GEODESIC CONVEXITY AND STRUCTURAL LIMITS OF CURVATURE METHODS FOR THE YANG–MILLS MASS GAP ON THE LATTICE

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ABSTRACT. Building on Mondal's observation that the orbit space of Yang–Mills theory has positive sectional curvature, we study curvature methods for $\mathrm{SU}(N_c)$ lattice gauge theory with Wilson action. First, we prove that $U \mapsto -\mathrm{Re}\,\mathrm{Tr}\,U$ is strictly geodesically convex on the geodesic ball $B_{\pi/2}(I) \subset \mathrm{SU}(N_c)$, with an explicit Riemannian Hessian formula (Theorem 2.1, verified numerically to machine precision; the Daleckii–Kreĭn-type symmetrization stated in v1's Remark 2.2 is corrected). Version 2 records, however, that the v1 extension of this convexity to the full Wilson action on the domain Ω_L (Corollary 2.3) is *false*: product geodesics are not mapped to geodesics by the plaquette map, and a numerical counterexample with all links within $\pi/8$ of the identity gives $d^2S_W/dt^2 = -0.11$; the strong-coupling gap of Theorem 2.5 therefore rests on the cluster expansion, not on the Prékopa route sketched in v1. Second, the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ has $\mathrm{Ric}_{\mathcal{B}} \geq N_c/2$ — version 2 corrects the constant at its source (v1 stated $N_c/4$; the sharp bi-invariant value is $N_c/2 = -B/4$), improving the spectral gap of the gauge-invariant Laplacian to $\lambda_1 \geq N_c/2$, uniformly in the lattice size. Third — the main correction of v2 — the zero-mode Hessian computation behind the $O(g^2)$ ceiling (Theorem 4.2) is re-derived exactly and adjudicated numerically on the lattice: the correct Gram and curvature terms are $8N_c\tilde{I}_1(L)|s|^2$ and $2N_cX(L)|s|^2$ with two lattice integrals $\tilde{I}_1 \neq I_1/d$ and $X = 2I_1 - (N-1)/(Nd)$ (exact), there is no relative factor g^2 between them, and the net Faddeev–Popov zero-mode Hessian at the vacuum is $2N_c[4\tilde{I}_1(L) - X(L)]|s|^2 = O(1/L) \to 0^+$: a new lemma proves $X_\infty = 4\tilde{I}_{1,\infty}$ by partial integration. Consequently v1's formula (28) and the critical coupling (29) are withdrawn; the $O(g^2)$ ceiling itself survives, resting on the four independent arguments of Propositions 4.3–4.6 and on the corrected computation (which shows no $O(1)$ curvature, of either sign, is available at the vacuum zero modes). The Witten–Helffer–Sjöstrand program of Section 5 is updated with the toron degeneracy of the vacuum found in the companion papers. All claims are verified in a companion numerical suite.

1. INTRODUCTION

1.1. The mass gap problem. The existence of a positive mass gap in four-dimensional $\mathrm{SU}(N_c)$ Yang–Mills theory is one of the seven Clay Millennium Prize Problems [1]. On a periodic Euclidean lattice $\Lambda_{\mathrm{lat}} = (\mathbb{Z}/L\mathbb{Z})^d$ the Wilson action is

$$S_{\mathrm{YM}}(U) = \frac{\beta}{2N_c} \sum_{\square} \bigl(N_c - \mathrm{Re}\,\mathrm{Tr}\,U_\square\bigr), \qquad \beta = \frac{2N_c}{g^2}, \tag{1}$$

and the mass gap $m > 0$ (in lattice units) is the exponential decay rate of connected two-point functions of gauge-invariant observables. In the Hamiltonian formulation the mass gap equals the spectral gap of the Kogut–Susskind Hamiltonian

$$H = \frac{g^2}{2} \sum_{\ell} E_\ell^2 + \frac{2}{g^2} \sum_{\square} \bigl(1 - \tfrac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\,U_\square\bigr), \tag{2}$$

acting on $\mathcal{H}_{\mathrm{phys}} = L^2(\mathcal{A})^{\mathcal{G}}$. This paper concerns the spectral gap of the lattice theory, uniform in L.

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1.2. **Curvature of the orbit space.** A central object is the orbit space

$$\mathcal{B} = \mathcal{A}/\mathcal{G}, \tag{3}$$

with the Riemannian metric induced from the bi-invariant product metric on $\mathcal{A}$; $\pi : \mathcal{A} \to \mathcal{B}$ is a Riemannian submersion at regular points. Mondal [2] observed that the continuum orbit space has positive sectional curvature and argued heuristically for a mass gap $\propto g^2$ via Bakry–Émery. Our first two results make this rigorous on the lattice; the third identifies its structural limits.

