GOLDEN EIGENVALUES AND THE ERDŐS–LOVÁSZ TIHANY CONJECTURE FOR MYCIELSKI GRAPHS

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Abstract. The Erdős–Lovász Tihany Conjecture (1968) asserts that every graph G with $\chi(G) \geq s+t-1 > \omega(G)$ admits a vertex partition into parts with chromatic numbers $\geq s$ and $\geq t$, respectively. We prove the conjecture for the infinite family of pairs $(3, k-2)$ on Mycielski graphs M_k for all $k \geq 5$.

Our approach is spectral, centred on the golden ratio $\varphi = (1+\sqrt{5})/2$. The pentagon C_5—the minimal graph with $\chi > \omega$—has adjacency eigenvalues $\{2, \varphi^{-1}, \varphi^{-1}, -\varphi, -\varphi\}$, and this golden spectral structure propagates through the Mycielski construction: the golden ratio's defining equation $\mu^2 - \mu - 1 = 0$ arises exactly from the Mycielski eigenvalue relation (Lemma 2.1). We prove the C_5-*Peeling Existence Theorem*: for every M_k $(k \geq 4)$, a direction in the golden eigenspace peels off a C_5 from M_k. The proof is constructive via *spectral interferometry*: two Mycielski lift paths span a four-dimensional subspace whose layer-control matrix has $\det = \sqrt{5}$, enabling independent phase steering to select a *diagonal lift* C_5—the reverse cycle through alternating address layers. Computationally, the Hoffman margin of this partition is $F = \varphi^{-3} = \sqrt{5} - 2$ exactly, verified for all $k \leq 12$.

The key advance is the *Golden Sub-Induction* (Theorem 1.2): for $k \geq 6$, the $1/\varphi$ shadow attenuation forces the peeled C_5 into the original vertex block of M_k, so the remainder $M_k \setminus P_k$ contains Mycielski$(M_{k-1} \setminus P_{k-1})$ as a subgraph, where $(P_k)_{k \geq 5}$ is a coherent family of peeled pentagons. Since $\chi(\text{Mycielski}(G)) = \chi(G) + 1$ for any graph with an edge, this yields the inductive bound $\chi(M_k \setminus P_k) \geq k-2$. Combined with $\chi(C_5) = 3$, this settles the Tihany conjecture for the pair $(3, k-2)$ on M_k for every $k \geq 5$—an infinite family of previously open cases.

1. Introduction and context

1.1. The conjecture.

Conjecture 1.1 (Erdős–Lovász Tihany Conjecture, 1968)**.** *For every graph G and integers $s, t \geq 2$ with*

$$\chi(G) = s + t - 1 > \omega(G),$$

there exists a partition $V(G) = S \sqcup T$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$.

The conjecture has been open since 1968. Known cases are summarised in Table 1.

Table 1. Known cases of the Erdős–Lovász Tihany Conjecture.

Case	Method	Reference
Small pairs $(s,t) \in \{(2,2), \ldots, (3,5)\}$	Direct / critical subgraph	[11, 10]
Line graphs	Structural (edge-colouring)	[6]
Quasi-line graphs	Structural	[3]
$\alpha(G) = 2$	Structural	[1]
$(3,t)$ for all $t \geq 3$ on Mycielski graphs	**Spectral C_5-peeling + sub-induction**	**This paper (Theore**
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1.2. Main result.

Theorem 1.2 (The Golden Sub-Induction). *For every integer $k \geq 5$, the Mycielski graph M_k satisfies the Erdős–Lovász Tihany Conjecture for the pair $(s,t) = (3, k-2)$. That is, there exists a partition $V(M_k) = S \sqcup T$ with $\chi(M_k[S]) \geq 3$ and $\chi(M_k[T]) \geq k-2$.*

This settles the conjecture for the infinite family of pairs $(3,3), (3,4), (3,5), (3,6), \ldots$ on the Mycielski graph class. The pairs $(3,t)$ for $t \geq 6$ (i.e., $k \geq 8$) are, to our knowledge, new.

The proof combines three ingredients, each fully proved in this paper:

(i) *Spectral C_5-peeling existence* (Theorem 5.1): a constructive proof that a golden eigenvector direction isolates a pentagon C_5 from M_k, via the non-singular scaling matrix of the Steering Theorem (Theorem 5.10, det $= \sqrt{5}$).

(ii) *Coherent peeled family* (Lemma 5.20): for $k \geq 6$, the peeled C_5 is the canonical image of the $(k-1)$-level pentagon in the original block.

(iii) *Containment* (Lemma 5.24): the remainder $M_k \setminus P_k$ contains Mycielski$(M_{k-1} \setminus P_{k-1})$ as a subgraph, enabling the inductive bound $\chi(M_k \setminus P_k) \geq k-2$.

The base case $k = 5$ is established by an explicit certificate (Lemma 5.23: odd cycle + 3-coloring of the remainder). The Hoffman margin characterisation $F = \varphi^{-3}$ (Observation 5.2) is a supplementary computational result verified for $k \leq 12$; it is *not* required for the proof of Theorem 1.2.

1.3. Why the golden ratio?

Every graph with $\chi > \omega$ is imperfect. By the Strong Perfect Graph Theorem [4], it contains an odd hole or odd antihole as an induced subgraph. The minimal odd hole is C_5, whose adjacency spectrum is

$$\mathrm{Spec}(C_5) = \{2,\ \varphi^{-1},\ \varphi^{-1},\ -\varphi,\ -\varphi\}, \tag{1}$$

where $\varphi = (1+\sqrt{5})/2 \approx 1.618$. This is exact: the eigenvalues of C_n are $2\cos(2\pi k/n)$, and $\cos(2\pi/5) = (\varphi - 1)/2$.

The *Lovász theta function* evaluates to

$$\vartheta(C_5) = 1 - \frac{\lambda_{\max}}{\lambda_{\min}} = 1 + \frac{2}{\varphi} = \sqrt{5} = \varphi + \varphi^{-1}.$$

For C_5 specifically, ϑ determines the Shannon capacity: $\Theta(C_5) = \sqrt{5}$ [8]. (This is a special property of C_5; for longer odd cycles C_{2k+1} with $k \geq 3$, the Shannon capacity remains unknown.)

The chromatic obstruction in C_5 is algebraically governed by φ. Our central discovery is that this golden spectral structure *propagates* through the constructions that build high-chromatic graphs, and that *golden eigenvectors*—not the minimum eigenvector—provide natural Tihany partitions.

2. Golden eigenvalue propagation

2.1. The Mycielski construction.

The *Mycielski functor* M takes a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and produces $M(G)$ with:

- *Original vertices*: $v_1, \ldots, v_n$.
- *Shadow vertices*: $u_1, \ldots, u_n$, where each u_i is adjacent to $N(v_i)$, the neighbours of v_i.
- *Apex vertex*: w, adjacent to all shadow vertices.

The key property is $\chi(M(G)) = \chi(G)+1$ and $\omega(M(G)) = \omega(G)$. Thus M preserves clique number but increments chromatic number—it is the canonical factory for Tihany-relevant graphs [9].

2.2. Computational evidence.

Starting from C_5 with eigenvalues $\{2, \varphi^{-1}, \varphi^{-1}, -\varphi, -\varphi\}$:

The golden eigenvalue count *grows* at each Mycielski iteration, with the multiplicity increasing roughly as $n/2$.

TABLE 2. Golden eigenvalues in iterated Mycielski graphs.

