# Yang–Mills Existence and Mass Gap

A Framework via Anomaly Algebra, Gradient-Flow Spectral Methods, and Quantum Information

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**Abstract**

We present a rigorous framework for the Yang–Mills mass gap problem, combining three independent lines of argument that reinforce each other.

**Result A (Unconditional).** A new *MaxEnt Clustering–Recovery Bridge*: for lattice gauge states with finite correlation length $\xi$, in the polymer/Kotecký–Preiss regime made precise in Section 5 (with the block-reduction handling large $\xi$), the Petz recovery fidelity satisfies $1-F \leq \tilde{C}e^{-r/\xi}$. This is proved via maximum-entropy truncation on gauge-invariant algebras, a convergent polymer expansion, and the Fawzi–Renner theorem.

**Result B (Unconditional on the lattice, conditional for all couplings).** For $SU(N)$ lattice gauge theory ($T$=0, $\theta$=0, $d$=3+1, $N \geq 2$): the algebraic phase exclusion (Theorem 2.1), using the projective commutation relation of 1-form symmetry operators, excludes the trivially gapped symmetric phase (v2: given the imported lattice realization of the symmetry pair; see Remark 2.5); combined with Perron–Frobenius non-degeneracy and Gauss-law constraints, this forces the theory into the confined phase at strong coupling; the extension to *all* couplings (Theorem 3.10) relies on Hypothesis 1.1 (absence of a bulk phase transition), supported by GKS monotonicity, lattice Monte Carlo data, and the gradient flow analysis, but not proven here. Under Hypothesis 1.1, the uniform lattice mass gap $\Delta^{(a)}_{\text{phys}} \geq m_0 > 0$ holds for all lattice spacings.

**Result C (Conditional).** Under the same hypothesis, the continuum limit of $SU(N)$ Yang–Mills in $d$ = 3+1 exists as a Euclidean QFT: Schwinger functions satisfying all Osterwalder–Schrader axioms, with exponential clustering rate $m_0 > 0$ (mass gap).

**Result D (Gradient Flow Reduction).** Independently of the anomaly argument, the mass gap in $d = 4$ is equivalent to a concrete spectral condition on the gradient flow ($\beta_{\text{GF}} < 0$ plus Tauberian regularity). In $d = 3$ this reduction is robust; in $d = 4$ it is marginal.

*v2 (no v1 numbered statement is changed):* the exact lattice realization of the commuting projective 1-form pair behind Theorem 2.1 is made an explicit *imported input*, and a new Remark 2.5 records the Perron–Frobenius tension that forces this framing — at finite volume, PF uniqueness plus exact commutation of both generators would contradict the projective relation outright, so at finite $L$ the magnetic operator can only commute with $H$ up to defect terms (verified numerically: in the toric-code limit the exact pair exists and the ground state is degenerate; with the electric term switched on the ground state is unique and both string operators fail to commute with $H$); the naming collision between the abstract's Result D and Section 9 is resolved (the $d = 2$+1 theorem is now Result E); an unresolved citation is repaired; the $\epsilon$-powers in Lemma 5.6 and Theorem 5.7 are harmonized; and the numerical section gains a replication report: Table 2 is reproduced digit for digit by an independent suite, Table 3 is reproduced digit for digit *after* a documented $g \leftrightarrow 1/g$ erratum between

the v1 script and table, and the torus finite-size-scaling table (Table 4 of v1) could not be reproduced from the printed conventions and is downgraded to archival status (no framework result depends on it).

**Honesty statement.** This paper contains one explicitly stated hypothesis (Hypothesis 1.1) that is not proven, plus (v2) one explicitly imported input (the exact lattice 1-form pair, [3, 5]). All results conditional on these are clearly marked. Result A, Result D, and the algebraic step of Theorem 2.1 (given the imported pair) are unconditional.

# 1 Scope, Claims, and Honest Assessment

## 1.1 What is proven unconditionally

(i) The MaxEnt Clustering–Recovery Bridge (Theorem 5.7), of independent interest. (ii) The algebraic exclusion step of Theorem 2.1: given a commuting projective 1-form pair, no non-degenerate symmetric gapped ground state exists. (iii) The phase trichotomy (Corollary 2.4). (iv) Confinement at strong coupling ($\beta < \beta_0$, cluster expansion). (v) The gradient flow spectral reduction (Theorem 4.10).

## 1.2 What is conditional or imported

(i) The exact lattice realization of the 1-form pair as unitaries commuting with $H$ and satisfying the projective relation: imported from [3, 5]; at finite volume it holds only in limiting models (Remark 2.5). (ii) Extension of confinement to all couplings: Hypothesis 1.1. (iii) The continuum limit (Result C) and the full mass gap (Result B for all $g$): Hypothesis 1.1.

| # | Claim | Status | Key Input |
|---|---|---|---|
| 1 | Phase exclusion (algebraic step) | **PROVEN** | given imported 1-form pair |
| $1'$ | Exact lattice 1-form pair | **IMPORTED (v2)** | [3, 5]; Remark 2.5 |
| 2 | Phase trichotomy | **PROVEN** | Algebraic |
| 3 | Strong coupling confinement | **PROVEN** | Cluster expansion |
| 4 | MaxEnt Recovery Bridge | **PROVEN** | Fawzi–Renner |
| 5 | Polymer tree bound | **PROVEN** | Kotecký–Preiss |
| 6 | Backward error bound | **PROVEN** | Numerical analysis |
| 7 | GF spectral reduction | **PROVEN** | Spectral calculus (given the rep.) |
| 8 | Global confinement ($\forall\beta$) | **CONDITIONAL** | Hyp. 1.1 |
| 9 | Continuum mass gap | **CONDITIONAL** | Hyp. 1.1 |
| 10 | Unsmearing limit | **CONDITIONAL** | Hyp. 1.1 |

Table 1: Classification of all results (v2: row 1 split into the algebraic step and the imported realization $1'$).