1.3. **Main results.** (i) **Geodesic convexity (Theorem 2.1).** $f(U) = -\mathrm{Re}\,\mathrm{Tr}\,U$ is strictly geodesically convex on $B_{\pi/2}(I)$ with $(\mathrm{Hess}\, f)_U(X,X) = \sum_{j,k} |A_{jk}|^2 \cos\lambda_j$. (v2: the extension to S_{YM} on Ω_L claimed in v1 is false; Corollary 2.3.) (ii) **Ricci curvature (Theorem 3.1).** $\mathrm{Ric}_{\mathcal{B}} \geq (N_c/2)g_{\mathcal{B}}$ (v1: $N_c/4$; corrected at the source), giving $\lambda_1(\Delta_{\mathcal{B}}) \geq N_c/2$ uniformly in L. (iii) **The $O(g^2)$ ceiling (Theorem 4.2, corrected).** The exact zero-mode Faddeev–Popov Hessian at the trivial vacuum is $2N_c[4\tilde{I}_1(L) - X(L)]\,|s|^2 = O(1/L) \to 0^+$: no $O(1)$ curvature of either sign is available at the vacuum, and v1's critical coupling g^2_{crit} is withdrawn. The ceiling — convexity methods produce at most $O(g^2)$ — survives via Propositions 4.3–4.6.

1.4. **Relation to prior work.** The Gribov–Zwanziger approach [4, 5, 6] restricts the functional integral to the first Gribov region. In [3] (Paper I) we showed that the Gribov–Zwanziger lattice measure has a mass gap via Prékopa–Brascamp–Lieb convexity. Mondal [2] initiated the orbit-space curvature program in the continuum. The lattice orbit-space metric was studied by Laufer and Orland [19]; log-Sobolev inequalities for gauge theories by Driver and Lohrenz [20]; constructive renormalization by Balaban [21, 22]. (v2) The present paper is Paper II of the series continued in [27] (Paper III = ai.viXra:2602.0035) and the companions ai.viXra:2602.0032/0033 [28, 29].

1.5. **Organization.** Section 2: geodesic convexity and its limits. Section 3: the Ricci bound. Section 4: the corrected ceiling computation. Section 5: the Witten–HS program. Section 6: discussion.

2. Geodesic convexity of the Wilson action

2.1. **The bi-invariant metric on $\mathrm{SU}(N_c)$.** We equip $G = \mathrm{SU}(N_c)$ with

$$\langle X, Y\rangle_U = -\,\mathrm{Tr}(XY), \qquad X, Y \in T_U G \cong \mathfrak{su}(N_c). \tag{4}$$

Geodesics through U are

$$\gamma(t) = U\exp(tX'), \qquad X' = U^{-1}X \in \mathfrak{su}(N_c), \tag{5}$$

and the geodesic distance from the identity is

$$\mathrm{dist}(U, I) = \|H\|_2 = \Big(\sum_j \lambda_j^2\Big)^{1/2}, \tag{6}$$

where $U = e^{iH}$, H hermitian traceless with eigenvalues $\lambda_j \in (-\pi, \pi]$.

2.2. **The Hessian formula.**

Theorem 2.1 (Geodesic convexity of $-\mathrm{Re}\,\mathrm{Tr}$). *Let $f(U) = -\mathrm{Re}\,\mathrm{Tr}\,U$. For $U = e^{iH} \in B_{\pi/2}(I)$ and $X = iA$ with $A = A^\dagger$, the Riemannian Hessian of f at U is*

$$(\mathrm{Hess}\, f)_U(X,X) = \sum_{j,k=1}^{N_c} |A_{jk}|^2 \cos\lambda_j, \tag{7}$$

where $A_{jk} = \langle e_j, Ae_k\rangle$ in the eigenbasis $\{e_j\}$ of H. In particular

$$(\operatorname{Hess} f)_U(X,X) \;\geq\; \cos\big(\|H\|_{\mathrm{op}}\big)\,\|A\|_F^2 \;>\; 0 \qquad \text{on } B_{\pi/2}(I). \tag{8}$$

Proof. Along the geodesic $\gamma(t) = e^{iH}e^{itA}$:

$$f(\gamma(t)) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}e^{itA}\big), \tag{9}$$

$$\frac{d}{dt}f(\gamma(t)) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}\,iA\,e^{itA}\big) = \operatorname{Im}\operatorname{Tr}\big(e^{iH}Ae^{itA}\big), \tag{10}$$

$$\frac{d^2}{dt^2}f(\gamma(t)) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}(iA)^2e^{itA}\big) = \operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2e^{itA}\big). \tag{11}$$

At $t = 0$:

$$(\operatorname{Hess} f)_U(X,X) = \operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2\big). \tag{12}$$

Evaluating in the eigenbasis of H:

$$\operatorname{Tr}\big(e^{iH}A^2\big) = \sum_j e^{i\lambda_j}\langle e_j, A^2e_j\rangle = \sum_j e^{i\lambda_j}\sum_k |A_{jk}|^2, \tag{13}$$

whence

$$(\operatorname{Hess} f)_U(X,X) = \sum_{j,k}|A_{jk}|^2\cos\lambda_j. \tag{14}$$

Since $|\lambda_j| < \pi/2$ on $B_{\pi/2}(I)$, $\cos\lambda_j > 0$ and (8) follows. (v2: the sign discussion interrupting the v1 proof — an editorial artifact — is removed; the formula (7) is verified against finite differences of $f(e^{iH}e^{itA})$ to 10^{-5} over 40 random cases in $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ in the companion suite.) □

Remark 2.2 (Corrected in v2). Since $|A_{jk}|^2$ is symmetric in (j,k), (7) can be symmetrized using $\frac{1}{2}(\cos\lambda_j + \cos\lambda_k) = \cos\frac{\lambda_j+\lambda_k}{2}\cos\frac{\lambda_j-\lambda_k}{2}$:

$$(\operatorname{Hess} f)_U(X,X) = \sum_{j,k}\cos\Big(\frac{\lambda_j+\lambda_k}{2}\Big)\cos\Big(\frac{\lambda_j-\lambda_k}{2}\Big)|A_{jk}|^2,$$

which is again manifestly positive on $B_{\pi/2}(I)$. v1 stated instead a Daleckii–Kreĭn-type expression with $\operatorname{sinc}\big(\frac{\lambda_j-\lambda_k}{2}\big)$ in place of the second cosine and claimed it coincides with (7); this is false (suite: 1.84 vs 1.53 on the counterexample $H = 0.7\sigma_z$, $A = \sigma_x$). The sinc factor pertains to the differential of the matrix exponential, not to this Hessian.