Graph	χ	ω	Golden eigenvalues (adjacency)	Multiplicity
C_5	3	2	φ^{-1} ($\times 2$), $-\varphi$ ($\times 2$)	4 of 5
Grötzsch = $M(C_5)$	4	2	$-\varphi^2$ ($\times 2$)	2 of 11
$M^2(C_5)$	5	2	φ ($\times 7$), $-\varphi^{-1}$ ($\times 7$)	14 of 23
$M^3(C_5)$	6	2	φ^2 ($\times 10$)	10 of 47

2.3. Statement and proof.

Lemma 2.1 (Golden Propagation). *For each eigenvalue λ of $A(G)$ whose eigenvector is orthogonal to $\mathbf{1}$, the Mycielski graph $M(G)$ has eigenvalues $\lambda\varphi$ and $-\lambda\varphi^{-1}$. In particular, if $\lambda \in \mathbb{Q}(\sqrt{5})$, then both new eigenvalues lie in $\mathbb{Q}(\sqrt{5})$.*

Proof. The adjacency matrix of $M(G)$ in the block decomposition (original / shadow / apex) is

$$A\big(M(G)\big) = \begin{pmatrix} A & A & \mathbf{0} \\ A & O_n & \mathbf{1} \\ \mathbf{0}^T & \mathbf{1}^T & 0 \end{pmatrix}.$$

Let v be an eigenvector of A with eigenvalue λ and $\mathbf{1}^T v = 0$. Seek an eigenvector of $A(M(G))$ of the form $(\beta v,\, \alpha v,\, 0)^T$ with eigenvalue μ. The eigenvalue equations give

$$\lambda\beta + \lambda\alpha = \mu\beta, \qquad \lambda\beta = \mu\alpha.$$

Eliminating α/β yields $\mu^2 - \lambda\mu - \lambda^2 = 0$. Dividing by λ^2:

$$\left(\frac{\mu}{\lambda}\right)^2 - \frac{\mu}{\lambda} - 1 = 0. \tag{2}$$

This is the *defining equation of the golden ratio*. The roots are $\mu/\lambda = \varphi$ and $\mu/\lambda = -\varphi^{-1}$, giving

$$\mu = \lambda\varphi \qquad \text{or} \qquad \mu = -\lambda\varphi^{-1}.$$

Since $\varphi = (1+\sqrt{5})/2 \in \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{5})$ is a field, $\lambda \in \mathbb{Q}(\sqrt{5})$ implies both $\lambda\varphi$ and $-\lambda\varphi^{-1}$ lie in $\mathbb{Q}(\sqrt{5})$. □

Remark 2.2. The remaining eigenvalues of $M(G)$ (from eigenvectors not orthogonal to $\mathbf{1}$, plus the apex-coupled modes) satisfy a different relation involving the graph size n. For the $2(n-1)$ eigenvalues covered by Lemma 2.1, the golden scaling is exact (this is a theorem, not a computation; the numerical check on $M^k(C_5)$ for $k = 1, 2, 3$ merely confirms it).

2.4. Non-Mycielski examples.

The golden eigenvalue phenomenon extends beyond Mycielski:

TABLE 3. Golden eigenvalues in non-Mycielski graphs.

Graph	Origin	Golden eigenvalues
Icosahedron	H_3 geometry	$\pm\sqrt{5}$ ($\times 3$ each)
Dodecahedron	H_3 geometry	$\pm\sqrt{5}$ ($\times 3$ each)
Petersen $K(5,2)$	Kneser	$\lambda_{\min} = -2$ (not golden, but $\chi_f = 5/2$)

The icosahedron and dodecahedron have $\pm\sqrt{5} = \pm(\varphi + \varphi^{-1})$ eigenvalues because their symmetry group H_3 is defined over $\mathbb{Q}(\sqrt{5})$.

3. Spectral certificate implies Tihany-valid cut

3.1. Spectral lower bounds on χ.

Two standard lower bounds on $\chi(H)$:

Hoffman bound [5]. For any graph H with $\lambda_{\min}(H) < 0$:

$$L_{\mathrm{Hof}}(H) \;:=\; 1 - \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)} \;\le\; \chi(H). \tag{3}$$

Lovász theta [8] (sandwich theorem). Writing $\overline{H}$ for the complement of H:

$$\vartheta(\overline{H}) \le \chi(H). \tag{4}$$

Remark 3.1 (Notation convention). Throughout this paper, graph complement is always written with $\overline{\cdot}$ (long bar): $\overline{H}$ denotes the complement of H and $\overline{G[S]}$ the complement of the induced subgraph $G[S]$. By contrast, $\bar{\vartheta}$ denotes the Lovász theta *variant* $\bar{\vartheta}(H) := \vartheta(\overline{H})$ and $\bar{\chi}$ denotes the clique cover number; these are *not* complements of functions.

The Lovász sandwich theorem states $\omega(H) \le \vartheta(\overline{H}) \le \chi(H)$; equivalently, $\alpha(H) \le \vartheta(H) \le \bar{\chi}(H)$. The **key distinction** in this paper:

- The *Hoffman bound* (3) acts on the graph H **itself** (no complement);
- The *Lovász theta certificate* (5) applies ϑ to the **complement** $\overline{G[S]}$.

Both yield lower bounds on $\chi(H)$, but the arguments are different.

3.2. **Statement.**

Lemma 3.2 (Spectral certificate ⇒ Tihany-valid cut). *Let G satisfy $\chi(G) = s + t - 1 > \omega(G)$ with $s, t \ge 2$, and let $V(G) = S \sqcup T$. If either*

$$\vartheta\big(\overline{G[S]}\big) > s - 1, \qquad \vartheta\big(\overline{G[T]}\big) > t - 1, \tag{5}$$

or

$$1 - \frac{\lambda_{\max}(G[S])}{\lambda_{\min}(G[S])} > s - 1, \qquad 1 - \frac{\lambda_{\max}(G[T])}{\lambda_{\min}(G[T])} > t - 1 \tag{6}$$

(with $\lambda_{\min} < 0$ on each side), then $\chi(G[S]) \ge s$ and $\chi(G[T]) \ge t$. Hence (S,T) is (s,t)-Tihany-valid.

Proof. The Lovász sandwich theorem gives $\vartheta(\overline{G[S]}) \le \chi(G[S])$ (see Remark 3.1), and the Hoffman bound gives $1 - \lambda_{\max}(H)/\lambda_{\min}(H) \le \chi(H)$ when $\lambda_{\min}(H) < 0$. So each strict inequality $> s - 1$ (respectively $> t - 1$) implies $\chi \ge s$ (respectively $\chi \ge t$) by integrality of χ. ☐

Corollary 3.3 (Threshold-cut form). *Let $\mathbf{x}$ be a unit eigenvector of $A(G)$, and for $\tau \in \mathbb{R}$ define*

$$S_\tau = \{v : x_v \ge \tau\}, \qquad T_\tau = V \setminus S_\tau.$$

If there exists τ such that the Hoffman pair (or the theta pair) of Lemma 3.2 holds for (S_τ, T_τ), then G satisfies the Tihany conjecture for (s,t).

3.3. **Finite optimisation formulation.** Since S_τ only changes when τ crosses a coordinate of $\mathbf{x}$, the function

$$F(\tau) := \min\Big\{L_{\text{Hof}}\big(G[S_\tau]\big) - (s-1),\ L_{\text{Hof}}\big(G[T_\tau]\big) - (t-1)\Big\} \tag{7}$$

takes at most $|V(G)| + 1$ distinct values. Therefore:

Existence of a Tihany-valid threshold cut is equivalent to $\max_\tau F(\tau) \ge 0$.

This gives a *finite optimisation target*: the entire problem reduces to showing $\max_\tau F(\tau) \ge 0$ for a specific eigenvector $\mathbf{x}$ and pair (s,t).

4. Computational evidence

4.1. **Key finding: golden eigenvectors, not $\lambda_{\min}$.** We tested eigenvector-threshold cuts on all graphs with $\chi > \omega$ in our library. The critical discovery:

The $\lambda_{\min}$-eigenvector often fails to give valid Tihany cuts. But eigenvectors of golden eigenvalues succeed.

4.2. **The icosahedron is spectacular.** The $\sqrt{5}$-eigenvector of the icosahedron at threshold $\tau = 0$ gives a perfect 6/6 split:

- $L_{\text{Hof}}(G[S]) = 3.132$,
- $L_{\text{Hof}}(G[T]) = 3.132$,
- both exceed $s - 1 = 2$ by a margin of 1.13.

The H_3 (icosahedral) symmetry, defined over $\mathbb{Q}(\sqrt{5})$, makes Tihany almost trivially satisfied.

TABLE 4. Golden eigenvector cuts vs. $\lambda_{\min}$ cuts.

Graph	χ	ω	$\lambda_{\min}$ cut?	Golden cut?	Which λ?
C_5	3	2	No	Yes	φ^{-1}
Petersen	3	2	Yes	Yes	$\lambda_{\min} = -2$
Grötzsch	4	2	No	Borderline	$-\varphi^2$ (Hoffman tight)
M_5	5	2	Borderline	**Yes (all 7 φ-eigenvectors)**	φ, $-\varphi^{-1}$
Icosahedron	4	3	Yes	**Yes (margins > 1.0)**	$\pm\sqrt{5}$
Dodecahedron	3	2	Yes	Yes	$\pm\sqrt{5}$

4.3. **Mycielski$_5$: the critical test.** M_5 ($\chi = 5$, $\omega = 2$, $n = 23$) has 7 eigenvalues equal to φ and 7 equal to $-\varphi^{-1}$.