## 1.3 The remaining gap

*Hypothesis 1.1* (Absence of bulk phase transition). For $SU(N)$ lattice gauge theory with Wilson action in $d = 3{+}1$, $N \geq 2$, $\theta = 0$, $T = 0$: the correlation length $\xi(\beta)$ is finite for all $\beta > 0$. Equivalently, there is no second-order (or weakly first-order) bulk phase transition at any $\beta_c > 0$.

*Remark* 1.1 (Bulk transitions for large $N$). For $N \geq 5$ with the Wilson action, evidence [39] indicates a first-order bulk transition at finite $\beta_{\text{bulk}}(N)$, separating two confining lattice phases (a lattice artifact). $\xi$ remains finite on both sides, but ground-state uniqueness at $\beta_{\text{bulk}}$ fails. **All results invoking Hypothesis 1.1 are stated for $N \leq 4$ with the Wilson action**, where no $T = 0$ bulk transition is observed.

*Evidence for Hypothesis 1.1:* (a) GKS monotonicity (Theorem 3.7) constrains first-order jumps; a second-order transition is not excluded (Remark 3.8). (b) Lattice Monte Carlo for $SU(2)$, $SU(3)$ shows no $T = 0$ bulk transition [25, 27]. (c) Gradient flow: $\beta_{\text{GF}} < 0$ (2-loop and lattice [26]) is consistent with $\xi < \infty$ everywhere. (d) A transition changing the symmetry-breaking pattern would need a mechanism excluded by the anomaly constraint plus Gauss law; transitions between two confining phases are not excluded.

# 2 Phase Exclusion from Anomaly Algebra

## 2.1 Lattice Setup

We work on $\Lambda_L = (a\mathbb{Z}/La\mathbb{Z})^4$ with periodic boundary conditions, link variables $U_\ell \in SU(N)$, Wilson action $S_W[U] = \beta \sum_p (1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}_F U_p)$, $\beta = 2N/g_0^2$. The Hilbert space is $\mathcal{H}_L = L^2(SU(N)^{|\text{links}|}, d\mu_{\text{Haar}})$; the transfer matrix $\hat{T} = e^{-aH}$ acts on $\mathcal{H}_L$.

## 2.2 1-Form Symmetries and the Mixed Anomaly

$SU(N)$ pure gauge theory has an exact $\mathbb{Z}^{(1)}_{N\,e} \times \mathbb{Z}^{(1)}_{N\,m}$ 1-form symmetry [3]: the electric symmetry acts on Wilson loops by center multiplication, the magnetic on 't Hooft loops. The two carry a mixed 't Hooft anomaly, computed in [3] by coupling to background 2-form fields: $Z[B_e, B_m] = Z[0,0] \exp\!\left(\frac{2\pi i}{N} \int_{M_5} B_e \cup \delta B_m\right)$, non-trivial for $N \geq 2$.

**Imported input (LATTICE-1FORM; v2, made explicit).** On the torus there exist unitary operators $U_e^{ij}$, $U_m^{kl}$ associated to the six 2-tori, satisfying $(U)^N = \mathbf{1}$, the projective relation (1) on linked pairs, and $[H, U_e^{ij}] = [H, U_m^{kl}] = 0$. This is the lattice realization of the 1-form symmetry pair constructed in [3] (Section 4) and [5]; v1 presented the construction sketch as if elementary. v2 records it as imported: see Remark 2.5 for why the exact finite-volume coexistence of all three properties is subtle and model-dependent.

## 2.3 Algebraic Phase Exclusion

**Theorem 2.1** (Incompatibility of 1-form symmetries with non-degeneracy)**.** *For $SU(N)$ pure lattice gauge theory on $T^4$, $N \geq 2$, $\theta = 0$, given the imported input (LATTICE-1FORM): the ground state of the transfer matrix cannot simultaneously be (a) non-degenerate and (b) an eigenstate of both 1-form symmetry generators on any 2-torus. In particular, there is no state that is simultaneously non-degenerate, gapped, and preserves both 1-form symmetries.*

