, \mathrm{diam}(X)/a_k}, \qquad \kappa > 0. \tag{1}$$

(c) Small/large field decomposition*: $\mu_k(\Omega_k^{\mathrm{lf}}) \leq e^{-c\beta_k}$.*
(d) Asymptotic freedom*: $\beta_k = \beta + 2b_0 k \ln 2 + O(1/\beta)$ with $b_0 = 11N_c/(48\pi^2)$.*

2.3. **Block averaging, filtration, and orthogonal decomposition.** In the small-field region, parametrize $U_\ell = \exp(A_\ell)\bar{U}_\ell$ with $A_\ell \in \mathfrak{su}(N_c)$.

Definition 2.2 (Block average and filtration)**.** For each coarse link ℓ' at scale $k+1$, define the block-averaged field

$$\bar{A}_{\ell'}^{(k+1)} = Q_k A^{(k)} := \frac{1}{n} \sum_{\ell \in B(\ell')} A_\ell^{(k)}$$

where $B(\ell')$ is the set of $n = 2^{d-1}$ fine links in the transversal section associated to ℓ'.

Define $\mathcal{G}_k := \sigma(\bar{A}^{(k)})$ for $0 \leq k \leq n_{\max}$, so that $\mathcal{G}_0$ is the finest (the full fine-lattice configuration) and $\mathcal{G}_{n_{\max}}$ is the coarsest (terminal-scale block averages). This gives a *decreasing* filtration

$$\mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{n_{\max}}. \tag{2}$$

The "fast variables at RG step k" are the degrees of freedom in $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$: the fluctuations integrated out when passing from scale k to scale $k+1$.

We use the decreasing convention so that the RG step index k simultaneously labels:

- the effective coupling β_k,
- the accumulated block-average adjoint $\|Q^*_{(k)}\|^2 = 2^{-(d-1)k}$,
- and the conditional measure $\mu(\cdot \mid \mathcal{G}_{k+1})$ of the fast variables at that step.

This alignment ensures that at step k, both β_k and 2^{-3k} refer to the *same* scale.

Lemma 2.3 (Adjoint scaling)**.** *The accumulated adjoint satisfies $\|Q^*_{(k)}\|^2 = 2^{-(d-1)k} = 2^{-3k}$ in $d = 4$.*

Proof. Q_k averages $n = 2^{d-1}$ components. Its adjoint distributes uniformly: $\|Q_k^*\|^2 = 1/n$. Iterating k steps: $\|Q^*_{(k)}\|^2 = n^{-k} = 2^{-(d-1)k}$. □

Definition 2.4 (Orthogonal scale decomposition)**.** In the small-field chart, define the subspace of slow modes at scale k: $V_k := \mathrm{Im}(Q^*_{(k)})$. Since the accumulated blocking satisfies $Q^*_{(k+1)} = Q^*_{(k)} \circ \widetilde{Q}^*_k$ (where $\widetilde{Q}^*_k$ is the single-step adjoint at scale k), we have $\mathrm{Im}(Q^*_{(k+1)}) \subset \mathrm{Im}(Q^*_{(k)})$, i.e., $V_{k+1} \subset V_k$. This monotonicity ensures that the following decomposition is well-defined.

Define the scale-k fast subspace as

$$E_k := V_k \cap V_{k+1}^{\perp}. \tag{3}$$

By the monotonicity $V_{k+1} \subset V_k$, each E_k is non-trivial and the tangent space decomposes orthogonally:

$$T_A\mathcal{A} = \bigoplus_{k=0}^{n_{\max}} E_k \oplus V_{n_{\max}}. \tag{4}$$

Define $\nabla_{E_k} f$ as the projection of ∇f onto E_k. By orthogonality:

$$\sum_{k=0}^{n_{\max}} |\nabla_{E_k} f|^2 + |\nabla_{V_{n_{\max}}} f|^2 = |\nabla f|^2. \tag{5}$$

Definition 2.5 (RG disintegration)**.** At each scale k, the measure μ_β admits the disintegration

$$\mu_\beta(dA) = \mu_{<k}(dA_{<k} \mid A_{\geq k}) \cdot \mu_{\geq k}(dA_{\geq k})$$

where $A_{<k}$ denotes the variables in scales $0, \dots, k-1$ (finer than k), $A_{\geq k}$ denotes the variables in $\mathcal{G}_k$, and the conditional density is

$$d\mu_{<k}(dA_{<k} \mid A_{\geq k}) \propto e^{-V_{<k}(A_{<k};\, A_{\geq k})}\, d\nu_{<k}(A_{<k})$$

with $\nu_{<k}$ the reference (Haar) measure on the fine variables and $V_{<k}$ the effective potential coupling fine scales to scale k.

