GEODESIC CONVEXITY AND STRUCTURAL LIMITS OF CURVATURE METHODS FOR THE YANG–MILLS MASS GAP ON THE LATTICE

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Abstract. Building on Mondal's observation that the orbit space of Yang–Mills theory has positive sectional curvature, we establish three results for $SU(N_c)$ lattice gauge theory with Wilson action. *First*, we prove that the function $U \mapsto -\operatorname{Re}\operatorname{Tr} U$ is strictly geodesically convex on the geodesic ball $B_{\pi/2}(\mathbb{I}) \subset SU(N_c)$, with an explicit Riemannian Hessian formula involving the eigenvalues of $\log U$ (Theorem 2.1). Combined with the Riemannian Prékopa inequality this yields a rigorous mass gap for the full Wilson action at strong coupling. *Second*, we show that the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of lattice gauge configurations modulo gauge transformations has Ricci curvature $\mathrm{Ric}_{\mathcal{B}} \geq N_c/4$, giving a spectral gap $\lambda_1(\Delta_{\mathcal{B}}) \geq N_c/4$ for the gauge-invariant Laplacian, uniform in the lattice size (Theorems 3.1 and 3.2). This makes Mondal's heuristic argument rigorous in the lattice setting. *Third*, and most importantly, we prove that *every* convexity-based method—Brascamp–Lieb, Bakry–Émery, Dobrushin, Prékopa—applied to the Yang–Mills–Faddeev–Popov potential produces a mass gap of at most $O(g^2)$ in lattice units (Theorem 4.2). We derive an explicit critical coupling $g^2_{\mathrm{crit}} = (2dI_1 - 1)/I_1$, with I_1 the lattice tadpole integral, beyond which all such methods fail. We argue that the physical mass gap $O(\Lambda_{\mathrm{QCD}}) = O(e^{-c/g^2})$ requires the global topology of $\mathcal{B}$, accessible via the Witten–Helffer–Sjöstrand framework, and identify the sphaleron barrier as the controlling quantity.

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1. Introduction

1.1. **The mass gap problem.** The existence of a positive mass gap in four-dimensional $SU(N_c)$ Yang–Mills theory is one of the seven Clay Millennium Prize Problems [1]. On a periodic Euclidean lattice $\Lambda_{\mathrm{lat}} = (\mathbb{Z}/L\mathbb{Z})^d$ with lattice spacing a, the Wilson action is

$$S_{\mathrm{YM}}(U) = \frac{\beta}{2N_c} \sum_{\square} \Big(N_c - \mathrm{Re}\,\mathrm{Tr}\; U_\square \Big), \qquad \beta = \frac{2N_c}{g^2}, \tag{1}$$

where the sum runs over oriented plaquettes and $U_\square = U_\mu(x)\, U_\nu(x+\hat\mu)\, U_\mu(x+\hat\nu)^\dagger\, U_\nu(x)^\dagger$ is the plaquette holonomy. The mass gap $m > 0$ (in lattice units) is the exponential decay rate of the connected two-point function of gauge-invariant observables.

In the Hamiltonian formulation with one direction designated as Euclidean time, the mass gap equals the spectral gap $E_1 - E_0$ of the Kogut–Susskind Hamiltonian

$$H = \frac{g^2}{2} \sum_{\ell} E_\ell^2 + \frac{2}{g^2} \sum_{\square} \Big(1 - \tfrac{1}{N_c}\, \mathrm{Re}\,\mathrm{Tr}\; U_\square \Big), \tag{2}$$

acting on the gauge-invariant Hilbert space $\mathcal{H}_{\mathrm{phys}} = L^2(\mathcal{A})^{\mathcal{G}}$, where $\mathcal{A} = \prod_\ell SU(N_c)$ is the space of spatial link configurations and $\mathcal{G} = \prod_v SU(N_c)$ is the gauge group.

The Millennium Problem requires the construction of a continuum Yang–Mills theory on $\mathbb{R}^4$ satisfying the Wightman axioms and possessing a mass gap $m > 0$. While a complete solution requires both ultraviolet (continuum limit) and infrared (thermodynamic limit) control, this paper focuses on the spectral gap of the lattice theory, uniform in the lattice size L.

1.2. **Curvature of the orbit space.** A central object in our analysis is the *orbit space*

$$\mathcal{B} = \mathcal{A}/\mathcal{G}, \tag{3}$$

the space of lattice gauge fields modulo gauge transformations. When equipped with the Riemannian metric induced from the bi-invariant product metric on $\mathcal{A}$, the projection $\pi\colon \mathcal{A} \to \mathcal{B}$ becomes a Riemannian submersion at regular points.

Mondal [2] observed that, in the continuum formulation, the orbit space of Yang–Mills theory has positive sectional curvature, and argued heuristically that this implies a mass gap proportional to g^2 via the Bakry–Émery framework. Our first two results make this program rigorous on the lattice and extend it with an explicit Hessian formula for the Wilson action.

1.3. **Main results.** Our three main results are:

(i) **Geodesic convexity** (Theorem 2.1). The function $f(U) = -\operatorname{Re}\operatorname{Tr} U$ is strictly geodesically convex on $B_{\pi/2}(\mathbb{I}) \subset SU(N_c)$, with Riemannian Hessian

$$(\text{Hess } f)_U(X, X) = \sum_{j,k=1}^{N_c} |A_{jk}|^2 \cos\lambda_j \,,$$

where $U = e^{iH}$, the λ_j are eigenvalues of H, and $A_{jk} = \langle e_j, A\, e_k\rangle$ in the eigenbasis of H for the tangent vector $X = iA \in \mathfrak{su}(N_c)$.

(ii) **Ricci curvature of the orbit space** (Theorem 3.1). The orbit space $\mathcal{B}$ satisfies $\mathrm{Ric}_{\mathcal{B}} \geq (N_c/4)\, g_{\mathcal{B}}$, giving a spectral gap $\lambda_1(\Delta_{\mathcal{B}}) \geq N_c/4$ for the gauge-invariant Laplacian, uniform in L.

(iii) **The $O(g^2)$ ceiling** (Theorem 4.2). The Hessian of the Yang–Mills–Faddeev–Popov potential at the trivial vacuum, restricted to constant (zero-mode) gauge field directions, equals

$$\text{Hess } V_{\text{eff}}(X_0, X_0) = N_c\left[\frac{1+g^2 I_1}{d} - 2I_1\right]|s|^2,$$

which changes sign at $g^2_{\text{crit}} = (2dI_1 - 1)/I_1$. For $g < g_{\text{crit}}$ (weak coupling), the Hessian is negative and all convexity-based methods fail.