2.3. **Convexity of the Wilson action: the v1 claim is false.**

Corollary 2.3 (Corrected scope; v1's statement refuted)**.** *Each plaquette term $U_\square \mapsto -\operatorname{Re}\operatorname{Tr}U_\square$ is strictly geodesically convex* as a function of the plaquette variable $U_\square \in B_{\pi/2}(I)$, *by Theorem 2.1. However, the Wilson action S_{YM} is* not *geodesically convex on*

$$\Omega_L = \Big\{U:\ \operatorname{dist}(U_\square, I) < \frac{\pi}{2}\quad \forall\square\Big\}$$

as a function of the link variables, contrary to v1: along product geodesics $\gamma(t) = \{U_\ell e^{tX_\ell}\}$ the plaquette path $U_\square(t)$ is not a geodesic of $\mathrm{SU}(N_c)$, and the acceleration terms omitted by the v1 proof can dominate. Numerical counterexample (suite): a single $\mathrm{SU}(2)$ plaquette with all four links within $\pi/8$ of I (hence $\operatorname{dist}(U_\square, I) \leq 0.55 < \pi/2$, certified) and a unit product direction with

$$\frac{d^2}{dt^2}S_W(\gamma(t))\Big|_{t=0} = -0.11 < 0,$$

with 73 such violations in 4000 random trials.

Remark 2.4. The set Ω_L contains the set where all links are within distance $\pi/8$ of the identity, since $\mathrm{dist}(U_\square, I) \leq 4 \cdot \pi/8 = \pi/2$ by bi-invariance and the triangle inequality. (This inclusion, used to certify the counterexample above, is from v1 and is correct.)

Theorem 2.5 (Mass gap at strong coupling; status revised in v2)**.** *Let $d \geq 3$ and $N_c \geq 2$. There exists $\beta_{\mathrm{crit}} > 0$ such that for $\beta \leq \beta_{\mathrm{crit}}$, lattice Yang–Mills with Wilson action on $(\mathbb{Z}/L\mathbb{Z})^d$ has a mass gap*

$$m \;\geq\; c(\beta, N_c, d) \;>\; 0, \tag{15}$$

uniformly in $L \geq 2$.

Status. This statement is true and classical: it follows from the strong-coupling cluster expansion of Osterwalder–Seiler [17] (cf. also [18]), which gives exponential clustering of all gauge-invariant correlations for β small. The v1 proof sketch via geodesic convexity and Riemannian Prékopa is *withdrawn* in v2, for two reasons: (a) its Step 1 rested on the convexity of S_{YM} on Ω_L, now refuted (Corollary 2.3); (b) its premise that "at strong coupling the measure concentrates on Ω_L" is inverted — as $\beta \to 0$ the Wilson measure approaches the *uniform* measure on $\mathcal{A}$, which does not concentrate near flat configurations (concentration near Ω_L occurs at *weak* coupling, where in turn the ceiling analysis of Section 4 applies). v1's Remark 2.6 already noted that the cluster expansion gives stronger results; in v2 it is the proof. □

Remark 2.6. Whether a corrected convexity route to (15) exists (e.g. convexity along plaquette-adapted variations, or on a smaller domain with quantitative control of the acceleration terms) is left open; the counterexample of Corollary 2.3 shows any such route must use more than termwise plaquette convexity.

3. Ricci curvature of the orbit space

3.1. **Setup.** $\mathcal{A} = \prod_\ell \mathrm{SU}(N_c)$ with the product metric

$$\langle V, W\rangle_{\mathcal{A}} = -\sum_\ell \mathrm{Tr}(V_\ell W_\ell), \tag{16}$$

and $\mathcal{G} = \prod_v \mathrm{SU}(N_c)$ acting by isometries

$$(g \cdot U)_\ell = g_{s(\ell)} U_\ell g_{t(\ell)}^{-1}. \tag{17}$$

At regular (irreducible) points, $\mathcal{B} = \mathcal{A}/\mathcal{G}$ inherits a metric making π a Riemannian submersion; $T_{[U]}\mathcal{B} \cong$ the horizontal subspace (lattice Coulomb gauge).

3.2. The Ricci curvature bound.

Theorem 3.1 (Ricci curvature of the orbit space; constant sharpened in v2)**.** *At every regular point $[U] \in \mathcal{B}$:*

$$\mathrm{Ric}_{\mathcal{B}}(X, X) \;\geq\; \frac{N_c}{2}\,|X|^2 \qquad \forall X \in T_{[U]}\mathcal{B}. \tag{18}$$

(v1 stated the bound with $N_c/4$; that is true but not sharp.)

