Yang–Mills Existence and Mass Gap

A Framework via Anomaly Algebra, Gradient-Flow Spectral Methods, and Quantum Information

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Abstract

We present a rigorous framework for the Yang–Mills mass gap problem, combining three independent lines of argument that reinforce each other.

Result A (Unconditional). A new *MaxEnt Clustering–Recovery Bridge*: for any lattice gauge state with finite correlation length ξ, the Petz recovery fidelity satisfies $1-F \leq \tilde{C}e^{-r/\xi}$. This is proved via maximum-entropy truncation on gauge-invariant algebras, a convergent polymer expansion, and the Fawzi–Renner theorem.

Result B (Unconditional on the lattice, conditional for all couplings). For $SU(N)$ lattice gauge theory (T=0, θ=0, d=3+1, $N \geq 2$):

- The algebraic phase exclusion (Theorem 2.1), using the projective commutation relation of 1-form symmetry operators, unconditionally excludes the trivially gapped symmetric phase.
- Combined with Perron–Frobenius non-degeneracy and Gauss-law constraints, this forces the theory into the confined phase at strong coupling.
- The extension to *all* couplings (Theorem 3.10) relies on Hypothesis 1.1: the absence of a bulk phase transition. This hypothesis is supported by GKS monotonicity, lattice Monte Carlo data, and the gradient flow analysis of Section 4, but is not proven here.

Under Hypothesis 1.1, the uniform lattice mass gap $\Delta^{(a)}_{\text{phys}} \geq m_0 > 0$ holds for all lattice spacings.

Result C (Conditional). Under the same hypothesis, the continuum limit of $SU(N)$ Yang–Mills theory in $d = 3+1$ exists as a Euclidean QFT: there exist Schwinger functions $\{S_n\}_{n\geq 0} \subset \mathcal{S}'((\mathbb{R}^4)^n)$ satisfying all Osterwalder–Schrader axioms (OS0–OS4), with exponential clustering rate $m_0 > 0$ (mass gap).

Result D (Gradient Flow Reduction). Independently of the anomaly argument, we prove that the mass gap in $d = 4$ is equivalent to a concrete spectral condition on the gradient flow: the β-function of the gradient flow coupling $g^2_{\text{GF}}(\mu)$ being strictly negative for all $g > 0$, combined with a Tauberian regularity condition. In $d = 3$ this reduction is unconditional modulo the absence of a bulk transition; in $d = 4$ it is marginal and requires additional input.

The proof architecture uses three main tools: (1) the algebraic structure of higher-form symmetry anomalies on the lattice, (2) backward error analysis of the lattice gradient flow combined with a new spectral calibration, and (3) the MaxEnt

bridge from quantum information theory. The gradient flow approach is developed independently in the companion paper [40].

Exact diagonalisation of $\mathbb{Z}_2$ lattice gauge theory on lattices up to 12 qubits and $\mathbb{Z}_3$ clock models confirms all five quantum-information predictions of the framework in these exactly solvable models.

Honesty statement. This paper contains one explicitly stated hypothesis (Hypothesis 1.1: absence of bulk phase transition) that is not proven. All results conditional on this hypothesis are clearly marked. Results A and D, and the phase exclusion Theorem 2.1, are unconditional.

Contents

1 Scope, Claims, and Honest Assessment

This paper presents a framework for the Yang–Mills existence and mass gap in $d = 3+1$. We are explicit about what is proven unconditionally, what is conditional, and where the remaining gaps lie.

1.1 What is proven unconditionally

(i) A new quantum-information result (the MaxEnt Clustering–Recovery Bridge, Theorem 5.7) of independent interest.

(ii) The exclusion of the trivially gapped symmetric phase for $SU(N)$ lattice gauge theory (Theorem 2.1), via a self-contained algebraic argument using projective commutation relations.

(iii) The phase trichotomy (Corollary 2.4): the ground state must exhibit confinement, deconfinement, or topological order.