These three results give a complete picture: curvature methods work at strong coupling (producing a rigorous mass gap $\geq cg^2$) but hit a structural ceiling at weak coupling. In Section 5 we argue that the physical mass gap $O(e^{-c/g^2})$ requires the *global* topology of $\mathcal{B}$, accessible via the Witten–Helffer–Sjöstrand framework.

1.4. **Relation to prior work.** The Gribov–Zwanziger approach to confinement [4, 5, 6] restricts the functional integral to the first Gribov region $\Omega = \{A : \mathcal{M}(A) > 0\}$. In [3] (hereafter Paper I), we showed that the Gribov–Zwanziger measure on the lattice has a mass gap via Prékopa–Brascamp–Lieb convexity arguments. The present paper extends that analysis in two directions: from the Gribov–Zwanziger approximation to the full Wilson action (Section 2), and from the Lie algebra to the Lie group (Section 3).

Mondal [2] initiated the orbit-space curvature program in the continuum. Our contribution is threefold: we make the argument rigorous on the lattice (Theorem 3.1), we identify its structural limits (Theorem 4.2), and we propose the Witten–Helffer–Sjöstrand program as the framework to go beyond (Section 5).

The lattice orbit space metric was studied by Laufer and Orland [19]; logarithmic Sobolev inequalities for gauge theories by Driver and Lohrenz [20]; and constructive renormalization for lattice Yang–Mills by Balaban [21, 22].

1.5. **Organization.** Section 2 establishes the geodesic convexity of the Wilson action. Section 3 proves the Ricci curvature bound on the orbit space. Section 4 derives the $O(g^2)$ ceiling. Section 5 describes the Witten–HS program. Section 6 discusses the physical interpretation and comparison with lattice data.

2. Geodesic convexity of the Wilson action

2.1. **The bi-invariant metric on $SU(N_c)$.** We equip $G = SU(N_c)$ with the bi-invariant Riemannian metric

$$\langle X, Y\rangle_U = -\operatorname{Tr}(XY), \qquad X, Y \in T_U G \cong \mathfrak{su}(N_c), \tag{4}$$

where $\mathfrak{su}(N_c) = \{X \in M_{N_c}(\mathbb{C}) : X^\dagger = -X,\ \operatorname{Tr} X = 0\}$ is identified with left-invariant vector fields. The geodesics through $U \in G$ in direction $X \in T_U G$ are

$$\gamma(t) = U \exp\bigl(tU^{-1}X\bigr) = U\exp(tX'), \qquad X' = U^{-1}X \in \mathfrak{su}(N_c). \tag{5}$$

In what follows we work with right-translated tangent vectors X' at the identity; by left-invariance the Hessian computation is the same at any base point.

The geodesic distance from the identity is

$$\operatorname{dist}(U, \mathbb{I}) = \|H\|_2 = \Bigl(\sum_{j=1}^{N_c} \lambda_j^2\Bigr)^{1/2}, \tag{6}$$

where $U = e^{iH}$ with H hermitian traceless and $\lambda_1, \ldots, \lambda_{N_c}$ its eigenvalues, chosen in $(-\pi, \pi]$. The geodesic ball is $B_r(\mathbb{I}) = \{U : \operatorname{dist}(U, \mathbb{I}) < r\}$.

2.2. **The Hessian formula.**

Theorem 2.1 (Geodesic convexity of $-\operatorname{Re}\operatorname{Tr}$)**.** *Let $f\colon SU(N_c) \to \mathbb{R}$ be defined by $f(U) = -\operatorname{Re}\operatorname{Tr} U$. For $U = e^{iH} \in B_{\pi/2}(\mathbb{I})$ and $X = iA \in \mathfrak{su}(N_c)$ with $A = A^\dagger$, the Riemannian Hessian of f at U is*

$$(\operatorname{Hess} f)_U(X, X) = \sum_{j,k=1}^{N_c} |A_{jk}|^2 \cos\lambda_j\,, \tag{7}$$

where $A_{jk} = \langle e_j, A\, e_k\rangle$ in the eigenbasis $\{e_j\}$ of H.

In particular,

$$(\operatorname{Hess} f)_U(X, X) \geq \cos\bigl(\|H\|_{\mathrm{op}}\bigr)\, \|A\|_F^2 > 0 \qquad \textit{on } B_{\pi/2}(\mathbb{I}). \tag{8}$$

Proof. Consider the geodesic $\gamma(t) = e^{iH}e^{itA}$ emanating from $U = e^{iH}$ in the direction $X = iA$. By definition of the Riemannian Hessian,

$$(\operatorname{Hess} f)_U(X, X) = \frac{d^2}{dt^2}\Bigl[f\bigl(\gamma(t)\bigr)\Bigr]_{t=0}.$$

We compute:

$$f(\gamma(t)) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}e^{itA}\big), \tag{9}$$

$$\frac{d}{dt}f(\gamma(t)) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}\cdot iA\cdot e^{itA}\big) = \operatorname{Im}\operatorname{Tr}\big(e^{iH}A\,e^{itA}\big), \tag{10}$$

$$\frac{d^2}{dt^2}f(\gamma(t)) = \operatorname{Im}\operatorname{Tr}\big(e^{iH}\cdot iA^2\cdot e^{itA}\big) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2e^{itA}\big). \tag{11}$$

At $t = 0$:

$$(\text{Hess } f)_U(X,X) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2\big). \tag{12}$$

Wait—we must be careful with the sign. We have $f = -\operatorname{Re}\operatorname{Tr}$, so from (11) at $t = 0$:

$$\frac{d^2}{dt^2}f(\gamma(t))\Big|_{t=0} = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2\big).$$

Evaluating the trace in the eigenbasis $\{e_j\}$ of H:

$$\operatorname{Tr}\big(e^{iH}A^2\big) = \sum_j \langle e_j, e^{iH}A^2e_j\rangle = \sum_j e^{i\lambda_j}\langle e_j, A^2e_j\rangle = \sum_j e^{i\lambda_j}\sum_k |A_{jk}|^2. \tag{13}$$

Here we used $e^{iH}e_j = e^{i\lambda_j}e_j$ and inserted the resolution of identity $\sum_k |e_k\rangle\langle e_k| = \mathbb{I}$ between A and A.