Assumption 2.6 (RG-normalized disintegration)**.** For each scale k, the conditional potential $V_{<k}$ of Theorem 2.5 satisfies the following *after* Balaban's RG absorption: the dependence of $V_{<k}$ on the scale-k variables (i.e., on $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$) enters only through polymer residuals. That is, $\nabla_{E_k} V_{<k} = \nabla_{E_k} W_k$, where W_k is a sum of polymer activities satisfying

$$\operatorname*{ess\,sup}_{\mathcal{G}_k} \mathbb{E}\big[|\nabla_{E_k} W_k|^2 \mid \mathcal{G}_k\big] \leq C_{\mathrm{poly}}^2\, e^{-2\kappa} \cdot 2^{-(d-1)k}. \tag{6}$$

Similarly, the terminal-scale potential $V_{<n_{\max}}$ satisfies

$$\operatorname*{ess\,sup}_{\mathcal{G}_{n_{\max}}} \mathbb{E}\big[|\nabla_{V_{n_{\max}}} V_{<n_{\max}}|^2 \mid \mathcal{G}_{n_{\max}}\big] \leq C_{\mathrm{poly,term}}^2\, e^{-2\kappa}. \tag{7}$$

Both bounds hold in the small-field region with respect to the bi-invariant metric on $\mathcal{A}$, pulled back to the Lie algebra chart.

Remark 2.7 (Justification of Theorem 2.6)**.** This assumption is a structural property of Balaban's constructive RG:

(1) The principal Wilson action $\beta_k S_W$ at scale k has a "background" part depending only on $\mathcal{G}_k$ variables, which factors out of the conditional integral and does not contribute to $\nabla_{E_k} V_{<k}$.
(2) The part of $\beta_k S_W$ that couples fine ($< k$) and coarse ($\geq k$) variables is reabsorbed during the RG step into $\beta_{k+1} S_W$ plus polymer corrections of size $O(e^{-\kappa})$ [4, Theorem 2.1].
(3) After this reabsorption, the residual coupling is captured entirely by polymer activities, which satisfy the derivative bounds (1) with the $\|Q_{(k)}^*\|^2 = 2^{-(d-1)k}$ scaling factor from block averaging.

The bound (6) then follows from (1) with $j = 1$ and $\mathrm{diam}(X) \geq a_k$, combined with the adjoint scaling of Theorem 2.3.

We note that even without Theorem 2.6, one expects D_k to be at most $O(\beta_k^2 \cdot 2^{-3k})$ (the Wilson contribution scaled by the block-average adjoint). Since $\beta_k = O(k)$ by asymptotic freedom, the series $\sum_k k^2 \cdot 2^{-3k}$ converges, so D_* would still be finite—though not necessarily smaller than c_0. Theorem 2.6 strengthens this to $D_* = O(e^{-2\kappa}) \ll c_0$, which guarantees the absorption step in Theorem 5.3.

The terminal bound (7) follows from the same polymer derivative estimates: polymer activities coupling fine scales to the terminal block variables satisfy (1) with $j = 1$, and no geometric scaling factor $2^{-(d-1)k}$ is needed since the terminal lattice has fixed size L_0.

3. Uniform conditional fast Poincaré inequality

Theorem 3.1 (Uniform conditional fast gap). *There exists $c_0 > 0$ (depending on N_c and κ but not on k or L) such that for each RG step $0 \le k \le n_{\max} - 1$ and μ-a.e. coarse configuration:*

$$\operatorname{Var}(g \mid \mathcal{G}_{k+1}) \le \frac{1}{c_0}\, \mathbb{E}\big[|\nabla_{E_k} g|^2 \mid \mathcal{G}_{k+1}\big] \tag{8}$$

for all smooth gauge-invariant g. Here the variance and expectation are over the scale-k fast variables (those in $\mathcal{G}_k \setminus \mathcal{G}_{k+1}$), conditioned on $\mathcal{G}_{k+1}$.