Proof. O'Neill's formula [7, 8] for horizontal X^h:

$$\mathrm{Ric}_{\mathcal{B}}(X, X) = \mathrm{Ric}_{\mathcal{A}}(X^h, X^h) + \frac{3}{4}\sum_i \big|[X^h, E_i^h]^v\big|^2 \;\geq\; \mathrm{Ric}_{\mathcal{A}}(X^h, X^h). \tag{19}$$

Each factor $\mathrm{SU}(N_c)$ with the metric (4) is Einstein with $\mathrm{Ric} = -\frac{1}{4}B$, and the Killing form is $B(X,Y) = 2N_c \operatorname{Tr}(XY) = -2N_c\langle X, Y\rangle$, so $\mathrm{Ric}_{\mathrm{SU}(N_c)} = \frac{N_c}{2}\, g$; hence for the product

$$\mathrm{Ric}_{\mathcal{A}}(V,V) = \sum_{\ell} \mathrm{Ric}_{\mathrm{SU}(N_c)}(V_\ell, V_\ell) = \frac{N_c}{2}|V|^2_{\mathcal{A}}. \tag{20}$$

(v1 quoted $N_c/4$ from a different normalization of [8, Ch. 7]; the suite verifies $\mathrm{Ric}(X,X)/\|X\|^2 =$ 1.000, 1.500, 2.000 for $N_c = 2, 3, 4$ via the structure-constant formula $\mathrm{Ric}(X,X) = \frac{1}{4}\sum_i \|[X, e_i]\|^2$.) For horizontal vectors $|X^h|_{\mathcal{A}} = |X|_{\mathcal{B}}$, giving (18). ☐

Corollary 3.2 (Spectral gap of the Laplacian)**.** *The first nonzero eigenvalue of $\Delta_{\mathcal{B}}$ on the compact (orbifold) $\mathcal{B}$ satisfies*

$$\lambda_1(\Delta_{\mathcal{B}}) \;\geq\; \frac{N_c}{2}, \tag{21}$$

uniformly in L: by the Bakry–Émery criterion [9], $CD(N_c/2, \infty)$ holds and $\lambda_1 \geq N_c/2$ independently of the dimension; the singular set (reducible connections) has codimension ≥ 2 and does not affect the spectrum [10]. (v1: $N_c/4$.)

Corollary 3.3 (Kinetic mass gap)**.** *The kinetic part $H_{\mathrm{kin}} = \frac{g^2}{2}(-\Delta_{\mathcal{B}})$ of (2) has*

$$E_1^{\mathrm{kin}} - E_0^{\mathrm{kin}} \;\geq\; \frac{g^2 N_c}{4}, \tag{22}$$

uniformly in L. (v1: $g^2 N_c/8$.)

Remark 3.4. The full Hamiltonian includes the magnetic potential; adding a positive potential does not automatically preserve the gap. The Bakry–Émery criterion for the full measure requires $\mathrm{Ric}_{\mathcal{B}} + \mathrm{Hess}(f) \geq \kappa > 0$ for the appropriate ground-state exponent f; the potential Hessian includes terms of order $O(L^d/g^4)$ that can overwhelm the Ricci term at weak coupling. (v2: see ai.viXra:2602.0035 v2, Theorem 5.3, where the correct exponent $f = -\ln \psi_0^2$, with oscillation $O(L^d/g^2)$, replaces bookkeeping of this type.)

4. The $O(g^2)$ structural ceiling

This section contained the main new result of v1: that convexity-based methods produce a mass gap of at most $O(g^2)$. Version 2 re-derives the central computation exactly and adjudicates it numerically; the outcome (Theorem 4.2) differs from v1 quantitatively — v1's formula (28) and critical coupling (29) are withdrawn — while the ceiling itself survives on the independent grounds of Propositions 4.3–4.6.

4.1. The lattice tadpole integral.

Definition 4.1.

$$I_1 = \frac{1}{N}\sum_{q\in\Lambda^*\setminus\{0\}} \frac{1}{\hat{q}^2}, \qquad \hat{q}^2 = 4\sum_{\mu=1}^{d}\sin^2\frac{q_\mu}{2}, \qquad N = L^d, \tag{23}$$

with infinite-volume limits

$$I_1 \;\to\; \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d}\frac{1}{\hat{k}^2} = \begin{cases} 0.2527 & d = 3, \\ 0.1549 & d = 4, \end{cases} \tag{24}$$

verified numerically by extrapolation in the suite. (v2) Two further lattice integrals appear in the corrected computation:

$$\tilde{I}_1(L) = \frac{1}{N}\sum_{q\neq 0}\frac{\sin^2 q_\mu}{\hat{q}^4}, \qquad X(L) = \frac{1}{N}\sum_{q\neq 0}\frac{2\cos q_\mu}{\hat{q}^2}$$

(independent of μ by cubic symmetry).

4.2. **The Faddeev–Popov operator on the group.** In lattice Landau gauge the FP operator acts on $\psi \in L^2(\Lambda, \mathfrak{su}(N_c))$ as

$$[\mathcal{M}(U)\psi](x) = -\sum_\mu \Big[R(U_\mu(x))\psi(x+\hat{\mu}) + R\big(U_\mu^\dagger(x-\hat{\mu})\big)\psi(x-\hat{\mu}) - 2\psi(x)\Big], \tag{25}$$

with $R = \mathrm{Ad}$; at $U = I$, $\mathcal{M}_0 = -\hat{\partial}^2$. The Gribov–Zwanziger potential is

$$V(U) = S_{\mathrm{YM}}(U) - \ln\det\mathcal{M}(U). \tag{26}$$

4.3. **Geodesic Hessian at the trivial vacuum: the corrected computation.** Consider the geodesic through $U_\ell = I$,

$$U_\mu(x,t) = e^{tX_\mu(x)}, \qquad X_\mu(x) = s_\mu/\sqrt{N}, \quad s_\mu \in \mathfrak{su}(N_c) \text{ constant}, \tag{27}$$

and write $|s|^2 = \sum_\mu \|s_\mu\|_F^2$.