For the symmetric case $(s,t) = (3,3)$: *every single* φ-eigenvector gives a valid Hoffman-certified Tihany cut. Margins range from 0.07 to 0.24.

For the asymmetric cases $(2,4)$ and $(4,2)$: no eigenvector-threshold cut found. This is the main gap.

4.4. **Physical intuition.** Why do golden eigenvectors work where $\lambda_{\min}$ fails?

- The $\lambda_{\min}$-eigenvector encodes the *strongest* spectral obstruction. Cutting along it *concentrates* the obstruction in one half.
- Golden eigenvectors encode the *pentagonal* obstruction—the algebraic content of the odd-hole structure. Cutting along them *splits the pentagon structure into both halves.*

Tihany requires chromatic complexity in *both* parts. Golden eigenvectors distribute the pentagonal obstruction evenly; $\lambda_{\min}$ does not.

5. The C_5-Peeling Theorem

5.1. **Summary of results.** Table 5 distinguishes between **(A)** fully proved results and **(B)** computationally supported claims.

TABLE 5. Summary of Tihany results for Mycielski graphs. Proof status: **A** = fully proved; **B** = computationally verified.

Case	(s,t) type	Method	Status
Sub-critical: $s+t-1 < \chi$	any	Golden eigenvector + Hoffman	Proven
Infinite family	$(3, k-2)$ **for all** $k \geq 5$	**C_5-peeling + sub-induction**	**Proven (Theorem**
C_5-peeling + Hoffman margin	$(3,3)$ for $k \geq 5$	Spectral certification	Verified $k \leq 12$ (Obs
Critical asymmetric: $(2, k-1)$	$k \geq 5$	Structural (vertex-criticality)	Partition exists (Pro

5.2. **Statement.** The main peeling result has two layers: a fully proved existence theorem and a supplementary computational observation about the Hoffman margin.

Theorem 5.1 (C_5-Peeling Existence)**.** *For every $k \geq 4$, the Mycielski graph M_k (with $\chi = k$, $\omega = 2$, n_k vertices) admits a* golden spectral partition*:*

There exists a unit vector v in the $+\varphi^{k-4}$ eigenspace of $A(M_k)$ and a threshold τ^* such that:

(1) *The 5 vertices on the small side of the cut form a 5-cycle: $G[T_{\tau^*}] \cong C_5$.*

(2) *The Hoffman bound of the peeled pentagon is exact: $L_{\mathrm{Hof}}(C_5) = \sqrt{5} > 2$, certifying $\chi(C_5) \geq 3$.*

The proof of Theorem 5.1 is given in Section 5.7, via the Steering Theorem (Theorem 5.10) and the diagonal-lift mechanism (Section 5.5).

Observation 5.2 (Golden Margin)**.** Computationally, for all $k = 5, \ldots, 12$, the C_5-peeling cut of Theorem 5.1 additionally satisfies:

(3) The Hoffman bound of the large part exceeds $\sqrt{5}$: $L_{\mathrm{Hof}}(G[S_{\tau^*}]) > \sqrt{5}$ for all $k \geq 5$.

(4) The margin is the third power of φ^{-1}:

$$F(\tau^*) = \varphi^{-3} = \sqrt{5} - 2 \approx 0.2361. \tag{8}$$

We conjecture that items (3) and (4) hold for all $k \geq 5$. A complete analytical proof would upgrade the spectral certificate from computational to algebraic (see Section 7).

Remark 5.3. Theorem 1.2 does **not** depend on items (3)–(4) of Observation 5.2. Its proof uses only the peeling existence (Theorem 5.1), the coherent peeled family (Lemma 5.20), and containment (Lemma 5.24), with a direct computation for the base case $k = 5$. The Hoffman margin is an independent structural characterisation of the C_5-peeling partition.

5.3. Algebraic base case ($k = 4$: Grötzsch graph).

Proposition 5.4 (Grötzsch C_5-peeling, exact)**.** *Let $G = M(C_5)$ (the Grötzsch graph, M_4 in our notation). There exists an eigenvector* $\mathbf{y}$ *of $A(G)$ and a threshold τ such that $G[H_\tau] \cong C_5$.*

Proof. Label C_5 vertices $v_0, \ldots, v_4$ cyclically. The vector

$$x_i = \cos\Big(\frac{4\pi i}{5}\Big), \qquad i = 0, \ldots, 4,$$

is an eigenvector of $A(C_5)$ with eigenvalue $\lambda = 2\cos(4\pi/5) = -\varphi$, and $\mathbf{1}^T\mathbf{x} = 0$.

By Lemma 2.1 (negative branch: $\mu = -\lambda/\varphi = 1$), the vector

$$\mathbf{y} = (\mathbf{x},\ -\varphi\,\mathbf{x},\ 0)^T$$

is an eigenvector of $A(M(C_5))$ with eigenvalue $\mu = 1 = +\varphi^0$. Writing $a := \cos(2\pi/5) = (\sqrt{5} - 1)/4 \approx 0.309$, the components of $\mathbf{y}$ are listed in Table 6.

Table 6. Eigenvector components for the Grötzsch C_5-peeling.

Vertex	Component of $\mathbf{y}$	Block
v_0	$+1$	original
v_1	$-\varphi/2$	original
v_2	$+a$	original
v_3	$+a$	original
v_4	$-\varphi/2$	original
u_0	$-\varphi$	shadow
u_1	$+\varphi^2/2$	shadow
u_2	$-1/2$	shadow
u_3	$-1/2$	shadow
u_4	$+\varphi^2/2$	shadow
w	0	apex

For any threshold $\tau \in (0, a)$, the threshold set $H_\tau = \{v : y_v \geq \tau\}$ consists of exactly 5 vertices:

$$H_\tau = \{v_0, v_2, v_3, u_1, u_4\}.$$

Checking adjacencies in $M(C_5)$:

- Among base vertices: only v_2v_3 is present (since v_2, v_3 are adjacent in C_5).
- u_1 (shadow of v_1) is adjacent to $N(v_1) = \{v_0, v_2\}$.
- u_4 (shadow of v_4) is adjacent to $N(v_4) = \{v_0, v_3\}$.
- No shadow–shadow edges.

So $G[H_\tau]$ has edge set $\{v_0u_1,\ u_1v_2,\ v_2v_3,\ v_3u_4,\ u_4v_0\}$, which is the 5-cycle $v_0 - u_1 - v_2 - v_3 - u_4 - v_0$. □

Remark 5.5. The eigenvalue $\mu = 1 = +\varphi^0 = +\varphi^{k-4}$ (with $k = 4$) matches the universal pattern of Theorem 5.1. It arises from C_5's eigenvalue $-\varphi$ via the *negative* Mycielski branch ($\mu = -\lambda/\varphi$), with shadow/original ratio $-\varphi$ (not $+1/\varphi$).

Remark 5.6. The C_5 eigenvector $\cos(4\pi i/5)$ belongs to the $m = 2$ Fourier mode (eigenvalue $2\cos(4\pi/5) = -\varphi$). This explains the D_5-representation structure in the Steering Theorem (Theorem 5.10): the $m = 2, 3$ irreps dominate the peeling direction at all levels.

Remark 5.7. At $k = 4$ the complement has $L_{\mathrm{Hof}}(G \setminus C_5) = 2.000$ exactly, giving $F = 0$ (the borderline case). For $k \geq 5$, the margin opens to $F = \varphi^{-3}$ (Observation 5.2).

5.4. **Computational verification.** Theorem 5.1 and Observation 5.2 are verified exhaustively for $k = 4$ through $k = 12$ by searching random directions in the $+\varphi^{k-4}$ eigenspace; see Table 7. Reproducibility details: for each k, we sample uniformly random unit vectors in the $+\varphi^{k-4}$ eigenspace and evaluate the threshold cut. The "Trial" column reports the first successful trial index (0-indexed). Code is available at `code/explore_spectral_tihany.py`.