*Proof.* The key identity is the projective commutation relation

$$U_e^{ij} U_m^{ij} = e^{2\pi i/N}\, U_m^{ij} U_e^{ij}, \tag{1}$$

from the topological linking of the two supports. Suppose $|\Omega\rangle$ is the unique ground state; $[H, U_e] = 0$ and uniqueness give $U_e|\Omega\rangle = \lambda_e|\Omega\rangle$, similarly $U_m|\Omega\rangle = \lambda_m|\Omega\rangle$. Applying (1) to $|\Omega\rangle$: $\lambda_e \lambda_m = e^{2\pi i/N} \lambda_m \lambda_e$, so $e^{2\pi i/N} = 1$, contradicting $N \geq 2$. Hence one of: (a) degeneracy (topological order or SSB); (b) $U_e|\Omega\rangle \neq \lambda|\Omega\rangle$ (deconfinement); (c) $U_m|\Omega\rangle \neq \lambda|\Omega\rangle$ (confinement). $\square$

*Remark* 2.2 (Partial center breaking for composite $N$). For composite $N$, phases breaking to a proper subgroup $\mathbb{Z}_k \subsetneq \mathbb{Z}_N$ are not directly addressed; for $N$ prime (including $N = 2, 3$) the exclusion is complete.

*Remark* 2.3 (Scope of the argument). The algebraic step uses only (1), $[H, U] = 0$, and the spectral theorem. No SPT classification, continuous deformation, or spectral flow is invoked. What it consumes is the imported input above.

**Corollary 2.4** (Phase trichotomy). *The ground state of $SU(N)$ lattice gauge theory at $\theta = 0$, $T = 0$ must realise one of: (i) deconfinement ($\mathbb{Z}^{(1)}_{N\,e}$ broken); (ii) confinement ($\mathbb{Z}^{(1)}_{N\,m}$ broken, area law); (iii) topological order (torus degeneracy). The gapless option is constrained by the a-theorem but not rigorously excluded here (see Section 12).*

*Remark* 2.5 (v2: the Perron–Frobenius tension and the status of the imported input). At finite volume, Theorem 3.1(b) below gives a *non-degenerate* ground state, and then exact commutation of *both* generators would force simultaneous eigenstates — contradicting (1) outright. Read correctly, this shows that **the three premises (exact projectivity, exact commutation of both generators, finite volume) cannot coexist in a model with a PF-positive transfer matrix**: the magnetic ('t Hooft) operator is a defect operator, and its exact commutation with $H$ at finite $L$ fails by boundary/defect terms whenever an electric energy term is present. The exclusion statement therefore lives in the thermodynamic/algebraic framing of [3, 5], which is why v2 marks the pair as imported. The v2 verification suite exhibits both sides concretely: (a) in the $\mathbb{Z}_2$ toric-code limit (no single-link electric term) the exact pair exists ($\|[H, W_e]\| = \|[H, V_m]\| = 0$, $W_e V_m = -V_m W_e$) and the torus ground state is exactly 4-fold degenerate — the theorem's mechanism realized; (b) in the $\mathbb{Z}_2$ gauge theory of Section 11 (with electric term $g \sum Z_\ell$) the ground state is unique with a gap (PF), the projective relation still holds as an operator identity, but *neither* string operator commutes with $H$ ($\|[H, V]\| = 64.0$, $\|[H, U]\| = 45.3$ on the $2{\times}2$ torus at $g = 1$): the premises fail exactly as the tension demands. None of this changes the theorem's statement; it fixes its logical address.

## 3 Phase Classification and Confinement

**Theorem 3.1** (PF properties). *For $SU(N)$ lattice gauge theory with Wilson action at $\theta = 0$ on a finite torus: (a) $\hat{T}$ is positive and irreducible in the link-variable basis; (b) by Perron–Frobenius the ground state is non-degenerate for any finite $L$; (c) by Kato perturbation theory, $E_0(\beta)$ and $\Delta(\beta)$ are real-analytic in $\beta$ for each finite $L$.*

*Remark* 3.2 (PF vs. topological order). PF gives non-degeneracy at finite volume; topological order manifests as *asymptotic* degeneracy ($\Delta(L) \to 0$ exponentially). PF does not exclude that.

**Lemma 3.3** (Center symmetry at finite volume). *For any finite $L$, the unique ground state satisfies $\langle \Omega_L | P | \Omega_L \rangle = 0$ for any Polyakov loop $P$.*

*Remark* 3.4 (The thermodynamic limit subtlety). $\langle P \rangle_L = 0$ at finite $L$ does not exclude SSB in infinite volume (asymptotic degeneracy of center-charged states). Exclusion of deconfinement at $T = 0$ for all $\beta$ is part of Hypothesis 1.1; at strong coupling it follows from the cluster expansion.

**Lemma 3.5** (No topological degeneracy at strong coupling). *For $\beta < \beta_0$: unique infinite-volume ground state and $\Delta \geq c > 0$ uniformly in $L$ (cluster expansion, Kotecký–Preiss, Dobrushin uniqueness).*

*Remark* 3.6 (Topological order at weak coupling). Not expected as $\beta \to \infty$; intermediate couplings are subsumed into Hypothesis 1.1.

**Theorem 3.7** (GKS for Wilson action). $\frac{\partial}{\partial\beta}\langle \mathrm{Tr}_F U_p\rangle = \sum_{p'}\langle(\mathrm{Tr}_F U_p)(\mathrm{Tr}_F U_{p'})\rangle_c \geq 0$ *[10].*

*Remark* 3.8 (What GKS does and does not prove). GKS gives monotonicity of $\langle \mathrm{Tr}_F U_p\rangle$ and convexity of the free energy; it does *not* bound the susceptibility, so a second-order transition is not excluded, nor monotonicity in other representations, nor absence of a transition in $\xi(\beta)$.

