Mass Gap for the Gribov–Zwanziger Lattice Measure: A Non-Perturbative Proof

Lluis Eriksson
Independent Researcher
lluiseriksson@gmail.com

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Abstract

We prove a volume-uniform bound on the zero-momentum gluon propagator of the quadratic Gribov–Zwanziger measure on a d-dimensional periodic lattice ($d \geq 2$) with gauge group $SU(N_c)$: $D(0) \leq C_{d,N_c}/g^2$ for all $L \geq 2$ and all $g > 0$, with the thermodynamic-limit value computed exactly. This is the necessary condition for a mass gap identified in Definition 4; under the massive-propagator interpretation it corresponds to the mass scale

$$m_{\text{gap}} = g\sqrt{\frac{(d-1)\,N_c\,\tilde{I}_1}{d}}, \qquad \tilde{I}_1 = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}\frac{\sin^2 k_\mu}{(\hat{k}^2)^2}.$$

(Version 1 claimed exponential decay of correlations, which does not follow from $D(0) < \infty$ alone; the claim is withdrawn.) Version 2 also corrects the model's operator and its constants, following a numerical adjudication: the forward-difference divergence form of v1's Faddeev–Popov operator is *not* symmetric ($\|\mathcal{M} - \mathcal{M}^T\| = 0.225$ measured, contradicting v1's "manifestly self-adjoint"), its zero-mode curvature is *negative* ($-0.039\,|s|^2$, matching to machine precision the phase-corrected lattice sum that v1's Lemma 12 omitted), and log-concavity fails for it (60/60 random directions) — so v1's proof collapses for the v1 operator. Version 2 redefines the Faddeev–Popov operator by the symmetric midpoint discretization, for which every step of the proof (Bhatia strict concavity, Prékopa reduction, Hessian computation, $1/N$ scaling) runs verbatim, with the corrected curvature $m_0^2 = g^2 N_c \tilde{I}_1 > 0$ — the same lattice integral found in the corrected Gram term of the companion paper ai.viXra:2602.0036 (v2). The predicted mass scale decreases accordingly (SU(3), $\beta = 6$: ≈ 0.37 GeV instead of v1's 0.59 GeV), and the comparison with lattice Monte Carlo is downgraded from quantitative agreement to order-of-magnitude consistency. No perturbative expansion in the coupling is employed. All claims are verified in a companion numerical suite.

Keywords: Gribov–Zwanziger, mass gap, lattice gauge theory, log-concave measures, Prékopa's theorem.

1 Introduction

The mass gap problem is one of the outstanding open questions in mathematical physics [1]. We do not solve the Clay Millennium Problem, which concerns the full Yang–Mills measure with Wilson action. Instead we establish a volume-uniform propagator bound for a simplified but well-defined lattice measure capturing the essential non-perturbative feature of the Gribov restriction: the quadratic Gribov–Zwanziger (GZ) measure [2, 3]. This replaces the Wilson weight by a Gaussian kinetic term while retaining the Faddeev–Popov (FP) determinant and the restriction to the first Gribov region Ω; the resulting measure is log-concave (with the operator of Definition/Eq. (1); see Remark 18 for why this fails for v1's operator), enabling convexity tools unavailable in the full theory.

Our main result (Theorem 5) states that the zero-momentum propagator $D(0)$ is bounded uniformly in the lattice volume. (v2 correction of framing: as v1's own Definition 4 records, boundedness of $D(0)$ is a *necessary* condition for a mass gap; v1's introduction and abstract asserted that it implies exponential decay of correlations, which is not proved here and is withdrawn. What is proved is the uniform bound, its exact thermodynamic value, and hence the mass *scale* of the would-be massive propagator.) The proof is non-perturbative in g; the only asymptotic parameter is the volume $N = L^d$.

Outline. Section 2 introduces the framework and the (corrected) GZ measure. Section 3 contains the proof in four steps, plus the v2 adjudication of the v1 operator (Section 3.5). Section 4 presents the corrected numerical values. Section 5 discusses scope and limitations.

2 Setup and Definitions

2.1 Lattice and fields

$\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$, $N = L^d$ sites; lattice Landau gauge, transverse fields $A^a_\mu(x)$, $\hat\partial_\mu A^a_\mu = 0$; $a = 1, \dots, N_c^2 - 1$; $\mu = 1, \dots, d$. Fourier components $\tilde A^a_\mu(k)$, hat-momenta $\hat k_\mu = 2\sin(k_\mu/2)$, $\hat k^2 = \sum_\mu \hat k_\mu^2$.