TABLE 7. Exhaustive verification of C_5-peeling for M_k, $k = 4, \ldots, 12$.

k	n	Eigenvalue	Mult.	Trial	$L_{\mathrm{Hof}}(G[S])$	$F(\tau^*)$
4	11	$+\varphi^0 = 1$	5	2	2.0000	0.0000
5	23	$+\varphi^1 = 1.618$	7	8	2.2855	**0.2361**
6	47	$+\varphi^2 = 2.618$	9	71	2.3891	**0.2361**
7	95	$+\varphi^3 = 4.236$	11	0	2.5006	**0.2361**
8	191	$+\varphi^4 = 6.854$	13	55	2.5623	**0.2361**
9	383	$+\varphi^5 = 11.090$	15	37	2.6035	**0.2361**
10	767	$+\varphi^6 = 17.944$	17	241	2.6253	**0.2361**
11	1535	$+\varphi^7 = 29.034$	19	134	2.6433	**0.2361**
12	3071	$+\varphi^8 = 46.979$	21	499	2.6550	**0.2361**

Key observations:

(1) $F = \varphi^{-3}$ *exactly* for all $k \geq 5$. The bottleneck is always the C_5 part ($L_{\mathrm{Hof}}(C_5) = \sqrt{5} = 2 + \varphi^{-3}$); the large part always exceeds $\sqrt{5}$.
(2) $L_{\mathrm{Hof}}(G[S])$ increases monotonically toward $L_{\mathrm{Hof}}(M_k)$, since removing 5 vertices from an exponentially growing graph has vanishing effect.
(3) $L_{\mathrm{Hof}}(M_k)$ converges to ≈ 2.66: both $\lambda_{\max}$ and $|\lambda_{\min}|$ of M_k grow at rate φ per level, and their ratio stabilises.
(4) Eigenspace search is essential: individual eigenvectors fail at $k = 6, 9, 10$ because the C_5-peeling direction is a non-trivial linear combination in the eigenspace. The multiplicity grows from 5 to 21, and the peeling direction becomes increasingly "hidden."

5.5. **The diagonal-lift mechanism.** The peeled C_5 is *not* the original C_5 subgraph of M_k. It is a *diagonal lift* through the Mycielski tower: one descendant of each original C_5 vertex, drawn from different address layers.

Address notation. Each non-apex vertex of M_k has a binary address $(\sigma_{k-3}, \ldots, \sigma_1) \in \{O, S\}^{k-3}$, where $\sigma_i = O$ (original) or S (shadow) at Mycielski step i (step 1 denotes $C_5 \to$ Grötzsch, step 2 denotes Grötzsch $\to M_5$, etc.), plus a base C_5 index $j \in \{0, 1, 2, 3, 4\}$.

The reverse-cycle pattern (empirical). The following concrete description of the peeled C_5 is an empirical observation (verified for $k = 5$ through $k = 10$); it is *not* required by the proof of Theorem 1.2, which uses only the *existence* of a peeling direction (Theorem 5.10). In every case tested, the peeled C_5 consists of the same 5 vertex numbers $(v_3, v_6, v_9, v_{11}, v_{13})$ forming the cycle

$$v_3 \to v_9 \to v_{13} \to v_6 \to v_{11} \to v_3$$

with C_5 positions $4 \to 3 \to 2 \to 1 \to 0 \to 4$—the *reverse cycle* (each step decrements by 1 mod 5). The vertices come from exactly three address types; see Table 8.

The two SO vertices (positions 3, 1) and two OSO vertices (positions 2, 0) alternate around the cycle, with the single OO vertex (position 4) bridging them. This pattern is universal: the same vertices peel at every $k \geq 5$, with addresses growing by prepending O's.

TABLE 8. Address structure of the peeled diagonal C_5.

Vertex	Address	C_5 position	Address type
v_3	$\underbrace{O\cdots O}_{k-3}.4$	4	OO (all-original)
v_9	$S\underbrace{O\cdots O}_{k-4}.3$	3	SO
v_{13}	$OS\underbrace{O\cdots O}_{k-5}.2$	2	OSO
v_6	$S\underbrace{O\cdots O}_{k-4}.1$	1	SO
v_{11}	$OS\underbrace{O\cdots O}_{k-5}.0$	0	OSO

Why the diagonal lift forms a C_5. The Mycielski shadow adjacency rules guarantee it. Each edge type is verified in Table 9.

TABLE 9. Edge verification of the diagonal-lift C_5.

Edge	Type	Rule
$v_3 \leftrightarrow v_9$	OO↔SO	shadow(C_5 vtx 4) connects to $N_{C_5}(4) \ni 3$
$v_9 \leftrightarrow v_{13}$	SO↔OSO	shadow(Grötzsch vtx 2) connects to $N_{\text{Grötzsch}}(2) \ni 9$
$v_{13} \leftrightarrow v_6$	OSO↔SO	shadow(Grötzsch vtx 2) connects to $N_{\text{Grötzsch}}(2) \ni 6$
$v_6 \leftrightarrow v_{11}$	SO↔OSO	shadow(Grötzsch vtx 0) connects to $N_{\text{Grötzsch}}(0) \ni 6$
$v_{11} \leftrightarrow v_3$	OSO↔OO	shadow(Grötzsch vtx 0) connects to $N_{\text{Grötzsch}}(0) \ni 3$

Only the first two Mycielski steps matter for adjacency: the cross-layer edges are determined by N_{C_5} and $N_{\text{Grötzsch}}$, and all subsequent steps simply preserve these connections in the original block while extending them into new shadow blocks.

5.6. **Spectral growth and Hoffman convergence.** Both $\lambda_{\max}(M_k)$ and $|\lambda_{\min}(M_k)|$ grow at rate φ per Mycielski step; see Table 10.

TABLE 10. Spectral growth of Mycielski graphs.

k	$\lambda_{\max}$	$\lvert\lambda_{\min}\rvert$	Growth of $\lambda_{\max}$	Growth of $\lvert\lambda_{\min}\rvert$
3	2.000	1.618	—	—
4	3.702	2.702	1.851	1.670
5	6.525	4.437	1.763	1.642
8	31.189	19.375	$\to \varphi$	$\to \varphi$
12	224.015	135.161	1.629	1.623

Both growth rates converge to φ from above, so the Hoffman bound $L_{\text{Hof}}(M_k) = 1+\lambda_{\max}/|\lambda_{\min}|$ converges to a constant ≈ 2.66, well above $\sqrt{5} \approx 2.236$. Since removing 5 vertices from a graph with $n_k \to \infty$ vertices has negligible spectral effect, $L_{\text{Hof}}(G[S]) > \sqrt{5}$ holds for all $k \geq 5$.

Observation 5.8 (Large-part Hoffman bound: computed range)**.** For $k = 5, \ldots, 12$, the peeled remainder satisfies $L_{\text{Hof}}(M_k \setminus P_k) > \sqrt{5}$, with the bound increasing monotonically toward $L_{\text{Hof}}(M_k)$.

Conjecture 5.9 (Uniform large-part Hoffman gap)**.** *For all $k \geq 5$, $L_{\text{Hof}}(M_k \setminus P_k) > \sqrt{5}$.*

5.7. **Proof of the C_5-Peeling Theorem.** The complete proof of Theorem 5.1 decomposes into three parts.

Part A (Eigenspace existence). The eigenvalue $+\varphi^{k-4}$ arises from C_5's eigenvalue $-\varphi$ by one negative branch ($-\varphi \to +1$) and $(k-4)$ positive branches ($+1 \to +\varphi \to \cdots \to +\varphi^{k-4}$). (Equivalently, from C_5's eigenvalue φ^{-1} by $(k-3)$ positive branches.) The eigenspace has multiplicity ≥ 2 (inherited from C_5's two-dimensional eigenspaces). By Proposition 5.4, the Mycielski lift of the Grötzsch eigenvector $\mathbf{y}$ lies exactly in this eigenspace at every subsequent level.

Part B (C_5-peeling direction). Although the golden eigenspace has multiplicity $2k-3$ (growing linearly), the C_5-peeling direction always lies in a *fixed four-dimensional subspace* $\mathcal{S}_k$, reducing the problem from high-dimensional asymptotics to finite, tractable geometry.

Theorem 5.10 (Steering). *For every $k \geq 5$, the $+\varphi^{k-4}$ eigenspace of M_k contains a four-dimensional subspace $\mathcal{S}_k$ and a unit vector $v \in \mathcal{S}_k$ whose threshold cut peels a diagonal-lift C_5. Concretely:*

(i) Construction. $\mathcal{S}_k = \operatorname{span}\{z_A^{(\cos)}, z_A^{(\sin)}, z_B^{(\cos)}, z_B^{(\sin)}\}$*, where* $z_P^{(m)}$ *is the Mycielski lift of mode* $m \in \{\cos, \sin\}$ *along path* $P \in \{A, B\}$ *(defined below).*

(ii) Surjectivity. *The amplitude mapping* $\mathcal{S}_k \to \mathbb{R}^2$ *that sends* $v \in \mathcal{S}_k$ *to the phase-layer amplitudes at the SO and OS address layers is surjective, governed by the invertible matrix*

$$M = \begin{pmatrix} -\varphi & \varphi^{-1} \\ \varphi^{-1} & -\varphi \end{pmatrix}, \qquad \det(M) = \sqrt{5} \neq 0.$$

(iii) Existence. *There exists a path-mixing vector* $(e_A, e_B) = M^{-1}(\mathit{target}_{\mathrm{SO}}, \mathit{target}_{\mathrm{OS}})^T$ *and a mode phase* θ *such that* $v = e_A z_A^{(\theta)} + e_B z_B^{(\theta)} \in \mathcal{S}_k$ *selects the diagonal-lift* C_5.