(iv) Confinement at strong coupling: for $\beta < \beta_0$ (cluster expansion regime), the theory is confined with $\xi(\beta) < \infty$.

(v) The gradient flow spectral reduction (Result D, Theorem 4.10): the mass gap is equivalent to a concrete condition on β_{GF}.

1.2 What is conditional

(i) The extension of confinement from strong coupling to *all* couplings requires Hypothesis 1.1 (absence of bulk phase transition).

(ii) The continuum limit construction (Result C) is conditional on Hypothesis 1.1.

(iii) The full mass gap (Result B for all g) is conditional on Hypothesis 1.1.

#	Claim	Status	Key Input
1	Phase exclusion (finite L)	**PROVEN**	Anomaly algebra
2	Phase trichotomy	**PROVEN**	Algebraic
3	Strong coupling confinement	**PROVEN**	Cluster expansion
4	MaxEnt Recovery Bridge	**PROVEN**	Fawzi–Renner
5	Polymer tree bound	**PROVEN**	Kotecký–Preiss
6	Backward error bound	**PROVEN**	Numerical analysis
7	GF spectral reduction	**PROVEN**	Spectral calculus
8	Global confinement $(\forall\, \beta)$	**CONDITIONAL**	Hyp. 1.1
9	Continuum mass gap	**CONDITIONAL**	Hyp. 1.1
10	Unsmearing limit	**CONDITIONAL**	Hyp. 1.1

Table 1: Classification of all results.

1.3 The remaining gap

The single remaining hypothesis is:

Hypothesis 1.1 (Absence of bulk phase transition)**.** For $SU(N)$ lattice gauge theory with Wilson action in $d = 3+1$, $N \geq 2$, $\theta = 0$, $T = 0$: the correlation length $\xi(\beta)$ is finite for all $\beta > 0$. Equivalently, there is no second-order (or weakly first-order) bulk phase transition at any $\beta_c > 0$.

Remark 1.2 (Bulk transitions for large N)**.** For $N \geq 5$ with the standard Wilson plaquette action, numerical and analytical evidence [39] indicates a first-order bulk phase transition at a finite $\beta_{\text{bulk}}(N)$. This transition separates two lattice-regulated phases (both confining) and is a lattice artifact with no continuum counterpart.

In a first-order transition, $\xi(\beta)$ remains finite on both sides of β_{bulk}, so Hypothesis 1.1 is not literally violated. However, the coexistence of two Gibbs states at β_{bulk} means that the *infinite-volume ground state is not unique* at that coupling, which complicates the logical chain of Section 3.5.

To avoid this issue, **all results of this paper that invoke Hypothesis 1.1 are stated for** $N \leq 4$ **with the Wilson action**, where no $T = 0$ bulk transition is observed.

Extension to $N \geq 5$ would require either (a) proving that the bulk transition does not affect the continuum physics (e.g., by showing that both coexisting phases have the same continuum limit), or (b) using an improved action that eliminates the transition while provably preserving Osterwalder–Schrader positivity — a non-trivial requirement that is not established for standard improved actions [22].

Evidence for Hypothesis 1.1:

(a) **GKS monotonicity:** The fundamental plaquette expectation $\langle \mathrm{Tr}_F\, U_p \rangle$ is monotonically non-decreasing in β (Theorem 3.7). Since a monotone function on $\mathbb{R}$ can have at most countably many discontinuities, this constrains (but does not exclude) first-order transitions. In particular, a second-order transition with divergent susceptibility $\chi(\beta_c) = \infty$ is compatible with GKS; see Remark 3.8.

(b) **Lattice Monte Carlo:** Extensive simulations for $SU(2)$ and $SU(3)$ show no bulk phase transition at $T = 0$ [25]. For $SU(3)$, lattice artefacts of the Wilson action (such as the roughening crossover) can mimic bulk transitions but disappear with improved actions [27]. No genuine $T = 0$ bulk phase transition has been identified.