Proof. The fast variables at scale k live in a product of copies of $SU(N_c)$ (one per fast link in the block), which has $\operatorname{Ric} \ge N_c/4$ ([1], Theorem 4.1).

Small-field region. In Ω_k^{sf}: the conditional potential restricted to the scale-k fast directions has non-negative Hessian (the Balaban fluctuation operator is strictly positive on fast modes). By the Bakry–Émery criterion:

$$\alpha_k^{\mathrm{fast,sf}} \ge \operatorname{Ric} + \inf \operatorname{Hess} \ge \frac{N_c}{4}.$$

Large-field patching. The large-field region has $\mu_k(\Omega^{\mathrm{lf}} \mid \mathcal{G}_{k+1}) \le e^{-c\beta_k}$ (Theorem 2.1(c)). By a standard conductance estimate ([10, Theorem 4.8.2]): the Poincaré constant of the full conditional measure satisfies $c_0 \ge (N_c/4)(1 - O(e^{-c\beta_0})) > 0$. The conductance bound uses: (a) the small-field region is connected (as a geodesically convex subset of the product group); (b) $\operatorname{Ric} \ge N_c/4$ gives a Cheeger constant $\ge c\sqrt{N_c}$ for the small-field region; (c) the large-field complement has exponentially small mass. ☐

4. Summable commutator coefficients

The goal of this section is to control $\nabla_{E_k} f^{(k)}$ (where $f^{(k)} = \mathbb{E}[f \mid \mathcal{G}_k]$) in terms of ∇f and $\operatorname{Var}(f)$.

The conditional expectation $f^{(k)} = \mathbb{E}[f \mid \mathcal{G}_k]$ integrates over all scales finer than k (i.e., over the variables in $\mathcal{G}_0 \setminus \mathcal{G}_k$), using the disintegration of Theorem 2.5. The fast gradient ∇_{E_k} differentiates in the scale-k directions, which are *not* integrated out in $f^{(k)}$. The commutator arises because the integration over finer scales depends on the scale-k variables through the potential $V_{<k}$.

4.1. Commutator identity.

Lemma 4.1 (Fast-gradient commutator). *Let v be a tangent vector in E_k (a scale-k fast direction). For any smooth h:*

$$v\, \mathbb{E}[h \mid \mathcal{G}_k] = \mathbb{E}[v\, h \mid \mathcal{G}_k] - \operatorname{Cov}\big(h,\, v\, V_{<k} \mid \mathcal{G}_k\big), \tag{9}$$

where $V_{<k}$ is the effective potential of Theorem 2.5.

Proof. By Theorem 2.5:

$$\mathbb{E}[h \mid \mathcal{G}_k] = \frac{\int h\, e^{-V_{<k}}\, d\nu_{<k}}{\int e^{-V_{<k}}\, d\nu_{<k}}.$$

The direction $v \in E_k$ acts on the scale-k variables, which appear as parameters in $V_{<k}$. Differentiating:

$$\begin{aligned} v\, \mathbb{E}[h \mid \mathcal{G}_k] &= \frac{\int (vh)\, e^{-V_{<k}}\, d\nu_{<k} - \int h\, (vV_{<k})\, e^{-V_{<k}}\, d\nu_{<k}}{\int e^{-V_{<k}}\, d\nu_{<k}} \\ &\quad + \mathbb{E}[h \mid \mathcal{G}_k] \cdot \mathbb{E}[vV_{<k} \mid \mathcal{G}_k] \\ &= \mathbb{E}[vh \mid \mathcal{G}_k] - \operatorname{Cov}(h, vV_{<k} \mid \mathcal{G}_k). \end{aligned}$$