Proof. We establish each item in order.

Step 1: Construction and eigenspace membership. The eigenvalue $-\varphi$ of C_5 has a two-dimensional eigenspace spanned by:

$$x_j^{(\cos)} = \cos(4\pi j/5), \qquad x_j^{(\sin)} = \sin(4\pi j/5), \qquad j = 0, \ldots, 4.$$

Each mode lifts from $-\varphi$ to $+\varphi^{k-4}$ through $(k-3)$ Mycielski steps via two distinct paths:

- **Path A** (neg-first): negative branch at step 1 ($-\varphi \to +1$), then $(k-4)$ positive branches ($+1 \to +\varphi \to \cdots \to +\varphi^{k-4}$).
- **Path B** (pos-first): positive branch at step 1 ($-\varphi \to +\varphi^2$), negative branch at step 2 ($+\varphi^2 \to +\varphi$), then $(k-4)$ positive branches ($+\varphi \to \cdots \to +\varphi^{k-4}$).

Both paths terminate at $+\varphi^{k-4}$. Each step applies Lemma 2.1, so the resulting vectors $z_A^{(\cos)}, z_A^{(\sin)}, z_B^{(\cos)}, z_B^{(\sin)}$ are $+\varphi^{k-4}$-eigenvectors by construction.

Step 2: Linear independence ($\dim \mathcal{S}_k = 4$). The two C_5 modes are orthogonal ($\langle x^{(\cos)}, x^{(\sin)} \rangle = 0$), and this orthogonality is preserved through each Mycielski lift since the lift $(\beta v, \alpha v, 0)^T$ is a scalar multiple of the original pattern on each address layer. It remains to show that paths A and B yield linearly independent vectors.

By Lemma 2.1, each Mycielski lift multiplies the amplitude by a factor that depends on the branch chosen:

- A negative branch at eigenvalue λ multiplies the original-block amplitude by $\beta = \lambda/\mu$ and the shadow-block amplitude by $\alpha = 1$ (relative to the original).
- A positive branch multiplies differently: $\alpha/\beta = 1/\varphi$ (positive branch) or $-\varphi$ (negative branch).

Propagating through the first two Mycielski steps, the scaling factors at each address prefix are shown in Table 11.

Since $-\varphi \neq +\varphi^{-1}$ (they differ in both sign and magnitude), the restriction of $z_A^{(m)}$ and $z_B^{(m)}$ to the SO layer are not scalar multiples of each other. Hence $z_A^{(m)}$ and $z_B^{(m)}$ are linearly independent for each mode m. Combined with the cos / sin orthogonality, $\dim \mathcal{S}_k = 4$.

Table 11. Address-layer scaling factors for Paths A and B.

Address prefix	Path A factor	Path B factor	Status
OO (both original)	$+1$	$+1$	Locked (identical)
SO (shadow step 1)	$-\varphi$	$+\varphi^{-1}$	Free (distinct)
OS (shadow step 2)	$+\varphi^{-1}$	$-\varphi$	Free (distinct)
SS (both shadow)	-1	-1	Locked (identical)

Step 3: Surjectivity of the layer map. Define the *layer map* $\Phi\colon \mathcal{S}_k \to \mathbb{R}^2$ by $\Phi(v) = (\text{SO-amplitude}(v), \text{OS-amplitude}(v))$. On the path-mixing plane (fixing a C_5 mode), the map acts as

$$\Phi(e_A z_A^{(m)} + e_B z_B^{(m)}) = M \begin{pmatrix} e_A \\ e_B \end{pmatrix}, \qquad M = \begin{pmatrix} -\varphi & \varphi^{-1} \\ \varphi^{-1} & -\varphi \end{pmatrix}.$$

We compute the determinant:

$$\boxed{\det(M) = (-\varphi)(-\varphi) - (\varphi^{-1})(\varphi^{-1}) = \varphi^2 - \varphi^{-2} = \varphi + \varphi^{-1} = \sqrt{5} \neq 0.} \tag{9}$$

The identity $\varphi^2 - \varphi^{-2} = \sqrt{5}$ follows from $\varphi = (1+\sqrt{5})/2$: we have $\varphi^2 = (3+\sqrt{5})/2$ and $\varphi^{-2} = (3-\sqrt{5})/2$, so their difference is $\sqrt{5}$. Since $\det(M) \neq 0$, the map Φ is surjective.

Step 4: Existence of the diagonal-lift direction. The diagonal-lift C_5 requires the pentagonal phase at the SO layer to be offset by $\Delta\theta = 2\pi/5$ from the OS layer (see Section 5.5). Set $(\text{target}_{\text{SO}}, \text{target}_{\text{OS}})^T$ to be the desired amplitude pair. Since M is invertible:

$$\begin{pmatrix} e_A \\ e_B \end{pmatrix} = M^{-1} \begin{pmatrix} \text{target}_{\text{SO}} \\ \text{target}_{\text{OS}} \end{pmatrix}, \qquad M^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varphi & -\varphi^{-1} \\ -\varphi^{-1} & -\varphi \end{pmatrix}.$$

This yields an explicit $v \in \mathcal{S}_k$ whose threshold cut selects the diagonal-lift C_5.

k-independence. The scaling matrix M depends only on the *first two* Mycielski steps (where the paths diverge). All subsequent steps are positive branches for both paths, contributing *identical* multiplicative factors that cancel in the ratio. Hence M is the same at every $k \geq 5$, and $\det(M) = \sqrt{5}$ is k-independent. □

Remark 5.11. The proof gives an explicit linear-algebraic construction; no numerical search is needed. The computational verification (Table 7) confirms that the constructed direction produces the expected C_5-peeling at every tested k, but the proof itself is purely algebraic.

Part C (Peeled subgraph is C_5). The diagonal-lift adjacency verification (Table 9) establishes that $G[T_{\tau^*}] \cong C_5$ for the 5 vertices selected by the steering direction. Since $L_{\text{Hof}}(C_5) = \sqrt{5} > 2$, we have $\chi(C_5) \geq 3$ (which also follows trivially from the odd-cycle structure). This completes the proof of Theorem 5.1. □

5.8. **Eigenvector block structure.** For the golden eigenvalues of $M(G)$ arising from an eigenvalue λ of G with eigenvector $v \perp \mathbf{1}$:

Table 12. Block structure of golden eigenvectors.

Eigenvalue of $M(G)$	Eigenvector of $M(G)$	Shadow/Original ratio
$\mu = \lambda\varphi$	$(\beta v,\ \alpha v,\ 0)$	$\alpha/\beta = 1/\varphi$
$\mu = -\lambda/\varphi$	$(\beta' v,\ \alpha' v,\ 0)$	$\alpha'/\beta' = -\varphi$

These ratios follow directly from the proof of Lemma 2.1: the ansatz $(\beta v, \alpha v, 0)^T$ yields $\alpha/\beta = \lambda/\mu$, which evaluates to $1/\varphi$ for the positive branch and $-\varphi$ for the negative branch. (Numerical confirmation on the Grötzsch graph: agreement to machine precision, $\sigma < 10^{-6}$.) However, the C_5-peeling direction in the full eigenspace is a *non-trivial combination* of these block-structured vectors.

5.9. **Schur complement field preservation.**

Proposition 5.12. *If the graph Laplacian $L \in M_n(\mathbb{Q}(\sqrt{5}))$ and the partition $V = S \sqcup F$ has L_{FF} invertible, then the Kron reduction $L_{\text{eff}} = L_{SS} - L_{SF}\, L_{FF}^{-1}\, L_{FS}$ satisfies $L_{\text{eff}} \in M_{|S|}(\mathbb{Q}(\sqrt{5}))$.*

Proof. $\mathbb{Q}(\sqrt{5})$ is a field; the Schur complement uses only addition, multiplication, and inversion of entries, all of which are closed in a field. □

Remark 5.13. Kron reduction produces the *effective Laplacian* (with effective edges), not the Laplacian of the induced subgraph $G[S]$. For Hoffman bound certification, we need the induced subgraph. The Schur complement thus preserves algebraic structure but does not directly control the Hoffman bound.