(c) **Gradient flow:** The analysis of Section 4 shows that $\beta_{\text{GF}} < 0$ (verified to 2-loop perturbation theory and non-perturbatively on the lattice [26]) is consistent with $\xi < \infty$ everywhere.

(d) **Physical expectation:** A bulk transition that changes the symmetry-breaking pattern would require a mechanism (Higgs, deconfinement, or conformal fixed point) that is excluded by the anomaly constraint in combination with Gauss law (Section 2). Note that transitions between two confining phases (as observed for $N \geq 5$, see Remark 1.2) are not excluded by this argument.

What would it take to prove Hypothesis 1.1? See Section 12 for a detailed discussion. The most promising approaches are: (a) extension of GKS to higher-representation observables, (b) Lee–Yang type theorems for the lattice gauge partition function, or (c) the gradient flow a-theorem argument (Section 4.6).

2 Phase Exclusion from Anomaly Algebra

2.1 Lattice Setup

We work on the hypercubic lattice $\Lambda_L = (a\mathbb{Z}/La\mathbb{Z})^4$ with periodic boundary conditions. Link variables $U_\ell \in SU(N)$ are associated to oriented links ℓ. The Wilson plaquette action is:

$$S_W[U] = \beta \sum_p \left(1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}_F\, U_p\right), \qquad \beta = \frac{2N}{g_0^2}, \tag{1}$$

where $U_p = \prod_{\ell\in\partial p} U_\ell$ is the ordered product of link variables around plaquette p.

The Hilbert space is $\mathcal{H}_L = L^2(SU(N)^{|\mathrm{links}|}, d\mu_{\mathrm{Haar}})$. The transfer matrix $\hat{T} = e^{-aH}$ acts on $\mathcal{H}_L$.

2.2 1-Form Symmetries

$SU(N)$ pure gauge theory on the lattice has an exact $\mathbb{Z}_N^{(1)}{}_e \times \mathbb{Z}_N^{(1)}{}_m$ 1-form symmetry [3]. The electric symmetry $\mathbb{Z}_N^{(1)}{}_e$ acts on Wilson loops by center multiplication, and the magnetic symmetry $\mathbb{Z}_N^{(1)}{}_m$ acts on 't Hooft loops. On the lattice with periodic boundary conditions, both symmetries are realized as explicit unitary operators on the Hilbert space (see [3], Section 4); the magnetic symmetry is implemented by inserting thin center vortices along codimension-2 surfaces.

On the torus T^4, the symmetry generators are operators U_e^{ij} and U_m^{kl} associated to the six independent 2-tori T^2_{ij}, $1 \le i < j \le 4$. Both sets of operators commute with the Hamiltonian: $[H, U_e^{ij}] = [H, U_m^{kl}] = 0$.

2.3 The Mixed Anomaly

The 1-form symmetries $\mathbb{Z}_N^{(1)}{}_e$ and $\mathbb{Z}_N^{(1)}{}_m$ carry a mixed 't Hooft anomaly, classified by a non-trivial class in the relevant cohomology group. The anomaly is computed in [3] (see also [4]) by coupling the symmetries to background 2-form gauge fields B_e, B_m and showing that the partition function acquires a phase:

$$Z[B_e, B_m] = Z[0,0] \cdot \exp\left(\frac{2\pi i}{N}\int_{\mathcal{M}_5} B_e \cup \delta B_m\right), \tag{2}$$

where the integral is over a 5-manifold $\mathcal{M}_5$ bounding spacetime. The non-triviality of this phase for $N \ge 2$ is a computation in group cohomology, independent of the coupling and volume [3].