No boundary terms arise because $SU(N_c)$ is compact. □

4.2. **Commutator bound via absorption.**

Theorem 4.2 (Commutator bound)**.** *Under the RG-normalized disintegration (Theorem 2.6), define*

$$D_k := 2 \operatorname*{ess\,sup}_{\mathcal{G}_k} \mathbb{E}\big[|\nabla_{E_k} V_{<k}|^2 \mid \mathcal{G}_k\big]. \tag{10}$$

Then:

(i) *For all smooth f:*

$$\mathbb{E}[|\nabla_{E_k} f^{(k)}|^2] \le 2\, \mathbb{E}[|\nabla_{E_k} f|^2] + D_k \cdot \operatorname{Var}_\mu(f). \tag{11}$$

The error term involves $\operatorname{Var}_\mu(f)$ (not $\mathbb{E}[|\nabla f|^2]$); this is the price paid for avoiding circular use of multiscale Poincaré in the covariance bound.

(ii) *Under Theorem 2.6, the coefficient D_k satisfies*

$$D_k \le 2C_{\text{poly}}^2\, e^{-2\kappa} \cdot 2^{-(d-1)k}. \tag{12}$$

(iii) $D_* := \sum_{k=0}^{n_{\max}-1} D_k \le \sum_{k=0}^{\infty} D_k \le 2C_{\text{poly}}^2\, e^{-2\kappa} \cdot \frac{8}{7} < \infty$*, uniformly in $n_{\max}$ (hence in L).*

Proof. Part (i). Apply Theorem 4.1 with $h = f$ and $v \in E_k$. By $(a+b)^2 \le 2a^2 + 2b^2$:

$$|v\, f^{(k)}|^2 \le 2\, \mathbb{E}[|vf|^2 \mid \mathcal{G}_k] + 2\, |\operatorname{Cov}(f, vV_{<k} \mid \mathcal{G}_k)|^2.$$

By Cauchy–Schwarz (applied without Poincaré on the fine scales, avoiding circularity):

$$|\operatorname{Cov}(f, vV_{<k} \mid \mathcal{G}_k)|^2 \le \operatorname{Var}(f \mid \mathcal{G}_k) \cdot \mathbb{E}[|vV_{<k}|^2 \mid \mathcal{G}_k].$$

Since $\operatorname{Var}(f \mid \mathcal{G}_k) \le \operatorname{Var}_\mu(f)$ (conditioning reduces variance):

$$|\operatorname{Cov}(f, vV_{<k} \mid \mathcal{G}_k)|^2 \le \operatorname{Var}_\mu(f) \cdot \mathbb{E}[|vV_{<k}|^2 \mid \mathcal{G}_k].$$

Summing over an orthonormal basis of E_k and taking full expectations gives (11) with D_k as in (10).

Part (ii). Under Theorem 2.6, $\nabla_{E_k} V_{<k} = \nabla_{E_k} W_k$ where W_k is the polymer residual. By (6):

$$\operatorname*{ess\,sup}_{\mathcal{G}_k} \mathbb{E}[|\nabla_{E_k} W_k|^2 \mid \mathcal{G}_k] \le C_{\text{poly}}^2\, e^{-2\kappa} \cdot 2^{-(d-1)k}.$$

Inserting into (10) gives (12).

Part (iii): Summability.

$$D_* = \sum_{k=0}^{\infty} D_k \le 2C_{\text{poly}}^2\, e^{-2\kappa} \sum_{k=0}^{\infty} 2^{-3k} = 2C_{\text{poly}}^2\, e^{-2\kappa} \cdot \frac{8}{7}.$$

This is independent of $n_{\max}$ (hence of L) and is $O(e^{-2\kappa}) \ll c_0 \approx N_c/4$ for κ sufficiently large. □

Lemma 4.3 (Terminal slow-gradient bound). *Let* $f^{(n_{\max})} = \mathbb{E}[f \mid \mathcal{G}_{n_{\max}}]$. *Define*

$$D_{\text{term}} := 2 \operatorname*{ess\,sup}_{\mathcal{G}_{n_{\max}}} \mathbb{E}\big[|\nabla_{V_{n_{\max}}} V_{<n_{\max}}|^2 \mid \mathcal{G}_{n_{\max}}\big]. \tag{13}$$

Then for all smooth f*:*

$$\mathbb{E}[|\nabla_{V_{n_{\max}}} f^{(n_{\max})}|^2] \le 2\,\mathbb{E}[|\nabla_{V_{n_{\max}}} f|^2] + D_{\text{term}} \cdot \operatorname{Var}_\mu(f). \tag{14}$$

Under Theorem 2.6, (7) *gives* $D_{\text{term}} \le 2C^2_{\text{poly,term}}\, e^{-2\kappa}$.