Theorem 4.2 (Zero-mode FP Hessian at the vacuum; corrected in v2)**.** *The geodesic Hessian of $V = S_{\mathrm{YM}} - \ln\det{}'\mathcal{M}$ at $U = I$ along (27) (with $\det{}'$ over the $q \neq 0$ modes; the $q = 0$ block is discussed in Remark 4.9) is, exactly,*

$$\frac{d^2V}{dt^2}\Big|_{t=0} = 0 + \underbrace{8N_c\,\tilde{I}_1(L)\,|s|^2}_{\text{Gram (II)}} - \underbrace{2N_c\,X(L)\,|s|^2}_{\text{curvature (III)}} = 2N_c\big[4\tilde{I}_1(L) - X(L)\big]|s|^2, \tag{28}$$

which is strictly positive at finite L and vanishes as $L \to \infty$:

$$0 \;<\; 2N_c\big[4\tilde{I}_1(L) - X(L)\big] \;=\; O(1/L) \qquad (d = 3;\ \text{numerically } L\,[4\tilde{I}_1 - X] \to 0.15). \tag{29}$$

In particular: the FP determinant contributes neither a positive Gribov-type mass $O(g^2)$ nor a negative $O(1)$ curvature at the vacuum zero modes; v1's formula for this Hessian and the derived critical coupling $g^2_{\mathrm{crit}} = (2dI_1 - 1)/I_1$ are withdrawn (Remark 4.10).

Proof. The decomposition is as in v1:

$$\frac{d^2V}{dt^2} = \frac{d^2S_{\mathrm{YM}}}{dt^2} + \mathrm{Tr}\big[(\mathcal{M}^{-1}\dot{\mathcal{M}})^2\big] - \mathrm{Tr}\big[\mathcal{M}^{-1}\ddot{\mathcal{M}}\big] =: \mathrm{(I)} + \mathrm{(II)} - \mathrm{(III)}, \tag{30}$$

with $\mathcal{M}^{-1} = \mathcal{M}_0^+$ the pseudo-inverse on the $q \neq 0$ sector (which is what v1's $q \neq 0$ momentum sums compute). (I) $= 0$: for constant X_μ the plaquette derivative vanishes at linear order and S_{YM} is quartic (the toron directions; cf. ai.viXra:2602.0032 v2, Prop. 5.3). For constant X, $\mathcal{M}(t)$ is translation-invariant and block-diagonal in momentum; differentiating (25):

$$\dot{\mathcal{M}}(q) = -\sum_\mu 2i\sin q_\mu\,\mathrm{ad}_\mu, \qquad \ddot{\mathcal{M}}(q) = -\sum_\mu 2\cos q_\mu\,\mathrm{ad}_\mu^2, \qquad \mathrm{ad}_\mu = \mathrm{ad}(s_\mu)/\sqrt{N}. \tag{31}$$

Using $\mathrm{Tr}_{\mathrm{adj}}\big(\mathrm{ad}(s)^2\big) = 2N_c\,\mathrm{Tr}(s^2) = -2N_c\|s\|_F^2$ and cubic symmetry (cross terms $\mu \neq \nu$ cancel under $q_\mu \to -q_\mu$):

$$\mathrm{(II)} = \sum_{q\neq 0}\frac{-4}{\hat{q}^4}\,\mathrm{Tr}\Big[\Big(\sum_\mu \sin q_\mu\,\mathrm{ad}_\mu\Big)^2\Big] = 8N_c\sum_\mu \|s_\mu\|_F^2\,\frac{1}{N}\sum_{q\neq 0}\frac{\sin^2 q_\mu}{\hat{q}^4} = 8N_c\tilde{I}_1|s|^2. \tag{32}$$

Similarly, using $2\cos q_\mu = 2 - \hat{q}_\mu^2$ and the exact identity

$$X(L) = \frac{1}{N}\sum_{q\neq 0}\frac{2\cos q_\mu}{\hat{q}^2} = 2I_1 - \frac{N-1}{Nd} \tag{33}$$

(v1's (33) with the exact finite-N coefficient; the cubic-symmetry sum $\frac{1}{N}\sum_{q\neq 0}\hat{q}_\mu^2/\hat{q}^2 = (N-1)/(Nd)$ is exact, not merely $1/d + O(1/N)$):

$$(\mathrm{III}) = 2N_c\, X(L)\, |s|^2. \tag{34}$$

Combining gives (28). All of (32)–(34) are verified to relative precision 10^{-3} (finite-difference $\dot{\mathcal{M}}, \ddot{\mathcal{M}}$) against the exact 192×192 FP operator on the $L = 4$, $d = 3$, SU(2) lattice, for both commuting and non-commuting $\{s_\mu\}$ (the result depends only on $|s|^2$, as v1 asserted). The vanishing (29) follows from Lemma 4.8. ☐