Remark 1 (Zero-mode counting). At $k = 0$ the transversality constraint is vacuous, so the zero-mode sector has dimension $K = d(N_c^2 - 1)$, independent of L.

2.2 Faddeev–Popov operator and vertex operators (corrected in v2)

The FP operator is defined in v2 by the *symmetric midpoint discretization*

$$\mathcal{M}(A) \;=\; -\hat\partial^2 \otimes \mathbf{1}_{\text{color}} \;-\; \frac{g}{2}\sum_\mu \{\, fA_\mu \,,\, \hat\partial^{\text{sym}}_\mu \}, \tag{1}$$

acting on $\mathcal{H} = L^2(\Lambda) \otimes \mathfrak{su}(N_c)$ (orthogonal complement of constants), where $(fA_\mu\psi)^a = f^{adb}A^d_\mu\psi^b$, $\hat\partial^{\text{sym}}_\mu\psi(x) = \frac{1}{2}[\psi(x+\hat\mu) - \psi(x-\hat\mu)]$, and $\{\cdot,\cdot\}$ is the anticommutator. $\mathcal{M}(A)$ is *symmetric for every real* A (each anticommutator term transposes into the other: $(\hat\partial^{\text{sym}})^T = -\hat\partial^{\text{sym}}$ and $(fA)^T = -fA$; verified exactly in the suite for random non-constant fields), affine in A, positive definite inside the first Gribov region, and reduces to the continuum $-\hat\partial \cdot D(A)$ up to $O(a^2)$.

v1 instead used the forward-difference divergence form $\mathcal{M}_{\text{v1}} = -\hat\partial_\mu(\delta^{ab}\hat\partial_\mu + gf^{adb}A^d_\mu\,\cdot\,)$ and asserted it to be "manifestly self-adjoint"; this is false, and the difference is not cosmetic — see Section 3.5.

Because $\mathcal{M}(A)$ is affine, $\mathcal{M}(A) = \mathcal{M}_0 + \sum_\alpha A_\alpha W_\alpha$ with $\mathcal{M}_0 = -\hat{\partial}^2 \otimes \mathbf{1}$. For the zero mode $\alpha = (\mu, a, k{=}0)$ the corrected vertex is, in the FP-momentum basis,

$$[W_{(\mu,a,0)}]_{(q,b),(q,c)} = -\frac{g}{\sqrt{N}}\, f^{bac}\, i \sin q_\mu, \tag{2}$$

which is Hermitian (v1's vertex $\propto i\hat{q}_\mu$ omitted the midpoint phase; equivalently, v1's operator carries the non-Hermitian factor $i\hat{q}_\mu e^{iq_\mu/2}$). It preserves FP momentum q; non-zero-mode vertices shift q by k.

2.3 Gribov region

$\Omega = \{A \in \mathcal{A}_T : \mathcal{M}(A) > 0\}$ (positive definite on the complement of constants; well posed since $\mathcal{M}$ is symmetric).

Proposition 2 (Dell'Antonio–Zwanziger [4])**.** *Ω is a convex subset of $\mathcal{A}_T$.*

2.4 Quadratic GZ measure

Definition 3.

$$d\mu(A) = \frac{1}{Z}\, \det \mathcal{M}(A)\; e^{-\frac{1}{2}(A,(-\hat{\partial}^2)A)}\; \mathbf{1}_\Omega(A)\, dA, \tag{3}$$

with potential

$$V(A) = -\ln \det \mathcal{M}(A) + \tfrac{1}{2}\big(A, (-\hat{\partial}^2)A\big). \tag{4}$$

2.5 Gluon propagator and mass gap

$$D(k) = \frac{1}{(d-1)(N_c^2-1)} \Big\langle \sum_{\mu,a} |\tilde{A}^a_\mu(k)|^2 \Big\rangle_\mu. \tag{5}$$

Definition 4. *The theory has a mass gap if the connected two-point function decays exponentially, uniformly in L. A* necessary *condition is $D(0) \le C < \infty$ uniformly in L. In this paper we prove this necessary condition. (v2: this definition is unchanged from v1; the abstract and introduction have been aligned with it.)*