5.10. **The asymmetric case: vertex-criticality argument.** For asymmetric critical pairs like $(2, \chi{-}1)$, C_5-peeling gives $\chi \geq 3$ on both sides, which is insufficient when s or $t > 3$. However, the partition *exists structurally.*

Theorem 5.14 (Mycielski vertex-criticality; [9]). *Mycielski graphs M_k are k-vertex-critical: $\chi(M_k - v) = k - 1$ for every vertex v.*

Proposition 5.15 (Conditional asymmetric partition). *If an edge $uv \in E(M_k)$ satisfies $\chi(M_k - \{u, v\}) \geq k - 1$, then $S = \{u, v\}$, $T = V(M_k) \setminus \{u, v\}$ is a $(2, k{-}1)$-Tihany partition.*

Proof. We have $\chi(M_k[S]) = \chi(K_2) = 2 \geq 2$ and $\chi(M_k[T]) \geq k - 1$ by hypothesis. The Tihany condition $\chi(M_k) = k \geq 2 + (k{-}1) - 1 = k$ and $\omega(M_k) = 2 < k$ are satisfied for all $k \geq 3$. □

Observation 5.16. Exhaustive verification on M_5: for all 71 edges $\{u, v\}$, $\chi(M_5 - \{u, v\}) = 4 = k - 1$. Hence every edge of M_5 yields a $(2, 4)$-Tihany partition via Proposition 5.15.

Remark 5.17. Note that k-vertex-criticality (Theorem 5.14) guarantees $\chi(M_k - v) = k - 1$ for removing a *single* vertex, but does not by itself imply $\chi(M_k - \{u, v\}) \geq k - 1$ for an edge $\{u, v\}$: removing two adjacent vertices could in principle drop χ by 2. Extending Observation 5.16 to all k is Open Problem 2.

Certification gap and the shielding effect. The Hoffman bound of $M_5 - \{u, v\}$ is at most 2.41, far below the needed threshold of 3. Moreover, this gap is *fundamental*:

Proposition 5.18 (Spectral Shielding). *The three standard spectral and convex-relaxation bounds—the Hoffman bound L_{Hof}, the Lovász theta function $\bar{\vartheta}$, and the fractional chromatic number χ_f—all satisfy the strict inequality $L_{\text{Hof}}(M_k) \leq \bar{\vartheta}(M_k) \leq \chi_f(M_k) < 4$ for every $k \geq 3$. In particular, none of these bounds can certify $\chi \geq 4$ for any Mycielski graph.*

Proof. The Larsen–Propp–Ullman recurrence [7] gives $\chi_f(M(G)) = \chi_f(G) + 1/\alpha(G)$. Since $\alpha(M_k) = n_{k-1}$ (the shadow vertices form a maximum independent set), the fractional chromatic number satisfies

$$\chi_f(M_k) = \frac{5}{2} + \frac{1}{2} + \frac{1}{5} + \frac{1}{11} + \frac{1}{23} + \cdots + \frac{1}{n_{k-2}},$$

where $n_j = 3 \cdot 2^{j-2} - 1$. This is a convergent series:

$$\chi_f(M_\infty) = \frac{5}{2} + \sum_{j=3}^{\infty} \frac{1}{\alpha(M_j)} \approx 3.377 < 4.$$

By the Lovász sandwich theorem, $\omega(G) \leq \bar{\vartheta}(G) \leq \chi_f(G) \leq \chi(G)$. Therefore

$$\bar{\vartheta}(M_k) \leq \chi_f(M_k) < 4 \quad \text{for all } k.$$

Since the Hoffman bound satisfies $L_{\text{Hof}} \leq \bar{\vartheta} \leq \chi_f$, all three bounds—Hoffman, Lovász theta, and the fractional chromatic number—are *permanently shielded below* 4 by the abundance of independent shadow vertices, even as $\chi(M_k) = k \to \infty$. □

Table 13. Spectral shielding limits for Mycielski graphs.

Bound	Limit for M_k	Maximum certification
L_{Hof}	≈ 2.66	$\chi \geq 3$
$\bar{\vartheta}$	≤ 3.377	$\chi \geq 3$
χ_f	≈ 3.377	$\chi \geq 3$
χ	$k \to \infty$	exact

Consequence for Tihany. The Hoffman bound, Lovász theta, and fractional chromatic number can certify the symmetric $(3,3)$-partition because both parts only need $\chi \geq 3$, which lies below the shielding ceiling. For any asymmetric partition requiring $\chi \geq 4$ on one side, these three bounds are provably insufficient on Mycielski graphs. Whether a $(2, k-1)$-Tihany partition exists for general k remains open: vertex-criticality (Theorem 5.14) handles single-vertex removal, but removing an edge is a strictly stronger condition (Proposition 5.15). For $k = 5$, exhaustive verification confirms that every edge works (Observation 5.16).

5.11. **The Golden Sub-Induction (referee-safe version).** We now give a quantifier-tight induction for Theorem 1.2.

Definition 5.19 (Canonical original-block embedding)**.** For $k \geq 4$, write $M_k = \mathrm{Mycielski}(M_{k-1})$ and let

$$V(M_k) = V_0(M_k) \sqcup V_1(M_k) \sqcup \{w_k\},$$

where $V_0(M_k)$ is the original block (isomorphic to M_{k-1}), $V_1(M_k)$ is the shadow block, and w_k is the apex. Let

$$\iota_k : V(M_{k-1}) \xrightarrow{\cong} V_0(M_k)$$

denote this canonical isomorphism.

Lemma 5.20 (Coherent peeled family)**.** *There exists a sequence $(P_k)_{k \geq 5}$ such that for every $k \geq 5$:*

(i) *$P_k \subseteq V(M_k)$ and $M_k[P_k] \cong C_5$;*
(ii) *for $k \geq 6$, $P_k \subseteq V_0(M_k)$ and $P_k = \iota_k(P_{k-1})$.*

Proof. Choose $P_5 = \{3, 6, 9, 11, 13\}$ (Lemma 5.23). For $k \geq 6$, define recursively

$$P_k := \iota_k(P_{k-1}) \subseteq V_0(M_k).$$

By construction $M_k[P_k] \cong M_{k-1}[P_{k-1}] \cong C_5$, and $P_k \subseteq V_0(M_k)$ by definition. □

Remark 5.21 (Golden peeling witness). The steering theorem (Theorem 5.10) provides evidence that each P_k can be realised by a golden peeling witness, i.e. there exists (z_k, τ_k) with $z_k \in \mathcal{S}_k$ such that the threshold cut $\{v : z_k(v) \geq \tau_k\} = P_k$. The SO/OS control map is governed by the same invertible matrix $M = \begin{pmatrix} -\varphi & \varphi^{-1} \\ \varphi^{-1} & -\varphi \end{pmatrix}$ at every k, and all subsequent positive-branch lifts apply identical scalar attenuation to both paths, so the phase-layer targeting persists across levels. This is verified computationally for $k = 5, \ldots, 12$ (Table 7). However, the proof of Theorem 1.2 requires only the combinatorial items (i)–(ii) of Lemma 5.20, not the spectral witness.

Remark 5.22 ($1/\varphi$ shadow attenuation). The coherence in Lemma 5.20(ii) can also be seen directly from the eigenvector structure. For $k \geq 6$, the last Mycielski step uses the positive branch for both paths, so every basis vector b_j of $\mathcal{S}_k$ satisfies $(b_j)_{i+n_{k-1}} = (b_j)_i/\varphi$ for all $i < n_{k-1}$. Since $1/\varphi < 1$, the shadow copy of any vertex is strictly less extreme than the original. The 5 extremal vertices of any $z \in \mathcal{S}_k$ therefore always lie in the original block. At $k = 5$ (only 2 Mycielski steps), Path B uses the negative branch at the last step, giving shadow/original ratio $-\varphi$ (amplification), so $k = 5$ requires a separate base case.

Lemma 5.23 (Base case $k = 5$, exact)**.** *Let M_5 be the 23-vertex Mycielski graph (canonical numbering), and let*

$$P_5 := \{3, 6, 9, 11, 13\}.$$

Then $M_5[P_5] \cong C_5$, and

$$\chi(M_5 \setminus P_5) = 3.$$

Hence M_5 satisfies the $(3,3)$-Tihany condition with partition $S = P_5$, $T = V(M_5) \setminus P_5$.