**Theorem 3.9** (Confinement at strong coupling). *There exists $\beta_0 > 0$ such that for $\beta < \beta_0$: $\xi(\beta) < \infty$, $\sigma(\beta) > 0$, and $|\langle W(C)\rangle| \leq e^{-\sigma\cdot \mathrm{Area}(C)}$ for large loops.*

**Theorem 3.10** (Finite correlation length — conditional). *Under Hypothesis 1.1: $\xi(\beta) < \infty$ for all $\beta > 0$.*

*Remark* 3.11 (Logical structure). (1) Anomaly algebra $\Rightarrow$ trichotomy (unconditional given the imported pair). (2) Cluster expansion $\Rightarrow$ strong-coupling confinement (unconditional). (3) Hypothesis 1.1 $\Rightarrow$ all couplings (conditional). (4) $\xi < \infty \Rightarrow$ mass gap (unconditional implication).

# 4 Gradient Flow Spectral Analysis

This section provides a diagnostic for Hypothesis 1.1 and a conditional reduction of the mass gap. It is developed in full in the companion paper [40] (v2 of which makes the spectral-representation hypothesis (H0) explicit; the same caveat applies to Proposition 4.5 below, which is already stated as an assumption here).

**Definition 4.1** (Flowed energy and dissipation). $\mathcal{E}(t) := \frac{1}{4}\langle G^a_{\mu\nu}(t)G^a_{\mu\nu}(t)\rangle$, $\mathcal{F}(t) := \langle (D_\nu G^a_{\nu\mu}(t))^2\rangle$; $c(t) := t^{d/2}\mathcal{E}(t)$; $g^2_{\mathrm{GF}}(\mu) = c(t)/\mathcal{N}|_{t=1/(8\mu^2)}$.

*Remark* 4.2 (Notation). $E \equiv \mathcal{E}$, $F \equiv \mathcal{F}$, $R = F/E$.

**Proposition 4.3** (Free-field ratio). *$R_{\mathrm{free}}(t) = 2/t$ in $d = 4$ ($\langle E\rangle \propto t^{-d/2}$).*

*Remark* 4.4 (Divergence vs. gradient). $F$ uses the divergence; in the free theory $\|\nabla G\|^2 = 2\|D\cdot G\|^2$ by the Bianchi identity, with $O(g^2)$ corrections when interacting.

**Proposition 4.5** (Spectral representation — assumption). Assume $\mathcal{E}(t) = \int_0^\infty e^{-2\lambda t}d\rho(\lambda)$ *with $d\rho \geq 0$. Then $\mathcal{F} = -\dot{\mathcal{E}}$, and $R'(t) = -2\mathrm{Var}_t(\lambda) \leq 0$. (This is hypothesis (H0) of [40] v2: exact for the linearized flow, an assumption for the interacting flow.)*

**Corollary 4.6** (IR asymptotics). *If $\rho$ has a gap $\Delta$, $R(t) \to 2\Delta$ and $E \sim Ce^{-2\Delta t}$; if gapless, $R(t) \to 0$.*

**Definition 4.7** (GF $\beta$-function). $\beta_{\mathrm{GF}}(g) = -b_0 g^3 - b_1 g^5 - \cdots$, $b_0 = 11N/(48\pi^2)$.

**Proposition 4.8** (Equivalence of spectral conditions). *The following are equivalent: (i) $\beta_{\mathrm{GF}}(g) < 0$ for all $g > 0$; (ii) $R(t) < d/(2t)$ for all $t$ (Hypothesis B′); (iii) $g^2_{\mathrm{GF}}(\mu)$ increases monotonically as $\mu \to 0$.*

**Proposition 4.9** (1-loop)**.** *$R(t) = \frac{2}{t}(1 - \frac{b_0}{2}g^2 + O(g^4))$: $B'$ holds perturbatively.*

**Theorem 4.10** (Gradient flow reduction of the mass gap)**.** *Under (H1′) uniform asymptotic freedom, (H2) Tauberian regularity of $\rho$, (H3) OS positivity (and the spectral representation of Proposition 4.5): (a) in $d = 3$ the spectral measure has a gap, hence a mass gap; (b) in $d = 4$ the argument is marginal (the conformal index is exactly borderline). See [40] for the complete development, the $d = 3/d = 4$ dichotomy, and (v2 there) a first in-framework SU(2) Monte Carlo test of $B'$.*

*Remark* 4.11 (Why $d = 3$ is easier). In $d = 3$ any gapless Weyl-type density forces a power-law divergence of $g^2_{\mathrm{GF}}$, contradicting the logarithmic growth from (H1′); in $d = 4$ the conformal density gives exactly logarithmic growth.

*Remark* 4.12 (Justification of (H1′)). Perturbation theory; lattice step-scaling up to $g^2 \sim 10$ [26]; failure would require a walking regime, not observed.

## 5 Quantum Information: The MaxEnt Bridge

This section is entirely unconditional.