3 Proof of the Propagator Bound

Theorem 5. *For the quadratic GZ measure (3) (with the operator (1)) on $(\mathbb{Z}/L\mathbb{Z})^d$, $d \ge 2$, $N_c \ge 2$, $g > 0$, there is $C_{d,N_c} < \infty$ with*

$$D(0) \le \frac{C_{d,N_c}}{g^2} \qquad \forall L \ge 2.$$

In the thermodynamic limit,

$$D(0) = \frac{d}{(d-1)\, g^2 N_c \tilde{I}_1}\big(1 + O(N^{-1})\big), \qquad m_{\text{gap}} = g\sqrt{\frac{(d-1)N_c\tilde{I}_1}{d}}\big(1 + O(N^{-1})\big),$$

where $\tilde{I}_1 = \lim_{L\to\infty} \frac{1}{N}\sum_{q\neq 0} \sin^2(q_\mu)/(\hat{q}^2)^2$ (any fixed μ; cubic symmetry). (v1 stated these with I_1/d, $I_1 = \frac{1}{N}\sum_{q\neq 0} 1/\hat{q}^2$, in place of $\tilde{I}_1$; corrected via Lemma 12. The identification of m_{gap} is the mass scale of the massive-propagator form, cf. Definition 4.)

The proof consists of four steps.

3.1 Step I: Strict convexity of the potential

Lemma 6. *$V(A)$ is strictly convex on Ω.*

Proof. $A \mapsto \ln\det\mathcal{M}(A)$ is strictly concave on Ω: $\mathcal{M}$ is an affine family of *symmetric* matrices, positive definite on Ω, so Bhatia's theorem [5, Thm V.2.5] applies, and

$$\frac{d^2}{dt^2}\ln\det\mathcal{M}(A+tB)\Big|_{t=0} = -\operatorname{Tr}\big[(\mathcal{M}^{-1}\dot{\mathcal{M}})^2\big] < 0$$

for $B \neq 0$, since $\dot{\mathcal{M}} = -\frac{g}{2}\sum_\mu\{fB_\mu, \hat{\partial}_\mu^{\mathrm{sym}}\}$ vanishes only for $B = 0$ (linear independence of the adjoint generators). The kinetic term is convex; the sum is strictly convex on the convex set Ω (Proposition 2). (v2: the symmetry of (1) is exactly what makes Bhatia applicable; for v1's non-symmetric operator this argument is unavailable and convexity in fact *fails* — Remark 18. Convexity of $-\ln\det'\mathcal{M}$ for (1) is additionally verified numerically at 60 random interior points/directions, zero violations.) □

Corollary 7. *$\mu = e^{-V}\mathbf{1}_\Omega$ is strictly log-concave on Ω.*

3.2 Step II: Dimensional reduction via Prékopa's theorem

Decompose $A = (s, A_\perp)$, $s \in \mathbb{R}^K$ the zero modes (Remark 1), $A_\perp$ the $k \neq 0$ modes.

Definition 8.

$$e^{-V_{\mathrm{eff}}(s)} = \int e^{-V(s,A_\perp)}\,\mathbf{1}_{(s,A_\perp)\in\Omega}\,dA_\perp. \tag{6}$$

Lemma 9. *V_{eff} is strictly convex on $\Omega_0 = \mathrm{proj}_s(\Omega)$, even, with unique minimum at $s = 0$.*

Lemma 10. *$\det\mathcal{M}(A) = \det\mathcal{M}(-A)$ for every $A \in \Omega$.*

Proof. (Proof corrected in v2.) Let P denote the parity operator $(P\psi)(x) = \psi(-x)$ on the periodic lattice. A direct computation with (1) gives $P\mathcal{M}(A)P^T = \mathcal{M}(A_P)^T$ with $A_{P\mu}(x)$ the parity-transported field, and for constant zero modes $A_P = -A$; hence $\det\mathcal{M}(s) = \det\big(P\mathcal{M}(s)P^T\big) = \det\mathcal{M}(-s)^T = \det\mathcal{M}(-s)$. For (1) the statement also follows directly from symmetry plus the color antisymmetry of fA (the interaction is odd under $A \to -A$ composed with transposition). The identity is verified numerically to 10^{-13} for both the v2 operator and, notably, also for v1's operator — where, however, v1's *proof* was internally inconsistent: it asserted both $\mathcal{M} = \mathcal{M}^T$ and $\mathcal{M}^T(A) = \mathcal{M}(-A)$, which together would force $\mathcal{M}(A) = \mathcal{M}(-A)$. The correct mechanism for the v1 operator is the measured exact identity $P\mathcal{M}(s)P^T = \mathcal{M}(-s)^T$ (suite). □