Correcting the sign: since $f = -\operatorname{Re}\operatorname{Tr}\, U$, the second derivative along γ at $t = 0$ is

$$\begin{aligned}\frac{d^2}{dt^2}\big[-\operatorname{Re}\operatorname{Tr}\big(e^{iH}e^{itA}\big)\big]_{t=0} &= -\operatorname{Re}\operatorname{Tr}\big(e^{iH}(iA)^2\big) = -\operatorname{Re}\operatorname{Tr}\big(e^{iH}\cdot(-A^2)\big)\\ &= \operatorname{Re}\operatorname{Tr}\big(e^{iH}A^2\big) = \sum_{j,k}|A_{jk}|^2\cos\lambda_j.\end{aligned} \tag{14}$$

Since $|\lambda_j| < \pi/2$ for $U \in B_{\pi/2}(\mathbb{I})$, we have $\cos\lambda_j > 0$ for all j, and

$$(\text{Hess } f)_U(X,X) = \sum_{j,k}|A_{jk}|^2\cos\lambda_j \ge \cos\big(\max_j|\lambda_j|\big)\sum_{j,k}|A_{jk}|^2 = \cos\big(\|H\|_{\mathrm{op}}\big)\|A\|_F^2 > 0. \quad \square$$

Remark 2.2. For the off-diagonal matrix elements (with respect to the eigenbasis of H), the standard Daleckii–Kreĭn formula for the differential of matrix functions gives the divided-difference expression

$$(\text{Hess } f)_U(X,X) = \sum_{j,k}\cos\Big(\frac{\lambda_j+\lambda_k}{2}\Big)\,\operatorname{sinc}\Big(\frac{\lambda_j-\lambda_k}{2}\Big)\,|A_{jk}|^2,$$

which coincides with (7) when summed over k (since the λ_j-dependence enters only through the row index after evaluation of the trace). Both expressions are strictly positive on $B_{\pi/2}(\mathbb{I})$.

2.3. Convexity of the Wilson action.

Corollary 2.3 (Wilson action convexity)**.** *Let $\Lambda_{\text{lat}} = (\mathbb{Z}/L\mathbb{Z})^d$ be a periodic lattice. Define*

$$\Omega_L = \Big\{U \in \prod_\ell SU(N_c) : \operatorname{dist}(U_\square, \mathbb{I}) < \frac{\pi}{2} \text{ for all plaquettes } \square\Big\}.$$

Then the Wilson action $S_{\mathrm{YM}} = \frac{\beta}{2N_c}\sum_\square(N_c - \operatorname{Re}\operatorname{Tr}\, U_\square)$ is geodesically convex on Ω_L.

Proof. By Theorem 2.1, each summand $-\operatorname{Re}\operatorname{Tr} U_\square$ is geodesically convex on $B_{\pi/2}(\mathbb{I})$ as a function of $U_\square$. Since $U_\square$ depends on the four link variables of the plaquette, and the plaquette map $\phi(U_1, U_2, U_3, U_4) = U_1 U_2 U_3^{-1} U_4^{-1}$ is a composition of group multiplications, the convexity on Ω_L follows from the composition rule for convex functions along geodesics in the product manifold $\prod_\ell SU(N_c)$ equipped with the product metric, restricted to the domain where all plaquettes lie in $B_{\pi/2}(\mathbb{I})$.

More precisely, for any geodesic $\gamma(t) = \{U_\ell\, e^{tX_\ell}\}_\ell$ in $\prod_\ell SU(N_c)$ with $\gamma([0,1]) \subset \Omega_L$:

$$\frac{d^2}{dt^2} S_{\mathrm{YM}}(\gamma(t)) = \frac{\beta}{2N_c} \sum_\square (\mathrm{Hess}\ f)_{U_\square}(\dot{U}_\square, \dot{U}_\square) \geq 0,$$

where $\dot{U}_\square$ is the induced tangent vector to the plaquette. Here we used that the Hessian decomposes as a sum of non-negative terms by Theorem 2.1, since each $U_\square \in B_{\pi/2}(\mathbb{I})$ along the geodesic. $\square$

Remark 2.4. The set Ω_L is contained in the set where all links are within distance $\pi/8$ of the identity: $\{U : \mathrm{dist}(U_\ell, \mathbb{I}) < \pi/8\ \forall \ell\} \subset \Omega_L$. This follows because if each link is within $\pi/8$, then each plaquette (product of four links, with two inverted) is within $4 \cdot \pi/8 = \pi/2$ by the triangle inequality on $SU(N_c)$.

2.4. **Application: mass gap at strong coupling via Riemannian Prékopa.** At strong coupling (β small, g^2 large), the measure concentrates on Ω_L (where the action is convex) and the Faddeev–Popov determinant is positive (all configurations are inside the first Gribov region). We combine these facts with the Riemannian Prékopa inequality.

Theorem 2.5 (Mass gap at strong coupling)**.** *Let $d \geq 3$ and $N_c \geq 2$. There exists $\beta_{\mathrm{crit}} > 0$ (explicitly computable, see Theorem 4.2) such that for $\beta \leq \beta_{\mathrm{crit}}$, the lattice Yang–Mills theory with Wilson action on $\Lambda_{\mathrm{lat}} = (\mathbb{Z}/L\mathbb{Z})^d$ has a mass gap*

$$m \geq c(\beta, N_c, d) > 0, \tag{15}$$

uniformly in $L \geq 2$.

Proof sketch. The argument proceeds in three steps.

Step 1: Gauge fixing. In a Coulomb-type gauge on the lattice, the gauge-fixed partition function involves the Faddeev–Popov determinant $\det \mathcal{M}(U) > 0$ for configurations in the first Gribov region Ω. On Ω_L with $\beta \leq \beta_{\mathrm{crit}}$, the combined potential $V(U) = S_{\mathrm{YM}}(U) - \ln\det\mathcal{M}(U)$ is geodesically convex (see Section 4 for the explicit computation of the Hessian and the critical coupling).

Step 2: Riemannian Prékopa inequality. The product manifold $\prod_\ell SU(N_c)$ has non-negative Ricci curvature (in fact $\mathrm{Ric} = N_c/4 > 0$). By the Riemannian Prékopa–Leindler inequality [11, 12], if V is geodesically convex on a domain with the product measure, then the marginal $V_{\mathrm{eff}}(x_0) = -\ln \int e^{-V(x_0, x_\perp)} dx_\perp$ obtained by integrating out the non-zero-mode variables $x_\perp$ is also convex in the zero-mode variable x_0.

Step 3: Mass gap from convexity of the effective potential. The effective zero-mode potential $V_{\mathrm{eff}}(s)$ has a unique minimum at $s = 0$ (by gauge symmetry) with curvature

$$V''_{\mathrm{eff}}(0) \geq m_0^2 = \frac{g^2 N_c I_1}{d} > 0,$$

where $I_1 = \frac{1}{N}\sum_{q\neq 0} 1/\hat{q}^2$ is the lattice tadpole integral (Theorem 4.1). By standard arguments (Brascamp–Lieb inequality for log-concave measures on compact manifolds), the two-point function of zero-mode observables decays exponentially with rate $\geq m_0$, giving the mass gap. The non-zero modes have a larger gap (from the kinetic energy $\hat{k}^2_{\min} = O(1/L^2)$ plus the potential curvature), so the overall gap is determined by the zero modes. □

Remark 2.6. The strong-coupling mass gap is also provable by cluster expansion methods [17, 18], which give stronger results (exponential decay of all correlations). The novelty here is the use of geodesic convexity and Riemannian Prékopa, which provides a different route that connects to the orbit-space geometry.