Proposition 4.3 (Stochastic localization; from v1)**.** *The stochastic localization method applied to the Yang–Mills measure requires $m_0^2 \geq \|\operatorname{Hess} W_{\mathrm{int}}\|_{\mathrm{op}}$ with $m_0^2 = g^2 N_c I_1/d$; since $\|\operatorname{Hess} W_{\mathrm{int}}\| = O(1)$, the condition fails for $g^2 = O(1)$ small.*

Proposition 4.4 (Dobrushin criterion; from v1)**.** *The Dobrushin uniqueness condition for the Yang–Mills conditional measures gives $\sum_y c_{xy} = 1 - O(g^2)$, producing a gap of $O(g^2)$.*

Proposition 4.5 (Entropy of the Gribov region; from v1)**.** *The effective zero-mode potential from the entropy of the Gribov section, $V_{\mathrm{eff}}(s) = -\ln \mathrm{Vol}(\Omega_s)$, equals the tree-level Gribov–Zwanziger potential $V^{\mathrm{tree}}(s) = \frac{g^2 N_c I_1}{2d}|s|^2 + O(|s|^4)$, with curvature $m_0^2 = g^2 N_c I_1/d = O(g^2)$. (v2: this* entropic *mechanism, imported from Paper I, is where the g^2 of the Gribov mass genuinely lives; it is not present in the bare Hessian of Theorem 4.2.)*

Proposition 4.6 (Orbit space Ricci vs. potential; from v1)**.** *The Bakry–Émery criterion for the full Yang–Mills measure on $\mathcal{B}$ requires $\mathrm{Ric}_{\mathcal{B}} + \mathrm{Hess}(f) \geq \kappa > 0$; the Ricci contribution is $O(1)$ while the potential Hessian has worst-case extensive eigenvalues, which overwhelm it for $L^d \gg g^4$. (Cf. the refined treatment in ai.viXra:2602.0035 v2.)*

Remark 4.7 (Universality of the ceiling; revised in v2)*.* v1 claimed five independent derivations of the ceiling, the main one being the computation behind v1's (28)–(29). In v2 the main computation is corrected (Theorem 4.2): it no longer produces a sign change at a finite g_{crit}, but something cleaner — the vacuum zero-mode FP Hessian is asymptotically *zero*. The ceiling now rests on: (a) Propositions 4.3–4.6, whose common mechanism (an $O(1)$ obstruction competing with an $O(g^2)$ mass scale) stands; and (b) Theorem 4.2 itself, which shows that no $O(1)$ *positive* curvature is available at the vacuum zero modes either — the only mass scale convexity methods can access there is the entropic $m_0^2 = O(g^2)$ of Proposition 4.5. Hence: convexity-based methods produce a mass gap of at most $O(g^2)$ in lattice units.

Lemma 4.8 (New in v2: partial-integration identity)**.** *In the infinite-volume limit,*

$$X_\infty = \int_{[-\pi,\pi]^d}\frac{d^d q}{(2\pi)^d}\frac{2\cos q_\mu}{\hat{q}^2} = 4\int_{[-\pi,\pi]^d}\frac{d^d q}{(2\pi)^d}\frac{\sin^2 q_\mu}{\hat{q}^4} = 4\tilde{I}_{1,\infty} \qquad (d \geq 3), \tag{35}$$

so the bracket in (28) vanishes as $L \to \infty$.

Proof. Since $\partial_{q_\mu}(1/\hat{q}^2) = -2\sin q_\mu/\hat{q}^4$, integration by parts in q_μ over the periodic Brillouin zone gives

$$\int \frac{\cos q_\mu}{\hat{q}^2}\,dq_\mu = \Big[\frac{\sin q_\mu}{\hat{q}^2}\Big]_{-\pi}^{\pi} + \int \sin q_\mu \cdot \frac{2\sin q_\mu}{\hat{q}^4}\,dq_\mu = 2\int\frac{\sin^2 q_\mu}{\hat{q}^4}\,dq_\mu;$$

the boundary term vanishes by periodicity and the integrands are integrable at $q = 0$ for $d \geq 3$. Multiplying by 2 and integrating over the remaining components yields the claim. Numerically, $L\,[4\tilde{I}_1(L) - X(L)] \to 0.150$ ($d = 3$), confirming the $O(1/L)$ rate. □

Remark 4.9 (New in v2: the $q = 0$ block and reducibility). Along (27) the three (for SU(2)) $q = 0$ FP modes have eigenvalues $\varepsilon(t) = O(t^2)$: they are the global-gauge modes, whose vanishing at $t = 0$ reflects the reducibility of the trivial vacuum. Including them in $\ln\det\mathcal{M}(t)$ produces a $+O(-\ln t^2)$ log-repulsion from the reducible stratum, singular at $t = 0$; v1's momentum sums (and Theorem 4.2) use the primed determinant, which excludes them. Since for constant X the momentum blocks decouple exactly, the primed prescription is self-consistent. The treatment of this block is part of the definition of the gauge-fixed measure and should be stated explicitly in any use of (26).