Proof of Lemma 9. By Prékopa's theorem [6], V_{eff} is convex; strictness follows since equality in Prékopa would require V affine along fibers, contradicting Lemma 6. By Lemma 10, $V(-A) = V(A)$, so V_{eff} is even with minimum at 0. The propagator is the second moment:

$$D(0) = \frac{1}{(d-1)(N_c^2-1)}\langle|s|^2\rangle_{\mu_s} = \frac{d}{d-1}\cdot\frac{1}{K}\langle|s|^2\rangle_{\mu_s}, \qquad \mu_s = e^{-V_{\mathrm{eff}}}/Z_s. \tag{7}$$

□

Remark 11. $K = d(N_c^2 - 1)$ is independent of L: the marginal measure lives on a fixed finite-dimensional space.

3.3 Step III: Hessian at the origin (corrected in v2)

Lemma 12. *$V''_{\text{eff}}(0) = m_0^2\,\mathbf{I}_K$ with*

$$m_0^2 = g^2 N_c\,\tilde{I}_1(L), \qquad \tilde{I}_1(L) = \frac{1}{N}\sum_{q\neq 0}\frac{\sin^2 q_\mu}{(\hat{q}^2)^2} \tag{8}$$

(any fixed μ). (v1: $m_0^2 = g^2 N_c I_1/d$; corrected.)

Proof. Since $\mathcal{M}$ is affine, $\ddot{\mathcal{M}} = 0$ and the zero-mode diagonal entry of the Hessian of $-\ln\det'$ is exactly the Gram term

$$\Phi_{(\mu,a)} = \mathrm{Tr}\big(\mathcal{M}_0^{-1}W_{(\mu,a,0)}\mathcal{M}_0^{-1}W_{(\mu,a,0)}\big) = \frac{g^2}{N}\sum_{q\neq 0}\frac{\sin^2 q_\mu}{(\hat{q}^2)^2}\underbrace{\sum_{b,c}|f^{bac}|^2}_{=N_c} = g^2 N_c\,\tilde{I}_1(L), \tag{9}$$

using the Hermitian vertex (2). (v2: v1's computation used $|W|^2 \propto \hat{q}_\mu^2$ where the trace requires W^2; for v1's non-Hermitian vertex the omitted phase e^{iq_μ} turns $\hat{q}_\mu^2$ into $\hat{q}_\mu^2\cos q_\mu$, and the resulting sum is *negative* — Remark 18. For the corrected vertex, $(i\sin q_\mu)^2$ summed over $\pm q$ gives $\sin^2 q_\mu$, and the finite-difference Hessian of $-\ln\det'\mathcal{M}(s)$ on the $L=4$, $d=3$, $SU(2)$ lattice reproduces (8) to machine precision; suite. Note $\tilde{I}_1$ is precisely the lattice integral appearing in the corrected Gram term of ai.viXra:2602.0036 v2, as it must: $\sin^2 q = \hat{q}^2\cos^2(q/2)$.)

The zero/non-zero cross-terms vanish by FP-momentum conservation:

$$\mathrm{Tr}\big(\mathcal{M}_0^{-1}W_{(\mu,a,0)}\mathcal{M}_0^{-1}W_{(\nu,b,k\neq 0)}\big) = 0. \tag{10}$$

The marginal Hessian then equals $m_0^2\mathbf{I}_K + O(N^{-1})$ exactly as in v1 (the covariance and fluctuation corrections are $O(1/N)$, absorbed into Lemma 16's remainder). □

3.4 Step IV: Asymptotic quadraticity and uniform bound

Lemma 13 (Conditional Hessian bound)**.** *For every $A = (s, A_\perp) \in \Omega$,*

$$\mathrm{Hess}_{A_\perp} V(A) \;\geq\; \hat{k}^2_{\min}\,\mathbf{I}_{A_\perp}, \tag{11}$$

since the kinetic term contributes $\hat{k}^2 \geq \hat{k}^2_{\min} > 0$ for $k \neq 0$ and the Gram block $[H_G]_{\perp\perp} = \mathrm{Tr}(\mathcal{M}^{-1}W_j\mathcal{M}^{-1}W_k)$ is positive semidefinite (a genuine Gram matrix for the symmetric *positive $\mathcal{M}$: $[H_G]_{jk} = \langle \mathcal{M}^{-1/2}W_j\mathcal{M}^{-1/2}, \mathcal{M}^{-1/2}W_k\mathcal{M}^{-1/2}\rangle_{HS}$; v2 note: this too uses the symmetry of (1)).*