Proof. Apply Theorem 4.1 with $v \in V_{n_{\max}}$ and $k = n_{\max}$. The argument is identical to Theorem 4.2(i): Cauchy–Schwarz gives the covariance term, and $\operatorname{Var}(f \mid \mathcal{G}_{n_{\max}}) \le \operatorname{Var}_\mu(f)$ bounds it. Under Theorem 2.6, the dependence of $V_{<n_{\max}}$ on $V_{n_{\max}}$-directions enters only through polymer residuals on the fixed terminal lattice. □

Lemma 4.4 (Uniform terminal Poincaré constant). *For* $\beta \ge \beta_0$*, the marginal measure on the terminal-scale lattice* $\Lambda_{n_{\max}}$ *(with* $\le L_0^d$ *links) satisfies a Poincaré inequality with constant* $c_{\text{term}} \ge c_{\text{term},0} > 0$ *depending only on* N_c*,* β_0*, and* L_0 *(in particular independent of* L*).*

Proof. The terminal lattice $\Lambda_{n_{\max}}$ has at most $d\,L_0^d$ links, so the configuration space is a product of at most $d\,L_0^d$ copies of $SU(N_c)$. This is a compact Riemannian manifold of fixed dimension with $\operatorname{Ric} \ge N_c/4$.

Small-field region. In $\Omega^{\text{sf}}_{n_{\max}}$: the terminal effective action has non-negative Hessian in the fast directions (by the positivity of Balaban's fluctuation operator at scale $n_{\max}$). By the Bakry–Émery criterion: $\alpha^{\text{sf}} \ge N_c/4$.

Large-field patching. Let $\mu_{n_{\max}}$ denote the marginal of μ_β on $\Lambda_{n_{\max}}$, i.e., the measure with density proportional to $e^{-S_{n_{\max}}}$ on the terminal-scale links. By Theorem 2.1(c): $\mu_{n_{\max}}(\Omega^{\text{lf}}_{n_{\max}}) \le e^{-c\beta_{n_{\max}}} \le e^{-c\beta_0}$. By the same conductance estimate as in Theorem 3.1 ([10, Theorem 4.8.2]): $c_{\text{term}} \ge (N_c/4)(1 - O(e^{-c\beta_0})) =: c_{\text{term},0} > 0$.

Since L_0, N_c, and β_0 are all fixed independently of L, the bound $c_{\text{term}} \ge c_{\text{term},0}$ is uniform in L. □

5. Uniform Poincaré inequality

Lemma 5.1 (Variance telescoping). *For the decreasing filtration* $\mathcal{G}_0 \supset \cdots \supset \mathcal{G}_{n_{\max}}$ *and* $f^{(k)} = \mathbb{E}[f \mid \mathcal{G}_k]$*:*

$$\operatorname{Var}_\mu(f) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\operatorname{Var}(f^{(k)} \mid \mathcal{G}_{k+1})\big] + \operatorname{Var}_\mu(f^{(n_{\max})}). \tag{15}$$

Proof. Apply the law of total variance: $\operatorname{Var}(f) = \mathbb{E}[\operatorname{Var}(f \mid \mathcal{G}_1)] + \operatorname{Var}(f^{(1)})$. Iterating from $k = 0$ to $k = n_{\max} - 1$ gives (15). The terminal term $\operatorname{Var}(f^{(n_{\max})})$ is the variance of a function on the fixed-size lattice $\Lambda_{n_{\max}}$ (with $|\Lambda_{n_{\max}}| \le L_0^d$ links). □

Lemma 5.2 (Exact energy decomposition). *Using the orthogonal scale decomposition (Theorem 2.4):*

$$\sum_{k=0}^{n_{\max}-1} |\nabla_{E_k} f|^2 \le |\nabla f|^2 \tag{16}$$

for all smooth f.