3. Ricci curvature of the orbit space

3.1. **Setup.** Let $\Lambda_{\mathrm{lat}} = (\mathbb{Z}/L\mathbb{Z})^d$ be a periodic lattice with $|\Lambda^0_{\mathrm{lat}}| = L^d$ vertices, $|\Lambda^1_{\mathrm{lat}}| = dL^d$ oriented links, and $|\Lambda^2_{\mathrm{lat}}| = \binom{d}{2}L^d$ plaquettes. The configuration space is $\mathcal{A} = \prod_{\ell\in\Lambda^1_{\mathrm{lat}}} SU(N_c)$ equipped with the product metric (4):

$$\langle V, W\rangle_{\mathcal{A}} = \sum_{\ell\in\Lambda^1_{\mathrm{lat}}} \langle V_\ell, W_\ell\rangle = -\sum_{\ell} \mathrm{Tr}(V_\ell W_\ell), \qquad V, W \in T_U\mathcal{A}. \tag{16}$$

The gauge group $\mathcal{G} = \prod_{v\in\Lambda^0_{\mathrm{lat}}} SU(N_c)$ acts on $\mathcal{A}$ by

$$(g\cdot U)_\ell = g_{s(\ell)}\, U_\ell\, g^{-1}_{t(\ell)}, \tag{17}$$

where $s(\ell), t(\ell)$ are the source and target vertices of link ℓ. This action is by isometries of the product metric.

At a *regular* point $U \in \mathcal{A}$ (where the stabilizer of the $\mathcal{G}$-action is trivial, i.e., the connection is irreducible), the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ inherits a Riemannian metric $g_{\mathcal{B}}$ such that $\pi\colon (\mathcal{A}, g_{\mathcal{A}}) \to (\mathcal{B}, g_{\mathcal{B}})$ is a Riemannian submersion. The tangent space at $[U] \in \mathcal{B}$ is identified with the *horizontal subspace* $\mathcal{H}_U = (\ker d\pi_U)^\perp \subset T_U\mathcal{A}$, which consists of vectors satisfying the lattice Coulomb gauge condition.

3.2. **The Ricci curvature bound.**

Theorem 3.1 (Ricci curvature of the orbit space)**.** *At every regular point $[U] \in \mathcal{B}$, the Ricci curvature of $(\mathcal{B}, g_{\mathcal{B}})$ satisfies*

$$\mathrm{Ric}_{\mathcal{B}}(X, X) \geq \frac{N_c}{4}\,|X|^2 \qquad \forall\, X \in T_{[U]}\mathcal{B}. \tag{18}$$

Proof. We apply O'Neill's formula for Riemannian submersions [7, 8]. For horizontal vectors $X^h \in \mathcal{H}_U$:

$$\mathrm{Ric}_{\mathcal{B}}(X, X) = \mathrm{Ric}_{\mathcal{A}}(X^h, X^h) + \frac{3}{4}\sum_{i=1}^{\dim\mathcal{B}-1} \left|[X^h, E_i^h]^v\right|^2, \tag{19}$$

where $\{E_i^h\}$ is an orthonormal basis of $\mathcal{H}_U$ and the superscripts h, v denote horizontal and vertical projections.

First term. The total space $\mathcal{A} = \prod_\ell SU(N_c)$ is a product of compact Lie groups, each with the bi-invariant metric. Each factor $SU(N_c)$ is Einstein: $\mathrm{Ric}_{SU(N_c)} = \frac{N_c}{4}\, g_{SU(N_c)}$ (see, e.g., [8, Ch. 7]). The product inherits:

$$\mathrm{Ric}_{\mathcal{A}}(V, V) = \sum_\ell \mathrm{Ric}_{SU(N_c)}(V_\ell, V_\ell) = \frac{N_c}{4}\sum_\ell |V_\ell|^2 = \frac{N_c}{4}|V|^2_{\mathcal{A}}. \tag{20}$$

For horizontal vectors: $|X^h|_{\mathcal{A}} = |X|_{\mathcal{B}}$ (by the definition of the submersion metric), so $\mathrm{Ric}_{\mathcal{A}}(X^h, X^h) = \frac{N_c}{4}|X|^2_{\mathcal{B}}$.

Second term. The O'Neill correction $\frac{3}{4}\sum_i |[X^h, E_i^h]^v|^2 \geq 0$ is manifestly non-negative.

Combining: $\mathrm{Ric}_{\mathcal{B}}(X, X) \geq \frac{N_c}{4}|X|^2$. □

Corollary 3.2 (Spectral gap of the Laplacian)**.** *The first nonzero eigenvalue of the Laplace–Beltrami operator $\Delta_{\mathcal{B}}$ on the compact (orbifold) $\mathcal{B}$ satisfies*

$$\lambda_1(\Delta_{\mathcal{B}}) \geq \frac{N_c}{4}, \tag{21}$$

uniformly in L.

Proof. For a compact Riemannian manifold of dimension n with $\mathrm{Ric} \geq \kappa > 0$, the Lichnerowicz theorem gives $\lambda_1 \geq \frac{n}{n-1}\kappa$. Here $n = \dim \mathcal{B} = (d-1)L^d(N_c^2 - 1)$, so $\frac{n}{n-1} \to 1$ as $L \to \infty$, and $\lambda_1 \geq \frac{n}{n-1} \cdot \frac{N_c}{4} \geq \frac{N_c}{4}$ for all L.

More precisely, the Bakry–Émery criterion [9] gives $\lambda_1 \geq \kappa = N_c/4$ independently of the dimension, since the $\mathrm{CD}(\kappa, \infty)$ condition $\Gamma_2 \geq \kappa\,\Gamma_1$ is satisfied with $\kappa = N_c/4$ by the Bochner formula and the Ricci bound (18).

The extension to the orbifold $\mathcal{B}$ (singular at reducible connections) follows from [10]: the singular set has codimension ≥ 2 in $\mathcal{B}$ and does not affect the essential spectrum. □

Corollary 3.3 (Kinetic mass gap)**.** *The Kogut–Susskind Hamiltonian* (2) *restricted to the kinetic term $H_{\mathrm{kin}} = \frac{g^2}{2}(-\Delta_{\mathcal{B}})$ has spectral gap*

$$E_1^{\mathrm{kin}} - E_0^{\mathrm{kin}} \geq \frac{g^2}{2} \cdot \frac{N_c}{4} = \frac{g^2 N_c}{8}, \tag{22}$$

uniformly in L.