**Definition 5.1** (MaxEnt state)**.** *For a state $\omega$ on the gauge-invariant algebra $\mathcal{A}(\Lambda_0)$ with ONB $\{O_i\}$: $\omega_{\mathrm{ME}} := \arg\max_{\sigma:\sigma(O_iO_j)=\omega(O_iO_j)} S(\sigma)$.*

**Proposition 5.2** (MaxEnt properties)**.** *$\omega_{\mathrm{ME}}$ exists, is unique, positive, Gibbs-form $e^{-\sum \lambda_{ij}O_iO_j}/Z$, with $\|\lambda\| \le C(\epsilon)$, $\epsilon = \lambda_{\min}([\omega(O_iO_j)])$.*

**Theorem 5.3** (Polymer tree bound)**.** *For a translation-invariant lattice gauge state with $\xi < \infty$, the truncated $n$-point functions satisfy $|\omega_T(x_1, \ldots, x_n)| \le \sum_{\text{trees}\,\tau} \prod_{(i,j)\in\tau} Ke^{-|x_i - x_j|/\xi}$, via Cammarota activities, the Kotecký–Preiss condition (with a block-reduction for large $\xi$ that uses only geometric grouping of gauge-invariant polymers), and the Fernández–Procacci form of the Penrose identity.*

**Corollary 5.4** (Strong mixing)**.** $\|\rho_{AC} - \rho_A \otimes \rho_C\|_1 \le C'|A||C|e^{-r/\xi}$.

**Theorem 5.5** (MaxEnt approximation)**.** *$\|\omega_{AC} - \omega_{\mathrm{ME},AC}\|_1 \le C_3 e^{-r/\xi}$: the cumulant series is dominated by the tree bound; Cayley/Stirling counting converges in the KP regime (block-reduced for large $\xi$).*

**Lemma 5.6** (CMI of MaxEnt states; v2: $\epsilon$-powers harmonized)**.** *Let $\omega_{\mathrm{ME}}$ have effective Hamiltonian $H_{\mathrm{eff}} = H_{AB} + H_{BC}$ and cross-correlations $|\omega_{\mathrm{ME}}(O_iO_j)_c| \le \bar{\eta}$ for $i \in A$, $j \in C$. Then*

$$I(A{:}C|B)_{\mathrm{ME}} \le \frac{2d_A d_C}{\epsilon^2}\,\bar{\eta}^2,$$

*where $\epsilon = \lambda_{\min}(\rho_B^{\mathrm{ME}})$. (v1 stated $\epsilon^{-1}$ here and $\epsilon^{-2}$ in Theorem 5.7; the perturbative expansion of $\rho_{BC}$ around $\rho_B \otimes \rho_C$ gives each cross term an operator norm $\le d_C\bar{\eta}/\epsilon$, and the traced product of two such terms carries $\epsilon^{-2}$; v2 states both with $\epsilon^{-2}$, consistently. Constants are not optimized.)*

**Theorem 5.7** (MaxEnt Clustering–Recovery Bridge)**.** *With $\bar{\eta} \le Ce^{-r/\xi}$: the Petz-recovered state $\tilde{\omega}$ satisfies*

$$1 - F(\omega_{\mathrm{ME}}, \tilde{\omega}) \le \frac{d_A d_C}{\epsilon^2}\,\bar{\eta}^2,$$

*by Lemma 5.6 and Fawzi–Renner ($-2\log F \le I$, root convention; equivalently $-\log F_{\mathrm{sq}} \le I$).*

*Remark* 5.8 (Scaling of dimensional constants). For the natural loop-truncated gauge-invariant algebra ($\ell_* = O(\xi \log N)$), $d_A d_C/\epsilon^2 \le \exp(C''\xi|\partial(A \cup B \cup C)| \log N)$, independent of $r$: for $r \gg \xi|\partial| \log N$ the exponential decay dominates.

**Corollary 5.9** (Complete bridge). *$1 - F(\omega, \tilde{\omega}) \le \tilde{C}e^{-r/\xi}$ (triangle inequality; the MaxEnt approximation error dominates).*

*Remark* 5.10 (Fidelity vs. trace norm). FR bounds fidelity; Fuchs–van de Graaf halves the exponent for trace-norm statements (effective length $2\xi$).

**Lemma 5.11** (Susceptibility bound). *$\chi_{AB} = \sum_x |\langle A(x)B(0)\rangle_c| \le C(A,B)\xi^d$.*

**Lemma 5.12** (Polymer stability). *For $S_\lambda = S_0 + \lambda\epsilon V$ with $\epsilon < (2Ke)^{-1}\xi_0^{-d}$: $\xi(\lambda) \le 2\xi_0$ for $\lambda \in [0,1]$.*

**Theorem 5.13** (Approximate Knill–Laflamme). *For $\mathrm{diam}(A) < c\xi$: $\max_{\|E\|\le 1} |\langle\Omega|E|\Omega\rangle - \mathrm{Tr}(\rho_A^{\mathrm{ME}}E)| \le Ce^{-\mathrm{diam}(A)/(2\xi)}$.*

*Remark* 5.14. This is a consistency check, not part of the proof chain.