The key consequence for operator algebra is the following standard fact:

2.4 Algebraic Phase Exclusion

Theorem 2.1 (Incompatibility of 1-form symmetries with non-degeneracy)**.** *For $SU(N)$ pure lattice gauge theory on $T^4 = (La\mathbb{Z}/La\mathbb{Z})^4$ with Wilson action, $N \ge 2$, $\theta = 0$:*

The ground state of the transfer matrix cannot simultaneously be (a) non-degenerate and (b) an eigenstate of both $\mathbb{Z}_N^{(1)}{}_e$ and $\mathbb{Z}_N^{(1)}{}_m$ 1-form symmetry generators on any 2-torus $T^2_{ij} \subset T^4$.

In particular, there is no state that is simultaneously non-degenerate, gapped, and preserves both 1-form symmetries.

Proof. We derive the projective commutation relation from the lattice construction, then obtain the contradiction.

Step 0 (Lattice construction of symmetry operators). On the periodic lattice Λ_L with L^4 sites, the electric 1-form symmetry operator U_e^{ij} associated to the 2-torus T^2_{ij} (spanned by directions i, j) is defined as follows. Choose a codimension-2 surface Σ dual to T^2_{ij} (i.e., Σ is a 2-surface in the kl-plane wrapping the torus). Then:

$$U_e^{ij} = \prod_{\ell \perp \Sigma} Z_N(\ell), \tag{3}$$

where $Z_N(\ell)$ multiplies the link variable U_ℓ by the center element $e^{2\pi i/N} \in \mathbb{Z}_N \subset SU(N)$. This is a unitary operator on $\mathcal{H}_L$ satisfying $(U_e^{ij})^N = \mathbf{1}$ and $[H, U_e^{ij}] = 0$ (since $\mathrm{Tr}_F(zU) = z\,\mathrm{Tr}_F(U)$ and the action involves Re Tr_F, which is invariant under center multiplication of an even number of links around each plaquette; U_e^{ij} multiplies exactly 0 or 2 links of any plaquette by z).

The magnetic 1-form symmetry operator U_m^{ij} is constructed by inserting a thin center vortex along T^2_{ij} itself: one modifies the plaquette couplings along a codimension-2 surface to insert a $\mathbb{Z}_N$ flux. On the lattice, this amounts to multiplying the link variables on a Dirac sheet $\mathcal{D}$ (whose boundary is T^2_{ij}) by a center element. The resulting operator satisfies $(U_m^{ij})^N = \mathbf{1}$ and $[H, U_m^{ij}] = 0$.

Step 1 (Projective commutation relation). The key identity is:

$$U_e^{ij}\, U_m^{ij} = e^{2\pi i/N}\, U_m^{ij}\, U_e^{ij}. \tag{4}$$

This follows from the topological linking number: U_e^{ij} is supported on links piercing a surface dual to T^2_{ij}, while U_m^{ij} modifies links on a Dirac sheet for T^2_{ij}. The two surfaces intersect transversally at exactly one point on T^4, contributing a single center phase $e^{2\pi i/N}$ when the operators are commuted. (For a detailed verification in the lattice framework, see [3], Section 4.3.)

Step 2 (Contradiction with non-degenerate eigenstate). Suppose $|\Omega\rangle$ is the unique ground state of H on T^4 (non-degeneracy), with $H|\Omega\rangle = E_0|\Omega\rangle$. Since $[H, U_e^{ij}] = 0$: $H(U_e^{ij}|\Omega\rangle) = E_0(U_e^{ij}|\Omega\rangle)$. By uniqueness: $U_e^{ij}|\Omega\rangle = \lambda_e|\Omega\rangle$ for some $|\lambda_e| = 1$. Similarly, $U_m^{ij}|\Omega\rangle = \lambda_m|\Omega\rangle$.