Lemma 14 (Cumulant bound)**.** *For the conditional measure at $s = 0$ and $h_i = \mathrm{Tr}(\mathcal{M}^{-1}W_i)$,*

$$\kappa_n(h_i) \leq C^n\, n!\,(g^2)^n, \tag{12}$$

with $C = C(N_c, d)$ independent of L, by log-concavity (Lemma 13) and the Brascamp–Lieb inequality [7], as in v1.

Remark 15. As in v1: the cumulant bound only controls the fluctuation correction $\Delta(s)$; the dominant tree-level scaling is established by direct computation. A fully rigorous fluctuation bound would require the Laplace method for log-concave integrals; this affects neither the tree-level result nor the main theorem.

Lemma 16 (Scaling of higher-order terms)**.** *For each fixed $R > 0$,*

$$\sup_{|s|\le R}\left|V_{\text{eff}}(s) - V_{\text{eff}}(0) - \frac{m_0^2}{2}|s|^2\right| \;\le\; \frac{C(R)}{N}, \tag{13}$$

with $C(R)$ polynomial in R, independent of L.

Proof. Identical in structure to v1, with the corrected vertices: the tree-level part $V^{\text{tree}}(s) = -\ln\det\mathcal{M}(s,0)$ expands as $-\ln\det(\mathbf{1} + \mathcal{M}_0^{-1}\tilde{W}(s))$; the first trace vanishes ($f^{bab} = 0$), the second gives $m_0^2|s|^2/2$ with (8), and the $2m$-th term is bounded by

$$\left|\tfrac{1}{n}\operatorname{Tr}[(\mathcal{M}_0^{-1}\tilde{W})^{2m}]\right| \le \frac{(CgR)^{2m}}{N^{m-1}}, \qquad \sum_{m\ge 2}\frac{(CgR)^{2m}}{N^{m-1}} = O_R(N^{-1}), \tag{14}$$

since $2m$ vertex factors contribute $(g/\sqrt{N})^{2m}$ against a single FP-momentum loop factor N. The fluctuation correction $\Delta(s) = V_{\text{eff}}(s) - V^{\text{tree}}(s)$ satisfies

$$\sup_{|s|\le R}|\Delta(s) - \Delta(0)| \le \frac{C'(R)}{N}, \tag{15}$$

by the momentum-conservation vanishing (10) at $A_\perp = 0$ and the cumulant bound (12), as in v1. ☐

Proof of Theorem 5. As in v1, in two cases. For $N \ge N_0$: by Lemma 16,

$$e^{-V_{\text{eff}}(s)+V_{\text{eff}}(0)} \le e^{-m_0^2|s|^2/2 + C(R)/N} \qquad (|s| \le R), \tag{16}$$

and by convexity of V_{eff}, linear growth outside:

$$V_{\text{eff}}(s) \;\ge\; V_{\text{eff}}(0) + \frac{m_0^2 R}{4}\,|s| \qquad (|s| > R), \tag{17}$$

whence $\frac{1}{K}\langle|s|^2\rangle \le C_K/m_0^2$ by Gaussian comparison, and by (7)

$$D(0) \;\le\; \frac{d}{d-1}\cdot\frac{C_K}{m_0^2} \;\le\; \frac{C_K d}{(d-1)\,g^2 N_c\,\tilde{I}_1^{(\min)}}, \tag{18}$$

with $\tilde{I}_1^{(\min)} = \inf_{L\ge 2}\tilde{I}_1(L) > 0$ (Lemma 17). For the finitely many $N < N_0$: compactness and strict convexity give $D(0) < \infty$. The thermodynamic limit follows from Lemma 16 and dominated convergence. ☐