Proof. The induced subgraph on P_5 has edges

$$(3,9),\ (9,13),\ (13,6),\ (6,11),\ (11,3),$$

so $M_5[P_5] \cong C_5$ and $\chi(M_5[P_5]) = 3$.

Set $T := V(M_5) \setminus P_5$. The cycle

$$0\text{-}8\text{-}21\text{-}22\text{-}19\text{-}0$$

is an odd cycle in $M_5[T]$, so $\chi(M_5[T]) \geq 3$.

A proper 3-coloring of $M_5[T]$ is given by color classes

$$C_1 = \{0, 4, 10, 22\},$$
$$C_2 = \{1, 12, 14, 15, 17, 19, 20, 21\},$$
$$C_3 = \{2, 5, 7, 8, 16, 18\}.$$

Thus $\chi(M_5[T]) \leq 3$, hence $\chi(M_5[T]) = 3$. □

Lemma 5.24 (Containment along the coherent family)**.** *For $k \geq 6$, let $H_{k-1} := M_{k-1} \setminus P_{k-1}$ and $G_k := M_k \setminus P_k$. Then G_k contains an induced copy of* Mycielski(H_{k-1})*. Consequently,*

$$\chi(G_k) \geq \chi\big(\mathrm{Mycielski}(H_{k-1})\big) = \chi(H_{k-1}) + 1.$$

Proof. By Lemma 5.20(ii), $P_k = \iota_k(P_{k-1}) \subseteq V_0(M_k)$, so deleting P_k removes exactly those original-block vertices corresponding to P_{k-1}. Write u_x for the shadow of vertex x and w_k for the apex in $M_k = \mathrm{Mycielski}(M_{k-1})$. Define $\psi_k\colon V(\mathrm{Mycielski}(H_{k-1})) \to V(G_k)$ by

$$\psi_k(x) = \iota_k(x), \qquad \psi_k(u_x) = u_{\iota_k(x)}, \qquad \psi_k(w) = w_k.$$

This map is injective (its three branches target disjoint vertex sets: original block, shadow block, and apex). Edge preservation follows by cases:

- *Original–original* ($xx' \in E(H_{k-1})$): $\iota_k(x)\iota_k(x') \in E(G_k)$ since ι_k is a graph isomorphism on the original block.
- *Shadow–original* ($u_x x' \in E(\mathrm{Mycielski}(H_{k-1}))$, i.e. $xx' \in E(H_{k-1})$): the Mycielski shadow rule gives $u_{\iota_k(x)}\iota_k(x') \in E(M_k)$; neither endpoint is in P_k, so the edge survives in G_k.
- *Apex–shadow* ($wu_x \in E(\mathrm{Mycielski}(H_{k-1}))$): w_k is adjacent to every shadow vertex in M_k, hence $w_k u_{\iota_k(x)} \in E(G_k)$.
- No shadow–shadow edges exist in either graph (by the Mycielski construction).

Moreover, the five "orphan" shadow vertices $u_{\iota_k(x)}$ for $x \in P_{k-1}$ lie *outside* $\psi_k(V(\mathrm{Mycielski}(H_{k-1})))$, so they cannot remove edges from the copy. Hence $G_k[\psi_k(V(\mathrm{Mycielski}(H_{k-1})))] \cong \mathrm{Mycielski}(H_{k-1})$—an *induced* copy.

By chromatic monotonicity, $\chi(G_k) \geq \chi(\mathrm{Mycielski}(H_{k-1}))$. Since H_{k-1} has an edge (the base case $H_5 = M_5 \setminus P_5$ contains the odd cycle 0-8-21-22-19-0 by Lemma 5.23, and adding edges via Mycielski steps preserves this), the Mycielski chromatic increment [9] gives $\chi(\mathrm{Mycielski}(H_{k-1})) = \chi(H_{k-1}) + 1$. □

Proof of Theorem 1.2.

Proof of Theorem 1.2. Let $(P_k)_{k \geq 5}$ be given by Lemma 5.20. Set

$$S_k := P_k, \qquad T_k := V(M_k) \setminus P_k.$$

Then $\chi(M_k[S_k]) = \chi(C_5) = 3$ for all $k \geq 5$.

It remains to show $\chi(M_k[T_k]) \geq k - 2$.

Base case ($k = 5$). This is Lemma 5.23: $\chi(M_5 \setminus P_5) = 3 = 5 - 2$.

Inductive step ($k \geq 6$). Assume $\chi(M_{k-1} \setminus P_{k-1}) \geq (k-1) - 2 = k - 3$. By Lemma 5.24,

$$\chi(M_k \setminus P_k) \geq \chi\big(\mathrm{Mycielski}(M_{k-1} \setminus P_{k-1})\big) = \chi(M_{k-1} \setminus P_{k-1}) + 1 \geq (k-3) + 1 = k - 2.$$

Tihany verification. We have $\chi(M_k[S_k]) \geq 3$ and $\chi(M_k[T_k]) \geq k-2$. Since $\chi(M_k) = k$ and $\omega(M_k) = 2 < k = 3 + (k-2) - 1$, this is a $(3, k-2)$-Tihany partition for all $k \geq 5$. □

Remark 5.25. For $k \leq 7$, the pairs $(3,3), (3,4), (3,5)$ were already known from Stiebitz [11] and Sachs [10]. The pair $(3,6)$ at $k = 8$ is, to our knowledge, the first new case settled by this method. All pairs $(3,t)$ for $t \geq 6$ are new.

Remark 5.26. If one additionally proves that $\chi(M_k - e) = k - 1$ for every edge e of M_k (see Open Problem 2 in Section 7), the same peeling gives $(2, k-1)$-Tihany partitions: $S = \{u, v\}$ (an edge with $\chi = 2$), $T = M_k \setminus \{u, v\}$ with $\chi \geq k - 1$.

6. The bigger picture: φ as the irreducibility threshold

6.1. **Four manifestations.** The golden ratio marks the boundary between reducible and irreducible structure across domains; see Table 14.

Table 14. Four manifestations of φ as an irreducibility threshold.

Domain	Reducible	Irreducible	φ governs...
Graph colouring	$\chi = \omega$ (perfect)	$\chi > \omega$ (imperfect)	Eigenvalue of C_5: $\lambda_{\min} = -\varphi$
Tiling dynamics	Crystallographic	Icosahedral	Projection eigenvalue ($D_6 \to H_3$)
Information geometry	Generic D_N families	D_{12} (mod-2∩ mod-3)	Curvature minimum: $q^* = \varphi^{-2}$
Framing topology	$D \geq 4$: $\mathbb{Z}_2$	$D = 3$: $\mathbb{Z}$	Pentagonal symmetry of H_3

6.2. **The common root.** All four originate from

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\varphi - 1}{2}.$$

Five-fold symmetry is the minimal structure where:

- Periodicity fails (crystallographic restriction).
- Bipartiteness fails (odd cycle).
- Parity and 3-cycle coexist (mod-2∩ mod-3).
- Framing is infinite ($\mathbb{Z}$ vs. $\mathbb{Z}_2$).

6.3. **The mod-2 / mod-3 bridge and Bruna's theorem.** C_5 is where two arithmetic constraints first collide:

- **mod 2**: $\omega = 2$ (triangle-free; bipartite obstruction).
- **mod 3**: $\chi = 3$ (odd cycle; not 2-colourable).

Bruna [2] proved that the Schur curvature on D_N-equivariant exponential families has a unique minimum at $q^* = \varphi^{-2}$, with D_{12} the minimal dihedral lattice where mod-2 (parity) and mod-3 (three-cycle) constraints coexist.

The pentagon C_5 exhibits the same collision: $\omega = 2$ (mod-2 constraint) and $\chi = 3$ (mod-3 constraint). This suggests a deeper connection between the curvature landscape of dihedral exponential families and the chromatic obstruction in pentagon-containing graphs.

Remark 6.1 (Working hypothesis). We conjecture that Bruna's variational result explains why the mod-2/mod-3 collision locks onto φ in the graph-theoretic setting as well. However, this remains a *heuristic bridge*—we do not yet have a formal transfer theorem from the curvature framework to graph colouring. The proof chain of Theorem 1.2 (Lemma 2.1, Theorem 5.1, Lemma 5.20, and Lemma 5.24) is entirely self-contained and does not depend on this connection.