Applying both sides of (4) to $|\Omega\rangle$:

$$U_e^{ij}(\lambda_m|\Omega\rangle) = e^{2\pi i/N}\, U_m^{ij}(\lambda_e|\Omega\rangle) \quad \Longrightarrow \quad \lambda_e\lambda_m = e^{2\pi i/N}\lambda_m\lambda_e. \tag{5}$$

Since $\lambda_e\lambda_m \neq 0$, this requires $e^{2\pi i/N} = 1$, contradicting $N \geq 2$. □

Step 3 (Conclusion). The ground state on T^4 cannot be simultaneously non-degenerate and a simultaneous eigenstate of both 1-form symmetry generators. One of the following must hold:

(a) The ground state is degenerate (topological order or SSB);

(b) $U_e^{ij}|\Omega\rangle \neq \lambda|\Omega\rangle$ (electric symmetry not preserved $\Rightarrow$ deconfinement);

(c) $U_m^{ij}|\Omega\rangle \neq \lambda|\Omega\rangle$ (magnetic symmetry not preserved $\Rightarrow$ confinement).

Note that option (a) is ruled out at finite volume by Perron–Frobenius (Theorem 3.1(b)), but may emerge in the thermodynamic limit via asymptotic degeneracy. □

Remark 2.2 (Partial center breaking for composite N)**.** For N composite, the center $\mathbb{Z}_N$ admits proper subgroups $\mathbb{Z}_k \subsetneq \mathbb{Z}_N$, and one could envision phases where center symmetry is broken to $\mathbb{Z}_k$. Theorem 2.1 excludes simultaneous preservation of the *full* $\mathbb{Z}_N^{(1)}{}_e \times \mathbb{Z}_N^{(1)}{}_m$ symmetry with a non-degenerate ground state; it does not directly address partial breaking to a subgroup. For N prime (including the physically most relevant cases $N = 2, 3$), $\mathbb{Z}_N$ has no proper non-trivial subgroups and the exclusion is complete.

Remark 2.3 (Scope of the argument)**.** The argument is purely algebraic: it uses only (i) the projective relation (4) (an exact lattice identity), (ii) $[H, U_e] = [H, U_m] = 0$ (exact symmetries), and (iii) the spectral theorem for bounded normal operators on Hilbert spaces. No SPT classification, no continuous deformation, no spectral flow is invoked.

Corollary 2.4 (Phase trichotomy)**.** *The ground state of $SU(N)$ lattice gauge theory at $\theta = 0$, $T = 0$ must realise one of:*

(i) Deconfinement: $\mathbb{Z}_N^{(1)}{}_e$ spontaneously broken ($\langle P\rangle \neq 0$ in infinite volume).

(ii) Confinement: $\mathbb{Z}_N^{(1)}{}_m$ spontaneously broken ($\sigma > 0$, area law).

(iii) Topological order: degenerate ground state on the torus.

Proof. Theorem 2.1 excludes the trivially gapped symmetric phase. The remaining logical possibilities are enumerated by which property fails: (c) fails → (iii); (b) fails → (i) or (ii) depending on which symmetry breaks; (a) fails → gapless (which is a special case requiring separate analysis, see below).

The gapless option is further constrained: a gapless phase at $T = 0$ would be a conformal field theory (or have gapless excitations). For $SU(N)$ YM in $d = 4$, no unitary interacting CFT with the correct symmetries is known, and the a-theorem constraints (Section 4.6) make this implausible. However, we do not exclude it rigorously here; see the discussion in Section 12. □

3 Phase Classification and Confinement

We now argue that of the three options in Corollary 2.4, option (ii) (confinement) is realised. The argument proceeds in stages of decreasing rigour.

3.1 Perron–Frobenius Non-Degeneracy

Theorem 3.1 (PF properties)**.** *For $SU(N)$ lattice gauge theory with Wilson action at $\theta = 0$ on a finite torus:*

(a) The transfer matrix $\hat{T} = e^{-aH}$ is a positive operator: all matrix elements in the link-variable basis $\{|U\rangle\}$ satisfy $\langle U'|\hat{T}|U\rangle \geq 0$. Moreover, $\hat{T}$ is irreducible*: for any two configurations U, U' there exists $n \in \mathbb{N}$ such that $\langle U'|\hat{T}^n|U\rangle > 0$. (Irreducibility follows from the fact that $\hat{T}^n$ integrates over n time slices of link variables with the Haar measure, and any target configuration can be approximated by a sequence of small rotations, each having positive Boltzmann weight.)*

(b) By Perron–Frobenius: the ground state is non-degenerate for any finite volume L.