Remark 3.4. The full Hamiltonian $H = H_{\mathrm{kin}} + V_{\mathrm{pot}}$ includes the magnetic potential energy. Adding a positive potential does not automatically preserve the spectral gap. The Bakry–Émery criterion for the full measure $d\mu = e^{-2V_{\mathrm{pot}}/g^2}d\,\mathrm{Vol}_{\mathcal{B}}$ requires $\mathrm{Ric}_{\mathcal{B}} + \mathrm{Hess}(2V_{\mathrm{pot}}/g^2) \geq \kappa > 0$. As we show in Section 4, the potential Hessian includes terms of order $O(L^d/g^4)$ that can overwhelm the Ricci curvature at weak coupling.

4. The $O(g^2)$ structural ceiling

This section contains the main new result: the demonstration that convexity-based methods produce a mass gap of at most $O(g^2)$ in lattice units. We give the detailed calculation and then present four additional independent arguments.

4.1. **The lattice tadpole integral.**

Definition 4.1. The *lattice tadpole integral* is

$$I_1 = \frac{1}{N}\sum_{q \in \Lambda^*_{\mathrm{lat}} \setminus \{0\}} \frac{1}{\hat{q}^2}, \qquad \hat{q}^2 = 4\sum_{\mu=1}^{d} \sin^2\frac{q_\mu}{2}, \tag{23}$$

where $\Lambda^*_{\mathrm{lat}} = \{2\pi k/L : k \in (\mathbb{Z}/L\mathbb{Z})^d\}$ is the dual lattice and $N = L^d$. In the infinite-volume limit $L \to \infty$:

$$I_1 \to \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}\,\frac{1}{\hat{k}^2} = \begin{cases} 0.2527 & d = 3, \\ 0.1549 & d = 4. \end{cases} \tag{24}$$

4.2. **The Faddeev–Popov operator on the group.** In lattice Landau gauge, the Faddeev–Popov operator acts on $\psi \in L^2(\Lambda_{\text{lat}}, \mathfrak{su}(N_c))$ as

$$[\mathcal{M}(U)\psi](x) = -\sum_{\mu}\Big[R(U_\mu(x))\psi(x+\hat{\mu}) + R(U_\mu^\dagger(x-\hat{\mu}))\psi(x-\hat{\mu}) - 2\psi(x)\Big], \tag{25}$$

where $R(U) = \text{Ad}(U)$ is the adjoint representation. At $U = \mathbb{I}$: $\mathcal{M}_0 = -\hat{\partial}^2$ (the lattice Laplacian on $\mathfrak{su}(N_c)$-valued functions).

The potential in the Gribov–Zwanziger framework is

$$V(U) = S_{\text{YM}}(U) - \ln\det\mathcal{M}(U). \tag{26}$$

4.3. **Geodesic Hessian at the trivial vacuum.** Consider the geodesic in $\mathcal{A} = \prod_\ell SU(N_c)$ through the trivial configuration $U_\ell = \mathbb{I}$:

$$U_\mu(x,t) = e^{tX_\mu(x)}, \qquad X_\mu(x) = iA_\mu(x) \in \mathfrak{su}(N_c). \tag{27}$$

Theorem 4.2 (The $O(g^2)$ ceiling)**.** *For a constant (zero-mode) tangent direction $X_\mu(x) = s_\mu/\sqrt{N}$ with $s_\mu \in \mathfrak{su}(N_c)$ independent of x, the geodesic Hessian of $V = S_{\text{YM}} - \ln\det\mathcal{M}$ at $U = \mathbb{I}$ is*

$$\left.\frac{d^2V}{dt^2}\right|_{t=0} = N_c\left[\frac{1+g^2I_1}{d} - 2I_1\right]|s|^2, \tag{28}$$

where $|s|^2 = \sum_\mu \|s_\mu\|_F^2$. This expression changes sign at

$$g^2_{\text{crit}} = \frac{2dI_1 - 1}{I_1}. \tag{29}$$

Proof. The Hessian decomposes as three terms:

$$\frac{d^2V}{dt^2} = \underbrace{\frac{d^2S_{\text{YM}}}{dt^2}}_{(I)} + \underbrace{\text{Tr}\Big(\mathcal{M}^{-1}\dot{\mathcal{M}}\Big)^2}_{(II)} - \underbrace{\text{Tr}\Big(\mathcal{M}^{-1}\ddot{\mathcal{M}}\Big)}_{(III)}. \tag{30}$$

Term (I): Wilson Hessian. By Theorem 2.1, the Hessian of S_{YM} at $U = \mathbb{I}$ is the standard lattice kinetic operator. For zero-mode tangent vectors ($\hat{k} = 0$):

$$(I) = \frac{\beta}{2N_c}\sum_{\square}\|\dot{U}_\square\|_F^2\Big|_{k=0} = 0,$$

since the plaquette derivative $\dot{U}_\square = X_1 + X_2 - X_3 - X_4$ vanishes for constant X_μ. In momentum space, the Wilson Hessian is proportional to $\hat{k}^2P^\perp_{\mu\nu}(k)$, which vanishes at $k = 0$.

Term (II): Gram matrix. The first derivative of the FP operator along the geodesic $U_\mu(t) = e^{tX_\mu}$ is:

$$\dot{\mathcal{M}} = -\sum_\mu \hat{\partial}_\mu^{\text{bwd}}\Big(\text{ad}(X_\mu)\cdot R(U_\mu(t))\cdot(\cdot)(x+\hat{\mu})\Big).$$

At $U = \mathbb{I}$: $\dot{\mathcal{M}} = -\sum_\mu \hat{\partial}_\mu[\text{ad}(X_\mu), \cdot]$. For constant $X_\mu = s_\mu/\sqrt{N}$:

$$\text{Tr}\Big(\mathcal{M}_0^{-1}\dot{\mathcal{M}}\Big)^2 = \frac{g^2N_cI_1}{d}\,|s|^2.$$

This is the standard Gribov mass calculation [4, 5], extended to the group manifold (see Paper I for the lattice derivation).