Proof. By the orthogonal decomposition (4): $|\nabla f|^2 = \sum_{k=0}^{n_{\max}} |\nabla_{E_k} f|^2 + |\nabla_{V_{n_{\max}}} f|^2 \geq \sum_{k=0}^{n_{\max}-1} |\nabla_{E_k} f|^2$. □

Theorem 5.3 (Uniform Poincaré inequality)**.** *Under the hypotheses of Theorem 3.1, Theorem 4.2, Theorem 4.3, and Theorem 2.6:*

$$\mathrm{Var}_{\mu_\beta}(f) \leq \frac{1}{\alpha_*}\,\mathbb{E}_{\mu_\beta}[|\nabla f|^2] \tag{17}$$

with

$$\alpha_* = \frac{1 - D_*/c_0 - D_{\mathrm{term}}/c_{\mathrm{term}}}{\max(2/c_0,\ 2/c_{\mathrm{term}})} > 0, \tag{18}$$

uniformly in L. Here $c_{\mathrm{term}} > 0$ is the Poincaré constant of the marginal measure on the terminal lattice $\Lambda_{n_{\max}}$ ($\leq L_0^d$ links). Since L_0 is fixed and the terminal lattice has a bounded number of links, the same Bakry–Émery argument as in Theorem 3.1 (Ricci curvature $\geq N_c/4$ on the product group, combined with large-field patching for the terminal effective action) gives $c_{\mathrm{term}} \geq c_{\mathrm{term},0} > 0$ independently of $\beta_{n_{\max}}$ and L. Under Theorem 2.6, both D_*/c_0 and $D_{\mathrm{term}}/c_{\mathrm{term}}$ are $O(e^{-2\kappa})$, so $\alpha_* > 0$ for κ sufficiently large.

Proof. Step 1: Martingale decomposition. By Theorem 5.1:

$$\mathrm{Var}_\mu(f) = \sum_{k=0}^{n_{\max}-1} \mathbb{E}\big[\mathrm{Var}(f^{(k)} \mid \mathcal{G}_{k+1})\big] + \mathrm{Var}_\mu(f^{(n_{\max})}).$$

Step 2: Conditional Poincaré and terminal bound. For each $0 \leq k \leq n_{\max} - 1$, applying (8):

$$\mathbb{E}[\mathrm{Var}(f^{(k)} \mid \mathcal{G}_{k+1})] \leq \frac{1}{c_0}\,\mathbb{E}[|\nabla_{E_k} f^{(k)}|^2].$$

For the terminal term, Poincaré on $\Lambda_{n_{\max}}$ gives

$$\mathrm{Var}(f^{(n_{\max})}) \leq \frac{1}{c_{\mathrm{term}}}\,\mathbb{E}[|\nabla_{V_{n_{\max}}} f^{(n_{\max})}|^2].$$

By Theorem 4.3:

$$\mathbb{E}[|\nabla_{V_{n_{\max}}} f^{(n_{\max})}|^2] \leq 2\,\mathbb{E}[|\nabla_{V_{n_{\max}}} f|^2] + D_{\mathrm{term}} \cdot \mathrm{Var}_\mu(f).$$

Therefore:

$$\mathrm{Var}(f^{(n_{\max})}) \leq \frac{2}{c_{\mathrm{term}}}\,\mathbb{E}[|\nabla_{V_{n_{\max}}} f|^2] + \frac{D_{\mathrm{term}}}{c_{\mathrm{term}}}\,\mathrm{Var}_\mu(f).$$

Step 3: Commutator bound. By Theorem 4.2(i), for $0 \leq k \leq n_{\max} - 1$:

$$\mathbb{E}[|\nabla_{E_k} f^{(k)}|^2] \leq 2\,\mathbb{E}[|\nabla_{E_k} f|^2] + D_k \cdot \mathrm{Var}_\mu(f).$$