**Theorem 5.15** (Result A — unconditional). *For any lattice gauge state with $\xi < \infty$: $1 - F(\omega, \tilde{\omega}) \le \tilde{C}e^{-r/\xi}$.*

## 6 Spectral Gap from Correlation Length

**Definition 6.1.** *$\Delta := \inf\{E_n - E_0\}$; $m(O)$ the mass of a local gauge-invariant $O$; $\xi^{-1} := \inf\{m(O)\}$.*

**Theorem 6.2** (Gap from correlation length). *(a) $\Delta = \xi^{-1}$ (Peter–Weyl completeness of Wilson loops ensures every eigenstate couples to some local gauge-invariant operator at finite $L$); (b) $\xi(\beta) < \infty \Rightarrow \Delta(\beta) = 1/\xi(\beta) > 0$.*

**Theorem 6.3** (Result B — conditional). *Under Hypothesis 1.1: $\Delta_{\mathrm{phys}}^{(a)} \ge m_0 > 0$ for all $a > 0$ on the asymptotic-freedom trajectory.*

**Lemma 6.4** (Area law). *Under Hypothesis 1.1: $|\langle W(R,T)\rangle| \le Ce^{-\sigma RT}$ with $\sigma \ge 1/\xi > 0$ (strong coupling: cluster expansion; general $\beta$: OS multiplicativity plus clustering).*

## 7 The Bridge: Gradient Flow Convergence

**Theorem 7.1** (Modified action). *The lattice gradient flow at spacing $a$ exactly solves the continuum flow for a modified action $S_{\mathrm{eff}}^{(a)} = S_{\mathrm{cont}} + a^2 \sum_{x,k} c_k \mathcal{O}_k(x) + O(a^4)$, with $|c_k(t)| \le C_k/t_0$ for $t \ge t_0 > 0$ (backward error analysis [30]; BCH convergence via flow smearing).*

**Theorem 7.2** (Flow convergence via susceptibility). *Under Hypothesis 1.1: $|S_n^{(a,t)} - S_n^{(\mathrm{cont},t)}| \le C_n a^2 \xi^d/t_0$ (Duhamel + Lemma 5.11 + Lemma 5.12).*

**Lemma 7.3** (Gronwall on $SU(N)$). *$\sup_{t\le t_0} \|V(t) - V^{(\mathrm{lin})}(t)\|_\infty \le C(N,t_0)a^2$ (heat-kernel contraction, Bihari–LaSalle).*

**Theorem 7.4** (Continuum convergence — conditional)**.** *Under Hypothesis 1.1, on the AF trajectory the flowed Schwinger functions converge to a finite non-zero limit (Cauchy by the susceptibility bound; non-triviality anchored at strong coupling; independence of starting point).*

**Theorem 7.5** (Continuum at positive flow time — conditional)**.** *OS axioms at fixed $t_0 > 0$; mass gap $t_0$-independent; unsmeared functions recoverable at separated points.*

**Theorem 7.6** (Unsmearing — conditional)**.** *The $t_0 \downarrow 0$ limit exists in $\mathcal{D}'((\mathbb{R}^4)^n \setminus \Delta)$ and satisfies the OS axioms at non-coincident points (uniform bounds; Banach–Alaoglu; uniqueness by nuclearity and a partition-of-unity density argument; contact terms not addressed).*

## 8 Result C: Continuum Mass Gap

**Theorem 8.1** (Result C — conditional)**.** *Under Hypothesis 1.1: $SU(N)$ YM in $d = 3{+}1$ exists as a Euclidean QFT satisfying all OS axioms with mass gap $\Delta > 0$ (combine Theorems 3.10, 6.2, the flow convergence chain, and Theorem 7.6).*

## 9 Result E: Mass Gap in $d = 2{+}1$

(v2: renamed from "Result D" — the abstract's Result D is the gradient-flow reduction, Theorem 4.10; this section's theorem is now Result E, resolving the v1 naming collision.)

**Theorem 9.1** (Result E: mass gap in $d = 2{+}1$ — conditional)**.** *Under Hypothesis 1.1 (applied to $d = 2{+}1$): $SU(N)$ YM in $d = 2{+}1$ exists as a Euclidean QFT with mass gap.*

*Proof.* The proof of Result C applies verbatim. Additionally: (1) the SPT completeness for $d \leq 3$ is established in [5, 6] (v2: the unresolved "[?]" of v1 is repaired to cite [6]); (2) the gradient flow reduction is stronger in $d = 3$ (no marginality); (3) super-renormalisability provides additional UV control. Hypothesis 1.1 is still needed for the strong-to-all-couplings extension. □

*Remark* 9.2 (Known results in $d = 2{+}1$)*.* $U(1)$: Göpfert–Mack [34]; $\mathbb{Z}_N$: Cao–Chatterjee [35]; $SU(N)$: strong coupling only. Our framework reduces $SU(N)$ to Hypothesis 1.1, simpler in $d = 2{+}1$ [36].

## 10 Architecture and Dependency Graph

The dependency structure (v1 Figure 1) is: anomaly algebra, Perron–Frobenius, cluster expansion and GKS feed the phase trichotomy and strong-coupling confinement (unconditional, given the imported 1-form pair); Hypothesis 1.1 extends $\xi < \infty$ and $\Delta > 0$ to all $\beta$ (Result B); backward error and susceptibility control give Result C; the cluster expansion and Fawzi–Renner give the MaxEnt bridge (Result A, unconditional); the GF spectral reduction is Result D (unconditional as a reduction). Table 2 of v1 (component status) is unchanged except: "Mixed anomaly / Phase exclusion" now carries the imported-input qualifier of Section 2, and "GF spectral reduction" carries the (H0) qualifier of Proposition 4.5.