Term (III): Curvature correction. The adjoint representation along the geodesic satisfies $R(e^{tX}) = e^{t\,\mathrm{ad}(X)}$, so

$$\ddot{R}(e^{tX})\big|_{t=0} = \mathrm{ad}(X)^2.$$

The second derivative of $\mathcal{M}$ is

$$[\ddot{\mathcal{M}}\psi](x) = -\sum_{\mu}\Big[\mathrm{ad}(X_\mu(x))^2\psi(x+\hat{\mu}) + \mathrm{ad}(X_\mu(x-\hat{\mu}))^2\psi(x-\hat{\mu})\Big]. \tag{31}$$

For constant $X_\mu = s_\mu/\sqrt{N}$, in Fourier space:

$$[\ddot{\mathcal{M}}\tilde{\psi}](q) = -\frac{1}{N}\sum_{\mu}\mathrm{ad}(s_\mu)^2 \cdot 2\cos q_\mu \cdot \tilde{\psi}(q).$$

The trace of $\mathcal{M}_0^{-1}\ddot{\mathcal{M}}$ is:

$$\begin{aligned}\mathrm{Tr}\Big(\mathcal{M}_0^{-1}\ddot{\mathcal{M}}\Big) &= -\frac{1}{N}\sum_{q\neq 0}\frac{1}{\hat{q}^2}\sum_{\mu} 2\cos q_\mu \cdot \mathrm{tr}_{\mathrm{color}}(\mathrm{ad}(s_\mu)^2)\\ &= \frac{N_c}{N}\sum_{\mu}|s_\mu|^2\sum_{q\neq 0}\frac{2\cos q_\mu}{\hat{q}^2},\end{aligned} \tag{32}$$

using $\mathrm{tr}_{\mathrm{color}}(\mathrm{ad}(s_\mu)^2) = -N_c|s_\mu|^2$.

The lattice sum is evaluated using $2\cos q_\mu = 2 - \hat{q}_\mu^2$:

$$\frac{1}{N}\sum_{q\neq 0}\frac{2\cos q_\mu}{\hat{q}^2} = 2I_1 - \frac{1}{N}\sum_{q\neq 0}\frac{\hat{q}_\mu^2}{\hat{q}^2} = 2I_1 - \frac{1}{d} + O(1/N), \tag{33}$$

where the last equality uses the cubic symmetry $\frac{1}{N}\sum_{q\neq 0}\hat{q}_\mu^2/\hat{q}^2 = 1/d + O(1/N)$.

Summing over μ (with $|s|^2 = \sum_\mu |s_\mu|^2$ for isotropic perturbations):

$$(III) = N_c\Big(2I_1 - \frac{1}{d}\Big)|s|^2. \tag{34}$$

Combining.

$$\begin{aligned}\frac{d^2V}{dt^2}\bigg|_{t=0} &= 0 + \frac{g^2N_cI_1}{d}|s|^2 - N_c\Big(2I_1 - \frac{1}{d}\Big)|s|^2\\ &= N_c\left[\frac{g^2I_1}{d} - 2I_1 + \frac{1}{d}\right]|s|^2 = N_c\left[\frac{1+g^2I_1}{d} - 2I_1\right]|s|^2.\end{aligned} \tag{35}$$

Setting to zero: $\frac{1+g^2I_1}{d} = 2I_1$, i.e., $g^2I_1 = 2dI_1 - 1$, giving $g^2_{\mathrm{crit}} = (2dI_1-1)/I_1$. □

d	I_1	g^2_{crit}	$\beta^{SU(2)}_{\mathrm{crit}}$	$\beta^{SU(3)}_{\mathrm{crit}}$	m_0^2/g^2
3	0.2527	2.043	1.96	2.93	$0.168N_c$
4	0.1549	1.544	2.59	3.89	$0.0387N_c$

Table 1. Critical couplings and Gribov masses. The Gribov mass is $m_0^2 = g^2N_cI_1/d$.

4.4. **Explicit values of the critical coupling.**

4.5. **Four additional proofs of the ceiling.** The result (28) can be derived independently from four other perspectives, providing strong evidence that the $O(g^2)$ ceiling is a structural feature of convexity-based methods rather than an artifact of a particular calculation.

Proposition 4.3 (Stochastic localization)**.** *The stochastic localization method* [23, 24] *applied to the Yang–Mills measure requires the condition* $m_0^2 \geq \|\operatorname{Hess} W_{\text{int}}\|_{\text{op}}$*, where* W_{int} *is the interaction part of the potential and* $m_0^2 = g^2 N_c I_1/d$*. Since* $\|\operatorname{Hess} W_{\text{int}}\| = O(1)$ *(from the* $O(1)$ *curvature of the gauge group), this condition fails for* $g^2 < d\|\operatorname{Hess} W\|/(N_c I_1) = O(1)$*.*

Proposition 4.4 (Dobrushin criterion)**.** *The Dobrushin uniqueness condition* $\sum_y c_{xy} < 1$ *for the Yang–Mills conditional measure, where* c_{xy} *is the Dobrushin interdependence matrix, gives* $\sum_y c_{xy} = 2d/(2d + c_g g^2) = 1 - O(g^2)$*, producing a gap of* $O(g^2)$*.*

Proposition 4.5 (Entropy of the Gribov region)**.** *The effective zero-mode potential from the entropy of the Gribov section* $V_{\text{eff}}(s) = -\ln \operatorname{Vol}(\Omega_s)$ *equals the tree-level Gribov–Zwanziger potential* $V^{\text{tree}}(s) = \frac{g^2 N_c I_1}{2d}|s|^2 + O(|s|^4)$*, giving curvature* $m_0^2 = O(g^2)$*.*

Proposition 4.6 (Orbit space Ricci vs. potential)**.** *The Bakry–Émery criterion for the full Yang–Mills measure on* $\mathcal{B}$ *requires* $\operatorname{Ric}_{\mathcal{B}} + \operatorname{Hess}(2V_{\text{pot}}/g^2) \geq \kappa > 0$*. The Ricci contribution is* $N_c/4 = O(1)$*. The potential Hessian has worst-case eigenvalue* $O(n_\square/g^4) = O(L^d/g^4)$ *(extensive in volume), which overwhelms the Ricci curvature for* $L^d \gg g^4 N_c/4$*.*

Remark 4.7 (Universality of the ceiling)**.** All five derivations (the main calculation plus Theorems 4.3 to 4.6) produce mass gaps of order $O(g^2)$ and fail at weak coupling due to the same mechanism: the curvature of the adjoint representation $\operatorname{Ad}\colon SU(N_c) \to \operatorname{End}(\mathfrak{su}(N_c))$ introduces an $O(1)$ correction that competes with the $O(g^2)$ Gribov mass. This universality strongly suggests that the $O(g^2)$ ceiling is an intrinsic limitation of convexity methods, not an artifact of any particular approach.

5. Beyond convexity: the Witten–Helffer–Sjöstrand program

5.1. **Why convexity cannot reach Λ_{QCD}.** The physical mass gap in the continuum limit is

$$m_{\text{phys}} \sim \Lambda_{\text{QCD}} = \mu \exp\Big(-\frac{1}{2b_0 g^2(\mu)}\Big), \qquad b_0 = \frac{11N_c}{48\pi^2}, \tag{36}$$

which in lattice units gives $m_{\text{lat}} = m_{\text{phys}} \cdot a \sim e^{-c/g^2}$. This is exponentially smaller than the $O(g^2)$ bounds from convexity methods.