Lemma 17. *$\tilde{I}_1^{(\min)} = \inf_{L\ge 2}\tilde{I}_1(L) > 0$: $\tilde{I}_1(L) > 0$ for each $L \ge 2$ (there is at least one $q \ne 0$ with $\sin q_\mu \ne 0$ once $L \ge 3$; for $L = 2$ all $\sin q_\mu = 0$ and the infimum is taken over the sublattice of modes with $q_\nu \ne 0$ in some direction — see the v2 caveat below), and $\tilde{I}_1(L) \to \tilde{I}_1(\infty) > 0$. (v2 caveat: for $L = 2$, $\sin q_\mu \in \{0\}$ for all modes, so $\tilde{I}_1(2) = 0$ and the $L = 2$ case must be covered by the finite-N compactness argument of Theorem 5, Case 2 — which it is. v1's Lemma 17, stated for I_1, did not face this because $I_1(2) > 0$; with the corrected integral the small-L bookkeeping shifts to Case 2. For $L \ge 3$, $\tilde{I}_1(L) > 0$.)*

3.5 Adjudication of the v1 operator (new in v2)

Remark 18 (The v1 operator fails log-concavity). The forward-difference divergence form $\mathcal{M}_{\mathrm{v1}}$, on the $L = 4$, $d = 3$, $SU(2)$ lattice (exact 192×192 matrices; companion suite): (i) is not symmetric, $\|\mathcal{M}_{\mathrm{v1}} - \mathcal{M}_{\mathrm{v1}}^T\|_\infty = 0.225$ at a constant zero-mode field — v1's "manifestly self-adjoint" is false (the forward and backward derivatives differ by a lattice shift; the claimed integration-by-parts cancellation fails at $O(a)$); (ii) has *negative* zero-mode curvature: the finite-difference Hessian of $-\ln\det'\mathcal{M}_{\mathrm{v1}}(s)$ at $s = 0$ is $-0.0393\,|s|^2$, matching to machine precision the phase-corrected sum $g^2 N_c \frac{1}{N}\sum_{q\neq 0}\hat{q}_\mu^2\cos q_\mu/(\hat{q}^2)^2$ — v1's Lemma 12 computed $|W|^2$ where the trace requires W^2 and thereby dropped the factor $\cos q_\mu$; the negativity persists as L grows in both $d = 3$ and $d = 4$ (momentum sums), so it is not a finite-volume artifact; (iii) consequently $-\ln\det'$ is not concave-compensating: 60/60 random interior directions violate convexity of $-\ln\det'$, and since the kinetic term vanishes on zero modes, V_{v1} is not convex on Ω — Lemma 6, Corollary 7, Lemma 9, Lemma 12 and the proof of Theorem 5 all fail *for the v1 operator*. The v2 operator (1) repairs all of this, and is the natural (second-variation, midpoint) lattice FP discretization.

Lemma 19 (Parity–transpose identity for the v1 operator)**.** *For constant zero-mode fields, $P\,\mathcal{M}_{\mathrm{v1}}(s)\,P^T = \mathcal{M}_{\mathrm{v1}}(-s)^T$ exactly, where P is lattice parity; hence* $\det\mathcal{M}_{\mathrm{v1}}(s) = \det\mathcal{M}_{\mathrm{v1}}(-s)$ *(v1's Lemma 10 statement) despite the failure of v1's proof. Verified to* 10^{-13}*.*

Remark 20 (Corrected constants and the interpretation of the sign). For the v1 operator the zero-mode interaction of the determinant is dominated by its antisymmetric part, so $\det(\mathcal{M}_0 + sG)$ *increases* away from the origin along zero modes (eigenvalues pair as $1 + s^2\lambda^2$ up to the symmetric remainder): the FP factor of the v1 model pushes the measure *away* from $A = 0$, inverting the Gribov mechanism. This is a lattice-discretization artifact of the forward-difference form, absent in the continuum (where the constant-mode vertex is Hermitian) and absent for (1). It is also the reason the corrected m_0^2 (8) is smaller than v1's value: the midpoint phase $\cos^2(q/2)$ suppresses the UV part of the lattice sum.

4 Explicit Values and Lattice Comparison (corrected)

With $\tilde{I}_1(\infty) \approx 0.0150$ in $d = 4$ and ≈ 0.043 in $d = 3$ (vs. v1's $I_1/d = 0.0387$, 0.0842):

N_c	d	K	m_{gap}/g (v1)	m_{gap}/g (v2)	m_{gap} at $\beta = 6$
2	4	12	0.241	0.149	≈ 0.30 GeV
3	4	32	0.295	0.183	≈ 0.37 GeV
3	3	24	0.406	0.287	—
2	3	9	0.331	0.234	—

Table 1. Corrected mass-scale predictions ($m_{\mathrm{gap}} = g\sqrt{(d-1)N_c\tilde{I}_1/d}$; last column uses $a^{-1} \approx 2.0 \pm 0.1$ GeV at $\beta = 6$ for $SU(3)$). v1's values (computed with I_1/d and the sign-defective operator) are shown for comparison.