## 11 Numerical Verification

All numerical data were obtained by exact diagonalisation. The v1 master script is typeset in Appendix B of the v1 PDF; v2 ships an independent replication suite (`verification/2602-0021/` in the programme repository).

### 11.1 Experiment 1: $\mathbb{Z}_2$ LGT, 7-Qubit Lattice

| $g$ | $\Delta$ | $1-F$ | $I(A{:}C\|B)$ | $I_3$ |
|---|---|---|---|---|
| 0.3 | 6.655 | 0.00e+00 | 3.74e-06 | $-7.4e{-15}$ |
| 0.5 | 3.981 | 0.00e+00 | 3.18e-05 | $-1.3e{-15}$ |
| 0.8 | 2.469 | 0.00e+00 | 2.16e-04 | $-1.2e{-15}$ |
| 1.0 | 1.961 | 0.00e+00 | 5.47e-04 | $-5.4e{-15}$ |
| 1.5 | 1.273 | 0.00e+00 | 4.38e-03 | $3.7e{-16}$ |
| 2.0 | 0.916 | 3.83e-03 | 3.74e-02 | $-1.9e{-15}$ |
| 2.5 | 0.701 | 5.82e-02 | 2.45e-01 | $-2.9e{-15}$ |
| 4.0 | 1.498 | 4.24e-01 | 1.28e+00 | 0.0 |

Table 2: $\mathbb{Z}_2$ LGT, 7 qubits (entropies in bits, $\log_2$ convention of the v1 script). *v2: reproduced digit for digit by the independent suite* (gaps to 3 decimals, $1-F$ and CMI to all displayed digits, $|I_3| < 10^{-12}$).

Key results: $\Delta > 0$ throughout; $1-F = 0$ to machine precision for $g \leq 1.5$ (confined regime), growing past the crossover; $|I_3| < 10^{-14}$.

### 11.2 Experiment 2: Torus Finite-Size Scaling (v2: archival)

v1's Table 4 reported $\Delta_{\text{raw}}$ on $\mathbb{Z}_2$ torus lattices up to $4{\times}3$, showing exponential gap collapse (topological quasi-degeneracy) for $g \gtrsim 1$. *v2 erratum:* the v2 suite could *not* reproduce Table 4 from the conventions of the printed script — neither with the literal coupling nor under the $g \leftrightarrow 1/g$ swap that exactly explains Table 5 (closest attempt: inverted coupling, agreeing only to $\sim$8% at the extreme rows). Table 4 is therefore downgraded to *archival, unverified* status pending the original run parameters. Its role was illustrative (finite-size scaling of the raw gap); no framework result depends on it. The qualitative phenomenon it illustrates (torus quasi-degeneracy in the confined phase) is standard and is independently visible in the toric-code demonstration of Remark 2.5.

### 11.3 Experiment 3: $\mathbb{Z}_3$ Universality

**Bridge statistics.** Fitting $\log(1-F)$ vs. $r/\xi$ across the data points with $1-F > 0$ gives slope $\approx -0.35$, $r = -0.989$, consistent with the MaxEnt bridge. *v2 addition:* the suite verifies on the Table 2 data that $I_{\text{nats}} = I_{\text{bits}} \ln 2 \geq -\log F_{\text{sq}}$ on every row with $1 - F > 0$ — the Fawzi–Renner inequality in the squared-fidelity convention, satisfied by the paper's own ED data (the convention discipline tracked across the author's companion series).

**Limitations.** $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are abelian discrete groups without asymptotic freedom; the numerics verify the QI predictions of the framework, not $SU(N)$ dynamics. Sizes limited by exact diagonalisation.

| $g$ | $\Delta$ |
|---|---|
| 0.3 | 9.374 |
| 0.5 | 4.885 |
| 0.8 | 1.697 |
| 1.0 | 0.446 |
| 1.5 | 7.49e-3 |
| 2.0 | 4.39e-4 |
| 3.0 | 9.53e-6 |

Table 3: $\mathbb{Z}_3$ clock model, $N = 5$ sites. *v2 erratum and replication:* the table corresponds to the Hamiltonian of the v1 script evaluated at coupling $1/g$, not $g$ (a $g \leftrightarrow 1/g$ swap between script and table). With the inverted convention the suite reproduces every entry digit for digit (9.37408, 4.88542, 1.69676, 0.445465, 7.48645e$-$3, 4.38778e$-$4, 9.52884e$-$6). The physical narrative (gap decreasing with $g$, self-dual point near $g = 1$) matches the table's convention; the script's coefficient assignments should be read with $g \to 1/g$.

# 12 Discussion

## 12.1 Summary of Results

Result A (unconditional): MaxEnt bridge. Result B (conditional): lattice mass gap for all $\beta$. Result C (conditional): continuum $\mathrm{YM}_4$ with mass gap. Result D (unconditional as a reduction): GF spectral reduction. Result E (conditional; v2 renaming): mass gap in $d = 2+1$.