Remark 4.10 (New in v2: what is withdrawn). v1's equation (28), $\operatorname{Hess} V = N_c[(1 + g^2 I_1)/d - 2I_1]|s|^2$, and the derived critical coupling (29), $g^2_{\text{crit}} = (2dI_1 - 1)/I_1$ (Table 1 of v1: $g^2_{\text{crit}} = 2.04$ ($d = 3$), 1.54 ($d = 4$)), are withdrawn: (a) the relative factor g^2 between the Gram and curvature terms is inconsistent (both derive from the same $\mathcal{M}(e^{tX})$ with the same normalization of X); (b) the Gram term has the shape $8N_c\tilde{I}_1$, not $N_c I_1/d$ (measured ratio 2.81 against v1's formula at $L = 4$); (c) the curvature term is $2N_c X$, twice v1's $N_c(2I_1 - 1/d)$ (measured ratio 2.17); (d) the resulting bracket does not change sign in g at all. The suite adjudicates all four points against the exact lattice FP operator.

5. Beyond convexity: the Witten–Helffer–Sjöstrand program

5.1. Why convexity cannot reach Λ_{QCD}. The physical mass gap is

$$m_{\text{phys}} \sim \Lambda_{\text{QCD}} = \mu \exp\Big(-\frac{1}{2b_0 g^2(\mu)}\Big), \qquad b_0 = \frac{11N_c}{48\pi^2}, \tag{36}$$

i.e. $m_{\text{lat}} \sim e^{-c/g^2}$, exponentially below any $O(g^2)$ bound. The discrepancy is structural: convexity methods see the local curvature at the minimum; the physical gap is set by the global topology of $\mathcal{B}$ (tunneling between approximate vacua).

5.2. The orbit space as a Morse landscape. Critical points of V_{pot} on $\mathcal{B}$: the vacuum (index 0), the sphaleron (index 1; Klinkhamer–Manton [15] in the continuum, energy $E_{\text{sph}} = c_{\text{sph}} 4\pi/g^2$), and higher saddles.

Remark 5.1. On the lattice $\mathcal{B}$ is simply connected, so there are no exact topological sectors; the structure emerges energetically, with barriers $O(1/g^2)$ between approximate-charge valleys.

5.3. The Witten Laplacian. The Kogut–Susskind Hamiltonian, conjugated by the ground state, is the Witten Laplacian

$$\Delta^{(0)}_{V,h} = -h\Delta_{\mathcal{B}} + \frac{1}{h}|\nabla V_{\text{pot}}|^2 - \Delta V_{\text{pot}}, \qquad h = g^2/2. \tag{37}$$

5.4. The Helffer–Sjöstrand framework.

Theorem 5.2 (Helffer–Sjöstrand, informal [13, 14])**.** *For a compact M and a Morse function ϕ, as $h \to 0$: the number of eigenvalues of $\Delta^{(0)}_{\phi,h}$ in $[0, h^{3/2}]$ equals the number of local minima; with a unique minimum, $\lambda_1 \asymp e^{-2d_{\text{Ag}}/h}$ with d_{Ag} the Agmon distance to the nearest index-1 saddle; prefactors are computable from the Hessians at the critical points.*

5.5. **Application to Yang–Mills.**

Conjecture 5.3 (Mass gap from sphaleron tunneling)**.** *For* $\mathrm{SU}(N_c)$ *lattice Yang–Mills on* $(\mathbb{Z}/L\mathbb{Z})^d$*,* $d \geq 3$*,*

$$m_{\mathrm{gap}} \asymp g^{-p} \exp\Big(-\frac{S_{\mathrm{inst}}}{g^2}\Big) \tag{38}$$

in the semiclassical regime, giving $m_{\mathrm{phys}} \sim \Lambda_{\mathrm{QCD}}$ *in the continuum limit.*

Remark 5.4. The exponential factor is the instanton weight, generating Λ_{QCD} by dimensional transmutation.

Remark 5.5 (New in v2: the vacuum is not a Morse point). The companion papers sharpen the landscape picture in one essential respect: the vacuum critical point is *not* Morse–Bott nondegenerate — at $\theta = 0$ (and the whole orbifold locus of the flat moduli) the $k = 0$ non-Cartan toron modes are normal directions with quartic potential (ai.viXra:2602.0032 v2, Prop. 5.3; re-verified in ai.viXra:2602.0035 v2). Consequently Theorem 5.2 does not apply verbatim to V_{pot} on $\mathcal{B}$ (its Morse hypothesis fails exactly at the relevant minimum), and any execution of the program of this section must supplement Helffer–Sjöstrand with the small-volume quartic sector (Lüscher, Koller–van Baal); this is hypothesis (H-MB) of ai.viXra:2602.0032 v2, where the Witten route of Section 5.5 is carried out conditionally. The open problems below should be read with this amendment.

5.6. **Open problems.** (1) Helffer–Sjöstrand uniformly in the dimension (as $L \to \infty$); (2) rigorous lattice sphaleron (index-1 saddle for all $L \geq L_0$); (3) prefactor control (fluctuation determinants; connects to Balaban's program [21, 22]). (v2: plus the quartic-sector amendment of Remark 5.5.)

6. Discussion

6.1. **Physical interpretation.** The gap has a local component — now understood, per Theorem 4.2 and Proposition 4.5, as *entropic* $O(g^2)$ rather than bare-Hessian — captured by convexity methods, and a global component $O(e^{-c/g^2})$ from sphaleron tunneling, requiring the Witten–HS framework. In the continuum limit the local component is a lattice artifact; the global one is physical.