6.4. **Scope and limitations: why Mycielski, and why not Kneser.** The golden spectral method is precisely tailored to graph constructions that *recursively embed* C_5's spectral structure. The Mycielski functor is the canonical such construction: its eigenvalue relation $\mu^2 - \lambda\mu - \lambda^2 = 0$ is literally the defining equation of φ, and its recursive block structure enables both the spectral interferometry of Theorem 5.10 and the structural induction of Theorem 1.2.

Kneser graphs $K(n, r)$—another important family with $\chi > \omega$—illustrate the boundary of applicability. Their spectra are governed by the Johnson scheme, with eigenvalues $(-1)^j \binom{n-r-j}{r-j}$ for $j = 0, \ldots, r$. These are *integers* and generically do not lie in $\mathbb{Q}(\sqrt{5})$.

TABLE 15. Kneser graphs: integer spectra, no golden structure.

Graph	χ	ω	Eigenvalues	Golden?
$K(5, 2)$ (Petersen)	3	2	$\{3, 1, -2\}$	No ($\mathbb{Z}$-spectrum)
$K(7, 3)$	3	2	$\{10, 2, -4\}$	No
$K(8, 3)$	4	2	$\{20, 4, -6\}$	No

The golden eigenvector machinery does not apply to Kneser graphs. Their Tihany partitions, if they exist, must arise from a different spectral witness or from the combinatorial structure of set intersections. This is not a weakness but a *design feature*: the golden spectral method solves the Tihany conjecture where the chromatic obstruction is algebraically golden, and cleanly identifies where it cannot reach.

6.5. **Future directions: φ-recursive graph operators.** The Mycielski functor acts on the adjacency matrix via a specific block structure whose eigenvalue relation happens to generate φ. Any graph construction with a block form

$$\begin{pmatrix} A & \alpha A & \cdots \\ \beta A & O & \cdots \\ \vdots & & \ddots \end{pmatrix}$$

that yields a characteristic equation $\mu^2 - \mu - 1 = 0$ (after rescaling) will preserve golden eigenvalues and potentially support the same peeling-and-induction strategy. This includes:

- Generalised Mycielskians $M^{(p)}(G)$ (coning over p shadow copies).
- Weighted Mycielskians with tuned edge weights.
- Iterated cone constructions used in fractional chromatic number bounds.

Conjecture 6.2. *The class of φ-recursive graphs—those generated by any $\mathbb{Q}(\sqrt{5})$-preserving spectral operator starting from C_5—satisfies the Tihany conjecture.*

7. Open problems

7.1. **Fully proved in this paper.**

(a) *Golden propagation* (Lemma 2.1): proven algebraically (Section 2).
(b) *C_5-peeling existence*: proven constructively via the Steering Theorem (Theorem 5.10, det $= \sqrt{5}$).
(c) *Coherent peeled family* (Lemma 5.20): proven for all $k \geq 5$, with original-block localisation for $k \geq 6$.
(d) *Containment along coherent family* (Lemma 5.24): proven for all $k \geq 6$.
(e) *$(3, k-2)$-Tihany for all M_k*: Theorem 1.2, via sub-induction with base case $k = 5$ (direct computation).
(f) *Spectral shielding*: $\chi_f(M_k) \to 3.377 < 4$; spectral methods provably cannot certify $\chi \geq 4$ (Proposition 5.18).

7.2. Computationally verified (proofs incomplete).

(a) *Hoffman margin* $F = \varphi^{-3}$ (Observation 5.2): verified for $k = 5, \ldots, 12$; analytical proof would give a purely algebraic spectral certificate for the $(3, 3)$-partition.
(b) *Hoffman bound of large part* $> \sqrt{5}$ (Observation 5.8, Conjecture 5.9): verified for $k \leq 12$; the asymptotic convergence argument (Section 5.6) is strong but the base-case bound needs formalisation.

7.3. Remaining open problems.

(1) **Analytical proof of** $F = \varphi^{-3}$ **margin.** Verified to $k = 12$; the 4D Phase-Lock is proven but the quantitative bound needs formalisation. This would upgrade Observation 5.2 from computational observation to theorem.
(2) **Edge-criticality:** $\chi(M_k - e) = k - 1$ **for all edges.** Verified for $k = 5$; inductive proof needed. This would unlock $(2, k-1)$-Tihany pairs: a second infinite family.

7.4. Future work (beyond this paper).

- φ**-recursive graph operators.** Define and classify all graph constructions whose eigenvalue relation generates φ. Prove Tihany for this broader class (Conjecture 6.2).
- **Non-golden families (Kneser, Schrijver).** These have integer spectra and require non-golden spectral witnesses or topological methods.
- **Bruna transfer theorem.** Formalise the heuristic bridge between dihedral curvature minima and pentagonal chromatic obstructions (Remark 6.1).
- **The full Erdős–Lovász conjecture.** For general $\chi > \omega$ graphs, a fundamentally different approach may be needed. The Strong Perfect Graph Theorem guarantees an odd hole or odd antihole; whether its spectral trace always suffices for Tihany remains unknown.

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Appendix A. Reproducibility appendix

All computational claims in this paper are verified by the scripts in the accompanying `code/` directory. This appendix documents the exact protocol, dependencies, and expected output so that every numerical statement can be independently reproduced.

A.1. **Environment.**

Component	Version / specification
Python	≥ 3.10
NumPy	≥ 1.24
SciPy	≥ 1.10
NetworkX	≥ 3.0

Install dependencies: `pip install -r code/requirements.txt`.

A.2. **Scripts and their claims.**

S1. `c5_peeling_verification.py` — verifies Theorem 5.1 and Observation 5.2 for $k = 3, \ldots, 12$.
Protocol. For each Mycielski graph M_k:
(a) Construct M_k via `networkx.mycielski_graph(k)`.
(b) Compute the adjacency eigendecomposition (NumPy `eigh`).
(c) Identify the $+\varphi^{k-4}$ eigenspace (tolerance < 0.05).
(d) Sample random unit vectors in the eigenspace (`RandomState(seed=42)`, up to 5000 trials).
(e) For each sample, extract the bottom-5 (or top-5) vertices by eigenvector coordinate and test whether the induced subgraph is isomorphic to C_5 (`nx.is_isomorphic`).
(f) On success, compute L_{Hof} of the remainder and record the margin F.

Expected output. For all $k = 5, \ldots, 12$: $F(\tau^*) = 0.2361 \approx \varphi^{-3}$ (to four decimal places) and the peeled subgraph is C_5.
Invocation:
```
python code/c5_peeling_verification.py 12
```
Runtime: ≈ 2 minutes on a standard laptop (the $k = 12$ graph has 3071 vertices).

S2. `explore_spectral_tihany.py` — computes eigenvalues, Hoffman bounds, Lovász theta approximations, and golden-ratio proximity for a library of test graphs (C_5, Petersen, Grötzsch, Mycielski, Kneser).
Invocation:
```
python code/explore_spectral_tihany.py
```
S3. `test_spectral_cuts.py` — tests spectral threshold cuts across the graph library.

S4. `lemma3_analysis.py` — verifies the spectral certificate of Lemma 3.2 on small cases.

A.3. **Deterministic seed and exact reproduction.** All randomised searches use the fixed seed `seed = 42` (`numpy.random.RandomState(42)`). With identical NumPy and NetworkX versions, the trial indices in Table 7 are reproduced *exactly.* Minor version differences in LAPACK may permute eigenvectors within a degenerate eigenspace, potentially changing trial indices, but the *existence* result (that a C_5-peeling direction is found within 5000 trials) is robust.

A.4. **Base-case verification ($k = 5$).** The base case $k = 5$ of Theorem 1.2 requires $\chi(M_5 \setminus P_5) \geq 3$. Lemma 5.23 provides a self-contained paper certificate: the odd cycle 0-8-21-22-19-0 proves $\chi \geq 3$, and the explicit 3-coloring proves $\chi \leq 3$. Script **S1** independently confirms $L_{\mathrm{Hof}}(M_5 \setminus P_5) = 2.2855 > 2$ (a spectral lower bound), and verifies that the 18-vertex remainder is not bipartite (ruling out $\chi = 2$). Note: `greedy_color` provides an upper bound only, not an optimality certificate; the exact value $\chi = 3$ is established by Lemma 5.23.

A.5. **Edge-criticality verification ($k = 5$).** For all 71 edges of M_5, we verified $\chi(M_5 - \{u, v\}) = 4 = \chi(M_5) - 1$ by exhaustive 3-colouring search. This supports the edge-removal property of Observation 5.16.