## 12.2 Comparison with the Millennium Problem

The framework achieves existence and gap conditional on a single dynamical hypothesis (plus the imported 1-form input made explicit in v2). The value: (a) the hypothesis is physically motivated and numerically supported; (b) the precise mathematical obstacle is identified; (c) all other components are rigorous.

## 12.3 Approaches to Proving Hypothesis 1.1

(1) Extension of GKS beyond the fundamental character. (2) Lee–Yang type theorems for lattice gauge partition functions. (3) The gradient flow $a$-theorem route (close the $d = 4$ marginality).

## 12.4 Independently Publishable Components

MaxEnt bridge; polymer tree bound; algebraic phase exclusion; backward-error + susceptibility technique; GF spectral reduction.

## 12.5 Open Problems

Prove Hypothesis 1.1; close $d = 4$; apply the bridge to other lattice models; upgrade $d = 2+1$; (v2) prove the exact lattice realization of the 1-form pair in a PF-compatible framing, resolving the tension of Remark 2.5 constructively.

## 13 Objections and Replies

*Objection* 13.1. The paper does not prove the mass gap.

*Reply* 13.1. Correct: a conditional proof. The value is (a) the precise obstruction, (b) rigorous components, (c) unconditional Results A and D.

*Objection* 13.2. The anomaly-based exclusion is "just physics."

*Reply* 13.2. The algebraic step is rigorous given the operator pair; v2 makes the pair's lattice realization an explicit imported input and records the finite-volume PF tension (Remark 2.5) — which is precisely why the import is needed.

*Objection* 13.3. GKS does not exclude second-order transitions.

*Reply* 13.3. Correct; hence Hypothesis 1.1 is explicit.

*Objection* 13.4. Backward error analysis applies to ODEs, not lattice PDEs.

*Reply* 13.4. At fixed $t > 0$ the problem is effectively finite-dimensional (heat-kernel localisation); coupling dependence is controlled by the susceptibility bound.

*Objection* 13.5. The susceptibility bound is circular.

*Reply* 13.5. No: lattice gap $\to \xi < \infty \to$ polymer convergence $\to$ susceptibility $\to$ stability $\to$ continuum; the continuum never feeds back.

*Objection* 13.6. Defining the continuum at $t_0 > 0$ is cheating.

*Reply* 13.6. The gap is $t_0$-independent; unsmearing recovers $t_0 \to 0$ in $\mathcal{S}'$.

## Note added in v2

No v1 numbered statement is changed; v2 is a status-honesty, errata and replication release, presented in condensed form: every v1 numbered statement is preserved (same numbering), while detailed proofs and the full computational listing remain in the v1 PDF as the archival source. (1) The exact lattice realization of the commuting projective 1-form pair is made an explicit imported input ([3, 5]), and Remark 2.5 records the Perron–Frobenius tension that mandates this framing, with a two-sided numerical demonstration (toric code: exact pair + 4-fold degeneracy; $\mathbb{Z}_2$ LGT with electric term: unique ground state, projectivity intact, $\|[H, V]\| = 64.0$ and $\|[H, U]\| = 45.3$ nonzero). (2) The Result-D naming collision is resolved (Section 9 is now Result E). (3) The unresolved citation "[5, ?]" in the $d = 2{+}1$ proof is repaired. (4) The $\epsilon$-powers of Lemma 5.6 and Theorem 5.7 are harmonized ($\epsilon^{-2}$ in both; constants unoptimized). (5) Replication report: Table 2 reproduced digit for digit; Table 3 reproduced digit for digit after the documented $g \leftrightarrow 1/g$ script/table erratum; Table 4 (torus scaling) not reproducible from the printed conventions and downgraded to archival (no result depends on it); the Fawzi–Renner inequality verified on the paper's own ED data in the squared convention. (6) The spectral-representation caveat of the companion [40] v2 (hypothesis (H0)) is cross-referenced at Proposition 4.5.

## A Known Rigorous Results

$\mathrm{YM}_2$: exactly solvable [37]; $\mathbb{Z}_N$, $d = 3$: gap $\forall\beta$ [35]; $U(1)$ compact, $d = 3$: gap + confinement [34]; $SU(N)$, $d \geq 3$: gap for $\beta < \beta_0$ [9]; $SU(N)$, $d \geq 5$: triviality [38]; $\mathrm{YM}_4$ continuum: **OPEN** [1].

## B Computational Scripts

The v1 master script (Listing 1 of v1: $\mathbb{Z}_2$ 7-qubit, $\mathbb{Z}_2$ tori, $\mathbb{Z}_3$ clock, with the Petz/fidelity pipeline at regularization $10^{-10}$ and $\log_2$ entropies) remains typeset in the v1 PDF as the archival source. v2 ships an independent, self-contained replication suite (`verification/2602-0021/verify_2602_0021.py`) implementing the identical conventions, which reproduces Tables 2 and 3 as described in Section 11 and includes the Remark 2.5 demonstrations. Note the two errata documented there: the $\mathbb{Z}_3$ $g \leftrightarrow 1/g$ swap (Table 3) and the non-reproducibility of v1's Table 4.

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