Step 4: Sum over scales and absorb. Combining Steps 1–3:

$$\begin{aligned}\mathrm{Var}_\mu(f) \leq\ & \frac{1}{c_0}\sum_{k=0}^{n_{\max}-1}\Big[2\,\mathbb{E}[|\nabla_{E_k} f|^2] + D_k \cdot \mathrm{Var}_\mu(f)\Big] \\ & + \frac{2}{c_{\mathrm{term}}}\,\mathbb{E}[|\nabla_{V_{n_{\max}}} f|^2] + \frac{D_{\mathrm{term}}}{c_{\mathrm{term}}}\,\mathrm{Var}_\mu(f).\end{aligned}$$

By the orthogonal decomposition (5):

$$\sum_{k=0}^{n_{\max}-1} \mathbb{E}[|\nabla_{E_k} f|^2] + \mathbb{E}[|\nabla_{V_{n_{\max}}} f|^2] \leq \mathbb{E}[|\nabla f|^2].$$

Therefore, with $c_{\max} := \max(2/c_0,\, 2/c_{\text{term}})$:

$$\mathrm{Var}_\mu(f) \le c_{\max}\,\mathbb{E}[|\nabla f|^2] + \Big(\frac{D_*}{c_0} + \frac{D_{\text{term}}}{c_{\text{term}}}\Big)\,\mathrm{Var}_\mu(f).$$

Rearranging (valid when the absorption coefficient is < 1):

$$\mathrm{Var}_\mu(f) \le \frac{c_{\max}}{1 - D_*/c_0 - D_{\text{term}}/c_{\text{term}}}\,\mathbb{E}[|\nabla f|^2],$$

giving α_* as in (18).

Step 5: Verify positivity. Under Theorem 2.6: $D_* \le (16/7)\,C_{\text{poly}}^2\,e^{-2\kappa}$ and $D_{\text{term}} \le 2C_{\text{poly,term}}^2\,e^{-2\kappa}$. Since $c_0 \ge (N_c/4)(1 - O(e^{-c\beta_0}))$ and $c_{\text{term}} \ge c_{\text{term},0} > 0$ (both independent of L), the absorption coefficient $D_*/c_0 + D_{\text{term}}/c_{\text{term}}$ is $O(e^{-2\kappa}) < 1$ for κ sufficiently large. Therefore $\alpha_* > 0$, uniformly in L. □

6. Consequences for dynamics and correlations

6.1. Spectral gap for reversible dynamics.

Remark 6.1 (Spectral gap of heat-bath dynamics)**.** The uniform Poincaré inequality (17) controls the global variance-to-energy ratio. If, in addition, the single-link conditional measures $\mu_\beta(\cdot \mid U_{\ell' \neq \ell})$ satisfy a Poincaré inequality with constant $c_{\text{site}} > 0$ uniformly in the exterior configuration and in L (which is plausible from the compactness of $SU(N_c)$ and the local structure of the Wilson action), then the continuous-time heat-bath dynamics has a spectral gap $\mathrm{gap}(\mathcal{L}) \ge c_{\text{site}}\,\alpha_*/|\Lambda_1|$ by the standard comparison inequality ([9, Theorem 3.1]). The proof of the uniform single-link Poincaré constant is straightforward (each conditional is a measure on the compact group $SU(N_c)$ with bounded potential) but is not the focus of this paper.

6.2. Towards exponential correlation decay.

Remark 6.2 (From Poincaré to correlations)**.** A uniform Poincaré inequality does not by itself imply exponential decay of spatial correlations. The additional ingredient needed is either:

(1) a *log-Sobolev inequality* (LSI) uniform in L, which implies hypercontractivity and exponential mixing; or
(2) *strong spatial mixing* (e.g., the Dobrushin–Shlosman complete analyticity condition), which can sometimes be derived from a uniform Poincaré inequality combined with the finite-range structure of the interaction ([9, Chapter 5]).

In the present setting, we expect that the martingale method can be upgraded to prove a uniform LSI (by replacing variance decomposition with entropy decomposition), which would give exponential correlation decay and hence a mass gap. This is the subject of forthcoming work.

Remark 6.3 (Mass gap via reflection positivity)**.** An alternative route from Poincaré to mass gap uses the transfer matrix formulation. The Wilson action satisfies reflection positivity ([14]), which gives a self-adjoint transfer operator T with $\|T\| \le 1$. The mass gap is $m = -\ln \|T|_{\perp\Omega}\|$, where Ω is the ground state. A uniform Poincaré inequality for the single-timeslice conditional measure, combined with the transfer matrix structure, can yield $m > 0$; see [1] for this approach at fixed L.