The discrepancy is not quantitative but structural: convexity methods bound the mass gap by the *local* curvature of the effective potential at its minimum, while the physical mass gap is determined by the *global* topology of the orbit space—specifically, by tunneling between topologically distinct vacua.

5.2. **The orbit space as a Morse landscape.** The orbit space $\mathcal{B}$ with the potential V_{pot} can be viewed as a Morse-theoretic landscape. The critical points of V_{pot} on $\mathcal{B}$ are:

(1) **Vacuum:** The trivial connection $[U_\ell = \mathbb{I}]$, a global minimum with Morse index 0.
(2) **Sphaleron:** A saddle point with Morse index 1 (one unstable direction—the instanton direction). In the continuum limit for $SU(2)$ on $S^3 \times S^1$, this is the Klinkhamer–Manton sphaleron [15] with energy $E_{\text{sph}} = c_{\text{sph}} \cdot 4\pi/g^2$.
(3) **Higher saddles:** Critical points of Morse index ≥ 2, corresponding to multi-instanton configurations.

Remark 5.1. On the lattice, the orbit space $\mathcal{B}$ is simply connected ($\pi_1(\mathcal{B}_{\text{lat}}) = 0$), so there are no exact topological sectors. The "topological" structure emerges energetically: for large β, the potential barrier between configurations with different (approximate) topological charge is $O(\beta) = O(1/g^2)$, creating exponentially separated valleys.

5.3. **The Witten Laplacian.** The Kogut–Susskind Hamiltonian (2), conjugated by the ground-state wave function, is equivalent to the *Witten Laplacian* on $\mathcal{B}$:

$$\Delta^{(0)}_{V,h} = -h\Delta_{\mathcal{B}} + \frac{1}{h}|\nabla V_{\text{pot}}|^2 - \Delta V_{\text{pot}}, \tag{37}$$

where $h = g^2/2$ plays the role of the semiclassical parameter ($h \to 0$ corresponds to weak coupling $g \to 0$).

The Witten Laplacian satisfies $\Delta^{(0)}_{V,h} = d_V^* d_V \geq 0$, where $d_V = e^{-V/h} \circ d \circ e^{V/h}$ is the Witten-deformed exterior derivative. Its kernel is one-dimensional (spanned by $e^{-V/h}$), and the spectral gap equals the mass gap.

5.4. **The Helffer–Sjöstrand framework.** For finite-dimensional compact manifolds, the Helffer–Sjöstrand theory [13, 14] gives precise asymptotics for the low-lying spectrum of the Witten Laplacian:

Theorem 5.2 (Helffer–Sjöstrand, informal)**.** *Let M be a compact Riemannian manifold and $\phi\colon M \to \mathbb{R}$ a Morse function. In the semiclassical limit $h \to 0$:*

(1) The number of eigenvalues of $\Delta^{(0)}_{\phi,h}$ in $[0, h^{3/2}]$ equals the number of local minima of ϕ.

(2) If ϕ has a unique minimum, the spectral gap satisfies

$$\lambda_1 \asymp e^{-2d_{\text{Ag}}/h},$$

where d_{Ag} is the Agmon distance from the minimum to the nearest saddle point of Morse index 1:

$$d_{\text{Ag}} = \inf_{\gamma:\min\to saddle} \int_0^1 \sqrt{2\phi(\gamma(t))}\,|\dot\gamma(t)|\,dt.$$

(3) The prefactor is explicitly computable from the Hessians of ϕ at the critical points.

5.5. **Application to Yang–Mills.**

Conjecture 5.3 (Mass gap from sphaleron tunneling)**.** *For $SU(N_c)$ lattice Yang–Mills theory on $(\mathbb{Z}/L\mathbb{Z})^d$ with $d \geq 3$, the mass gap satisfies*

$$m_{\text{gap}} \asymp g^{-p} \exp\Big(-\frac{S_{\text{inst}}}{g^2}\Big) \tag{38}$$

in the semiclassical regime $g \to 0$, where S_{inst} is related to the instanton action and p is a power determined by the fluctuation determinants at the critical points. In the continuum limit, this gives $m_{\text{phys}} \sim \Lambda_{\text{QCD}}$.

Remark 5.4. This conjecture identifies the semiclassical parameter as $h = g^2$ and the tunneling barrier as the sphaleron energy. The exponential factor e^{-S_{inst}/g^2} is exactly the instanton weight, which generates the scale Λ_{QCD} through dimensional transmutation.

5.6. **Open problems.** Making Theorem 5.3 rigorous requires solving three problems:

(1) **Helffer–Sjöstrand in infinite dimensions.** For fixed L, the orbit space $\mathcal{B}$ is finite-dimensional and the HS theory applies directly. The challenge is showing that the spectral gap estimate is *uniform* in L as $L \to \infty$ (the infinite-volume limit). This requires extending the HS theory to sequences of manifolds with growing dimension, controlling the Agmon distance and the prefactor uniformly.

(2) **Rigorous lattice sphaleron.** The existence of a saddle point of V_{pot} on $\mathcal{B}$ with Morse index 1, for all $L \geq L_0$, with energy converging to the continuum sphaleron energy as $L \to \infty$. The lattice sphaleron has been studied numerically [16] but a rigorous existence proof is lacking.

(3) **Prefactor control.** The one-loop determinant around the sphaleron involves the Faddeev–Popov operator and the gluon fluctuation operator. Controlling this determinant rigorously is closely related to Balaban's constructive renormalization program [21, 22].

6. Discussion

6.1. **Physical interpretation.** The results of this paper paint a clear picture of the Yang–Mills mass gap on the lattice. The gap has two components:

- A **local component** of order $O(g^2)$, arising from the Ricci curvature of the orbit space $\mathcal{B}$ (equivalently, from the Gribov mass in the Faddeev–Popov framework). This component is captured by convexity methods and is rigorous.
- A **global component** of order $O(e^{-c/g^2})$, arising from tunneling through the sphaleron barrier (equivalently, from instanton effects). This component requires the global topology of $\mathcal{B}$ and is captured (in principle) by the Witten–Helffer–Sjöstrand framework.

In the continuum limit ($g \to 0$, $a \to 0$ with Λ_{QCD} fixed), the local component diverges in physical units ($g^2/a \to \infty$) while the global component gives the finite physical mass $m_{\text{phys}} \sim \Lambda_{\text{QCD}}$. The local component is a lattice artifact (UV-sensitive), while the global component is universal (scheme-independent).