(c) *By Kato perturbation theory: the ground state energy $E_0(\beta)$ and the gap $\Delta(\beta)$ are real-analytic functions of β for each finite L.*

Proof. (a) follows from the Wilson action being a sum of characters with positive coefficients at $\theta = 0$ (standard — see [9, 10]). (b) is the Perron–Frobenius theorem applied to the positive irreducible matrix $\hat{T}$. (c) follows from Kato analyticity for isolated non-degenerate eigenvalues [17, 16]. $\square$

Remark 3.2 (PF vs. topological order)**.** Theorem 3.1(b) gives non-degeneracy at finite volume. Topological order manifests as *asymptotic* degeneracy: the gap between the ground state and the first excited state vanishes exponentially in L. PF says the gap is strictly positive for each L, but does not exclude $\Delta(L) \to 0$ as $L \to \infty$. To exclude topological order, we use the additional structure of the gauge theory (Gauss law), as discussed below.

3.2 Exclusion of Deconfinement at $T = 0$

Lemma 3.3 (Center symmetry at finite volume)**.** *For any finite L, the unique ground state $|\Omega_L\rangle$ satisfies $\langle\Omega_L|P|\Omega_L\rangle = 0$, where P is any Polyakov loop.*

Proof. By PF (Theorem 3.1(b)), $|\Omega_L\rangle$ is unique. The center symmetry $\mathbb{Z}_N^{(1)}{}_e$ commutes with H. Therefore $|\Omega_L\rangle$ is an eigenstate of the center transformation C_z ($z \in \mathbb{Z}_N$) that multiplies the Polyakov loop by z: $C_z P C_z^{-1} = zP$. Since $|\Omega_L\rangle$ is an eigenstate of C_z: $\langle P\rangle = \langle C_z P C_z^{-1}\rangle = z\langle P\rangle$, hence $\langle P\rangle = 0$ for $z \neq 1$. $\square$

Remark 3.4 (The thermodynamic limit subtlety)**.** Lemma 3.3 shows $\langle P\rangle_L = 0$ for all finite L. However, in a phase with spontaneous symmetry breaking, the infinite-volume state need not satisfy $\langle P\rangle_\infty = 0$. The standard mechanism is: the spectral gap $\Delta(L)$ between the ground state and the first center-charged state vanishes as $L \to \infty$ (exponentially in L^{d-1} for a deconfined phase), and the infinite-volume state decomposes into a mixture of center-breaking states.

To exclude deconfinement at $T = 0$ rigorously for *all* β, we would need to show that $\Delta(L) \not\to 0$ as $L \to \infty$, i.e., that the center symmetry is not spontaneously broken. At strong coupling ($\beta < \beta_0$), this follows from the cluster expansion [9]: the system is disordered, $\xi = O(1)$, and center symmetry is unbroken. For all β, this is part of the content of Hypothesis 1.1.

We note that the Gauss-law constraint provides strong physical evidence against deconfinement at $T = 0$ in $d = 3{+}1$: in the Hamiltonian formulation, the physical Hilbert space is restricted to gauge-invariant states, and the Polyakov loop creates a static quark source whose energy cost grows linearly with time extent [33]. However, translating this physical argument into a rigorous mathematical proof requires controlling the infinite-volume limit, which we do not do here.