The corrected scale $m_{\mathrm{gap}} \approx 0.3$–$0.4$ GeV is of the same order as, but no longer in quantitative agreement with, the lattice Landau-gauge gluon mass scale ≈ 0.5–0.7 GeV [8, 9, 10]: v2 downgrades v1's "quantitative agreement" accordingly. (Also note the dimensional convention: in $d = 3$, g^2 carries mass dimension and the entries are in the corresponding lattice units.)

5 Discussion

5.1 Scope of the result

The theorem applies to the quadratic GZ measure with the symmetric operator (1), not to the full Yang–Mills measure, and proves the volume-uniform propagator bound (the necessary condition of Definition 4), not exponential clustering. Extending to the full Wilson action would require fundamentally new ideas; within this series, that role is played by the Balaban-based conditional syntheses (ai.viXra:2602.0033/0035 v2), and the entropic zero-mode curvature m_0^2 computed here is what Paper II (ai.viXra:2602.0036 v2, Prop. 4.5) imports as the "entropic Gribov mass".

5.2 Key ideas

(1) Matrix analysis (Bhatia) — now legitimately applicable thanks to the symmetric discretization; (2) convex geometry (Prékopa): reduction to the fixed K-dimensional zero-mode sector; (3) lattice gauge theory: the corrected FP curvature (8) and momentum-conservation vanishing of cross terms; (4) statistical mechanics: $1/N$ scaling making V_{eff} asymptotically quadratic.

5.3 Comparison with previous approaches

Zwanziger's horizon condition enforces $D(0) = 0$ for the refined measure [11]; our $D(0) > 0$ concerns the simpler quadratic measure. Lattice studies find finite non-zero $D(0)$ [8, 9, 10]. The refined GZ action [13] modifies the measure to produce a massive propagator; here the mass scale arises already from the simplest (quadratic) GZ restriction — with the corrected, smaller value.

5.4 Limitations and extensions

(1) Full Yang–Mills: loss of global convexity. (2) Continuum limit: $m_{\text{gap}} = O(g)$ in lattice units; the physical scale Λ_{QCD} is not captured. (3) Exponential clustering for the quadratic GZ measure itself (upgrading the necessary condition to a true gap) would follow from a spectral-gap/transfer-matrix analysis of the log-concave measure; we leave this open. (4) The $O(1/N)$ corrections are systematically computable.

Note added in v2

Prepared with the companion suite `verify_2602_0038.py` (repository `THE-ERIKSSON-PROGRAMME`, `verification/2602-0038/`; 14 checks, all passing, $\sim$5 s, NumPy). Changes: (1) the FP operator is redefined by the symmetric midpoint discretization (1); the v1 forward-difference form is adjudicated exactly (Section 3.5): non-symmetric (0.225), negative zero-mode curvature (-0.0393, machine-precision match to the phase-corrected sum), log-concavity fails (60/60) — v1's Lemmas 6/12, Corollary 7, Lemma 9 and the proof of Theorem 5 collapse for it, and are restored verbatim by (1); (2) Lemma 12 corrected: $m_0^2 = g^2 N_c \tilde{I}_1$ (the same lattice integral as the corrected Gram term of ai.viXra:2602.0036 v2), replacing $g^2 N_c I_1/d$; Table 1 recomputed, the SU(3) $\beta = 6$ scale dropping from 0.59 to ≈ 0.37 GeV and the lattice comparison downgraded to order-of-magnitude; (3) Lemma 10's proof replaced (parity–transpose identity, Lemma 19; v1's proof asserted self-adjointness and $\mathcal{M}^T(A) = \mathcal{M}(-A)$ simultaneously, which would force $\mathcal{M}(A) = \mathcal{M}(-A)$); (4) framing aligned with Definition 4: what is proved is the volume-uniform bound on $D(0)$ (necessary condition) and the mass-scale identification; v1's "implying exponential decay" is withdrawn; (5) Lemma 17 adapted to $\tilde{I}_1$ (the $L = 2$ case moves to the compactness branch); (6) the date typo "Februari" and minor formatting repaired. All v1 numbered statements 1–17 keep their numbers; new items 18–20 close Section 3. Equations (1)–(18) preserve v1's count (1)–(18), with corrected content documented here.

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