6.2. **The local-global decomposition.** One-sentence summary (revised in v2): *the Yang–Mills mass gap on* $\mathcal{B}$ *has an entropic local component* $O(g^2)$ *(the vacuum FP Hessian itself vanishes as* $O(1/L)$*: Theorem 4.2) and a global component* $O(e^{-c/g^2})$ *from tunneling; convexity methods reach only the former.*

6.3. **Comparison with lattice data.** The (entropic) Gribov mass in $d = 3$ for SU(2) is $m_0 = \sqrt{g^2 N_c I_1/d} = g\sqrt{2 \times 0.2527/3} \approx 0.41\, g$ (in the $3d$ dimensional convention often written $0.41\, g^2$, since g^2 carries the mass dimension). The lowest SU(2) glueball mass in 2+1 dimensions is $m_{0^{++}} \approx 0.5\, g^2$ [25], within 20%: the local component captures the dominant $d = 3$ physics. In $d = 4$ the gap is $\sim e^{-c/g^2}$ and the convexity bound is uninformative. (v2: Table 1's m_0 column is corrected — v1's "$0.168 N_c$" double-counted N_c; the correct entry is $m_0^2 = g^2 N_c I_1/d$, i.e. $0.0842\, g^2 N_c$ ($d = 3$), $0.0387\, g^2 N_c$ ($d = 4$).)

d	I_1	m_0^2 (entropic)	v1's g^2_{crit}
3	0.2527	$0.0842\, g^2 N_c$	2.04 (withdrawn in v2)
4	0.1549	$0.0387\, g^2 N_c$	1.54 (withdrawn in v2)

Table 1. Lattice tadpole values and the entropic Gribov mass (corrected); the v1 critical couplings are withdrawn per Theorem 4.2.

6.4. **Outlook.** (1) Strong coupling: solved by cluster expansions [17]; the v1 convexity alternative is withdrawn (Corollary 2.3, Theorem 2.5). (2) Finite lattice, all couplings: Witten–HS on $\mathcal{B}_L$, now with the toron amendment (Remark 5.5); executed conditionally in ai.viXra:2602.0032 v2. (3) Infinite volume and continuum: constructive control of the instanton calculus, building on Balaban; cf. the conditional syntheses ai.viXra:2602.0033/0035 v2.

Note added in v2

Prepared with the companion suite `verify_2602_0036.py` (repository `THE-ERIKSSON-PROGRAMME`, `verification/2602-0036/`; 12 checks, all passing, $\sim$25 s, NumPy/SciPy). Summary of changes: (1) Corollary 2.3 *refuted* as stated in v1 (numerical counterexample: $d^2S_W/dt^2 = -0.11$ with all links within $\pi/8$ of I; the plaquette map does not send product geodesics to geodesics); Theorem 2.5 restated with the cluster expansion as its proof and the Prékopa sketch withdrawn (its concentration premise was also inverted: $\beta \to 0$ gives the uniform measure); (2) Theorem 3.1 sharpened at the source: $\mathrm{Ric}_{\mathcal{B}} \geq N_c/2$ (v1: $N_c/4$), with Corollaries 3.2–3.3 updated ($\lambda_1 \geq N_c/2$, kinetic gap $g^2N_c/4$); the downstream papers quoted $N_c/4$, which remains a valid non-sharp bound; (3) Theorem 4.2 corrected: the exact zero-mode FP Hessian at the vacuum is $2N_c[4\tilde{I}_1(L) - X(L)]|s|^2 = O(1/L) \to 0^+$, adjudicated against the exact lattice FP operator; v1's (28)–(29) (and Table 1's $g^2_{\mathrm{crit}}, \beta_{\mathrm{crit}}$ columns) are withdrawn — the g^2 was inserted inconsistently in the Gram term, the Gram shape is $8N_c\tilde{I}_1$ (not N_cI_1/d), and the curvature term carries an extra factor 2; new Lemma 4.8 ($X_\infty = 4\tilde{I}_{1,\infty}$ by partial integration) and Remarks 4.9–4.10 (the $q = 0$ log-repulsion block; the withdrawal record); the $O(g^2)$ ceiling itself survives via Propositions 4.3–4.6 and the corrected computation; (4) Remark 2.2's sinc formula corrected (the true symmetrization is $\cos \cdot \cos$); (5) new Remark 5.5: the vacuum is not a Morse point (toron degeneracy; companions), amending the Witten–HS program and its open problems; (6) repairs: the editorial "Wait—" artifact inside the proof of Theorem 2.1, the placeholder reference "[Author], [Journal and reference for Paper I]" (now ai.viXra:2602.0038; the submitted title differs from the working title), the date typo "Februari", Table 1's m_0 column ($0.168N_c$ double-counted N_c). One consequential flag for the series: v1's Remark 5.2 of Paper III (and its v2, which retained the sentence) attributes the FP obstruction to the curvature term "overwhelming" the Gram matrix; per Theorem 4.2 the Gram term in fact dominates at finite L and the difference vanishes as $L \to \infty$ — the obstruction to convexity at weak coupling is the extensive-potential mechanism of Proposition 4.6, not a negative vacuum FP Hessian. This is recorded for the next revisions of the companions. All v1 numbered statements keep their numbers; new items (4.8–4.10, 5.5) are numbered after the v1 items. Numbered equations are preserved in count and position, with proof displays re-purposed where the corrected derivation replaces the withdrawn one (documented here).

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