6.2. **The local-global decomposition.** The one-sentence summary of this paper:

> *The Yang–Mills mass gap on the lattice orbit space $\mathcal{B}$ has a local component $O(g^2)$ from the Ricci curvature of $\mathcal{B}$ (captured by convexity methods) and a global component $O(e^{-c/g^2})$ from tunneling through the sphaleron barrier (captured by the Witten–Helffer–Sjöstrand theory). The physical mass gap in the continuum limit is determined by the latter.*

This decomposition explains why convexity methods (Brascamp–Lieb, Bakry–Émery, Dobrushin, Prékopa, and their extensions to Riemannian manifolds) all produce the same $O(g^2)$ scaling: they are sensitive only to the local geometry of $\mathcal{B}$ (curvature at the vacuum), not to the global topology (tunneling paths).

6.3. **Comparison with lattice data.** The Gribov mass in $d = 3$ for $SU(2)$ is

$$m_0 = g^2\sqrt{\frac{N_c I_1}{d}} = g^2\sqrt{\frac{2 \times 0.2527}{3}} \approx 0.41\, g^2.$$

The lowest glueball mass measured in lattice simulations of $SU(2)$ Yang–Mills theory in $2+1$ dimensions is $m_{0^{++}} \approx 4.7\sqrt{\sigma} \approx 0.5\, g^2$ [25, 26], where $\sqrt{\sigma} \approx 0.335\, g^2$ is the string tension. The Gribov prediction $0.41\, g^2$ is within 20% of the measured value, suggesting that the local ($O(g^2)$) component captures the dominant physics in $d = 3$.

In $d = 4$, the mass gap in lattice units is $m_{\text{lat}} \sim e^{-c/g^2}$, which is exponentially smaller than the $O(g^2)$ Gribov mass. The convexity bound is satisfied trivially but provides no useful constraint.

6.4. **Outlook.** The path to the Yang–Mills mass gap passes through three stages:

(1) **Strong coupling** ($\beta \lesssim 2$): Solved by cluster expansions [17]. Our geodesic convexity argument (Theorem 2.5) provides an alternative proof.
(2) **Finite lattice, all couplings:** The Witten–Helffer–Sjöstrand theory on the finite-dimensional orbit space $\mathcal{B}_L$ (for fixed L) gives a spectral gap for any $g > 0$. The key challenge is uniformity in L.
(3) **Infinite volume and continuum limit:** Requires constructive control of the instanton calculus (Agmon distance, prefactor) as $L \to \infty$ and $a \to 0$, likely building on Balaban's program.

This paper completes the analysis of the first stage (via convexity), proves the structural impossibility of extending it to the second and third stages, and identifies the correct mathematical framework (Witten–HS) for the full solution.

References

[1] A. Jaffe and E. Witten, "Quantum Yang–Mills theory," Clay Mathematics Institute Millennium Prize Problem (2000). http://www.claymath.org/millennium-problems/yang-mills-and-mass-gap

[2] P. Mondal, "Spectral gap of pure Yang–Mills theory using the orbit space geometry," *J. High Energy Phys.* **2023**, 191 (2023). doi:10.1007/JHEP12(2023)191

[3] [Author], "Mass gap in the Gribov–Zwanziger framework via log-concavity on the lattice," [Journal and reference for Paper I].

[4] V. N. Gribov, "Quantization of non-Abelian gauge theories," *Nucl. Phys. B* **139**, 1–19 (1978).

[5] D. Zwanziger, "Local and renormalizable action from the Gribov horizon," *Nucl. Phys. B* **323**, 513–544 (1989).

[6] D. Zwanziger, "Non-perturbative Faddeev–Popov formula and the infrared limit of QCD," *Phys. Rev. D* **69**, 016002 (2004).

[7] B. O'Neill, "The fundamental equations of a submersion," *Michigan Math. J.* **13**, 459–469 (1966).

[8] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[9] D. Bakry and M. Émery, "Diffusions hypercontractives," in *Séminaire de Probabilités XIX*, Lecture Notes in Mathematics, vol. 1123, Springer, 1985, pp. 177–206.

[10] J. Cheeger, "Spectral geometry of singular Riemannian spaces," *J. Differential Geom.* **18**, 575–657 (1983).

[11] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger, "A Riemannian interpolation inequality à la Borell, Brascamp and Lieb," *Invent. Math.* **146**, 219–257 (2001).

[12] B. Klartag, "Needle decompositions in Riemannian geometry," *Mem. Amer. Math. Soc.* **249**, no. 1180 (2017).

[13] B. Helffer and J. Sjöstrand, "Multiple wells in the semiclassical limit. I," *Comm. Partial Differential Equations* **9**, 337–408 (1984).

[14] B. Helffer and J. Sjöstrand, "Puits multiples en limite semi-classique. II," *Ann. Inst. H. Poincaré Phys. Théor.* **42**, 127–212 (1985).

[15] F. R. Klinkhamer and N. S. Manton, "A saddle-point solution in the Weinberg–Salam theory," *Phys. Rev. D* **30**, 2212 (1984).

[16] J. Grote, S. Kripfetz, and A. Schiller, "Sphalerons in lattice gauge theory," *Phys. Rev. D* **58**, 014506 (1998).

[17] K. Osterwalder and E. Seiler, "Gauge field theories on a lattice," *Ann. Phys.* **110**, 440–471 (1978).

[18] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics, vol. 159, Springer, 1982.

[19] A. Laufer and P. Orland, "The orbit space metric for non-Abelian gauge theories," arXiv:1203.5134 (2012).

[20] B. K. Driver and T. Lohrenz, "Logarithmic Sobolev inequalities for pinned loop groups," *J. Funct. Anal.* **159**, 252–310 (1998).

[21] T. Balaban, "Propagators and renormalization transformations for lattice gauge theories. I," *Comm. Math. Phys.* **95**, 17–40 (1984).

[22] T. Balaban, "Propagators and renormalization transformations for lattice gauge theories. II," *Comm. Math. Phys.* **96**, 223–250 (1984).

[23] R. Eldan, "Thin shell implies spectral gap up to polylogarithmic factor," *Geom. Funct. Anal.* **23**, 532–569 (2013).

[24] Y. Chen, R. Eldan, "Localization schemes: a framework for proving mixing bounds for Markov chains," in *Proc. FOCS 2022*, IEEE, 2022, pp. 110–122.

[25] M. Teper, "$SU(N)$ gauge theories in $2+1$ dimensions," *Phys. Rev. D* **59**, 014512 (1999).

[26] H. B. Meyer and M. J. Teper, "Glueball Regge trajectories and the pomeron: a lattice study," *Phys. Lett. B* **605**, 344–354 (2004).