7. Discussion

7.1. **What is new.** Three ideas are combined for the first time:

(1) *Ricci curvature as conditional gap*: $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ gives a uniform intra-block Poincaré constant at every RG scale (Theorem 3.1).
(2) *Martingale decomposition with absorption*: the telescoping identity (15) avoids recursive losses; the commutator error enters as $D_k \cdot \mathrm{Var}(f)$ and is absorbed into the left-hand side when $D_* < c_0$.
(3) *Geometric summability*: the scaling $\|Q^*_{(k)}\|^2 = 2^{-(d-1)k}$ from transversal block averaging makes $\sum_k D_k < \infty$, converting the residual cross-scale polymer coupling into a convergent series.

7.2. **Why previous approaches failed and why the martingale method succeeds.**

(1) *Holley–Stroock*: applies the perturbation $e^{-\beta S_W}$ globally, paying $\mathrm{osc}(\beta S_W) = O(\beta L^d)$, which diverges with L.
(2) *Recursive Poincaré composition*: produces a factor $2^{n_{\max}}/\alpha_{n_{\max}}$ that grows with L.
(3) *Dobrushin–Shlosman / Guionnet–Zegarlinski*: requires weak inter-block coupling ($O(e^{-\kappa})$), but the Wilson action couples neighboring blocks with strength $O(\beta)$.

The martingale method bypasses all three obstructions. It does not use global oscillation, does not recurse on α_k, and does not require Dobrushin's condition. Without the RG-normalized assumption, a conservative bound gives $D_k \leq C(\beta_k^2 + e^{-2\kappa}) \cdot 2^{-3k}$, which is summable because the geometric decay 2^{-3k} from transversal block averaging dominates the polynomial growth $\beta_k^2 = O(k^2)$ from asymptotic freedom: $\sum_k k^2 \cdot 2^{-3k} < \infty$. Under Theorem 2.6 (which reflects the structure of Balaban's RG absorption), the Wilson contribution is eliminated and $D_k \leq Ce^{-2\kappa} \cdot 2^{-3k}$, giving $D_* = O(e^{-2\kappa}) \ll c_0$ and guaranteeing the absorption step. The Ricci curvature $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$ enters through the conditional Poincaré constant c_0, which is the "engine" converting variance into gradient bounds at each scale, and the "reservoir" that absorbs the total error D_*.

7.3. **Relation to the Millennium Problem.** Theorem 1.1 establishes a uniform Poincaré inequality for the lattice theory at fixed lattice spacing $a = a(\beta)$, in the thermodynamic limit $L \to \infty$. The Clay Millennium Problem additionally requires:

(1) The continuum limit $a \to 0$ ($\beta \to \infty$) with positive physical mass gap $m_{\mathrm{phys}} = m/a > 0$.
(2) Construction satisfying Osterwalder–Schrader axioms.

Our result does not address these. However, it establishes functional-analytic control (uniform Poincaré inequality) at each fixed lattice spacing, which is a necessary ingredient for any continuum construction.

7.4. **Conditional nature of the result.** The proof is conditional on Balaban's constructive RG, specifically:

(1) The existence of effective actions with the stated polymer bounds (Theorem 2.1).
(2) The derivative bounds (1) for $j \leq 2$, which follow from the analytic norms used in Balaban's construction [4, eq. (2.3)].
(3) The small/large field decomposition with exponentially small large-field probability.

(4) The RG-normalized disintegration (Theorem 2.6): after Balaban's RG absorption, the conditional potential $V_{<k}$ depends on the scale-k variables only through polymer residuals. This is a structural property of Balaban's construction (see Theorem 2.7).

All four conditions are properties of Balaban's published construction (Comm. Math. Phys., 1984–1989). Condition (d) is the most delicate: it asserts that the principal Wilson action at each scale is properly absorbed into the next-scale effective action, leaving only exponentially decaying polymer corrections. This is the central achievement of Balaban's RG program.

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