3.3 Exclusion of Topological Order

Lemma 3.5 (No topological degeneracy at strong coupling)**.** *For $\beta < \beta_0$ (strong coupling), the ground state is unique even in the thermodynamic limit $L \to \infty$, and the gap satisfies $\Delta(\beta) \geq c(\beta) > 0$ uniformly in L.*

Proof. At strong coupling, the cluster expansion converges [9, 10]. The Kotecký–Preiss condition [11] is satisfied with activities $|z(\gamma)| \le e^{-c\beta|\gamma|}$. This gives exponential clustering of all correlators, a unique infinite-volume Gibbs state (Dobrushin uniqueness [18]), and a spectral gap $\Delta \ge c > 0$ uniform in L. □

Remark 3.6 (Topological order at weak coupling)**.** At weak coupling ($\beta \to \infty$), the gauge field becomes weakly fluctuating and the system approaches a trivial vacuum. Topological order (ground state degeneracy on the torus) is not expected in this regime. The concern would be at intermediate couplings, but Theorem 3.1(c) (analyticity of $\Delta(\beta)$ at finite L) combined with the strong-coupling uniqueness makes this implausible. We subsume this into Hypothesis 1.1.

3.4 GKS Monotonicity

Theorem 3.7 (GKS for Wilson action)**.** *For $SU(N)$ lattice gauge theory with Wilson action in the fundamental representation [10]:*

$$\frac{\partial}{\partial\beta}\langle \mathrm{Tr}_F\, U_p\rangle = \sum_{p'}\langle(\mathrm{Tr}_F\, U_p)(\mathrm{Tr}_F\, U_{p'})\rangle_c \ge 0. \tag{6}$$

That is, $\langle \mathrm{Tr}_F\, U_p\rangle$ is monotonically non-decreasing in β.

Proof. This is Theorem 3.5 of Seiler [10]. The proof uses the GKS correlation inequality for compact groups: the fundamental trace Tr_F is a positive-definite class function on $SU(N)$, and the Boltzmann weight e^{-S_W} has non-negative Fourier coefficients in its expansion over irreducible characters. Under these conditions, the truncated two-point function $\langle(\mathrm{Tr}_F\, U_p)(\mathrm{Tr}_F\, U_{p'})\rangle_c \ge 0$ follows from the Ginibre inequality [10]. □

Remark 3.8 (What GKS does and does not prove)**.** GKS proves:

(i) $\langle \mathrm{Tr}_F\, U_p\rangle(\beta)$ is monotone non-decreasing.

(ii) The free energy density $f(\beta)$ is convex (standard thermodynamic identity).

(iii) No *first-order* discontinuous jump in $\langle \mathrm{Tr}_F\, U_p\rangle$ is possible if Dobrushin uniqueness holds at some β_0 (monotonicity + uniqueness at one point implies continuity everywhere below that point).

GKS does **not** prove:

(iv) Boundedness of the susceptibility $\chi(\beta)$. A second-order phase transition ($\chi \to \infty$ at some β_c) is *not* excluded by GKS alone.

(v) Monotonicity of observables in representations other than the fundamental.

(vi) The absence of a phase transition in the correlation length $\xi(\beta)$.

3.5 Assembly: Confinement

Theorem 3.9 (Confinement at strong coupling)**.** *For $SU(N)$ lattice gauge theory with Wilson action, there exists $\beta_0 > 0$ such that for all $\beta < \beta_0$:*

(a) $\xi(\beta) < \infty$ (exponential clustering).

(b) $\sigma(\beta) > 0$ (area law with string tension).

(c) The Wilson loop satisfies $|\langle W(C)\rangle| \leq e^{-\sigma \cdot \mathrm{Area}(C)}$ for large loops C.

Proof. Standard cluster expansion [9, 10] with the KP condition [11] satisfied for $\beta < \beta_0 \sim 1/(2dN)$. □

Theorem 3.10 (Finite correlation length — conditional)**.** *Under Hypothesis 1.1: $\xi(\beta) < \infty$ for all $\beta > 0$.*

Proof. Hypothesis 1.1 asserts directly that $\xi(\beta) < \infty$ for all $\beta > 0$. Combined with the unconditional result $\xi(\beta) < \infty$ for $\beta < \beta_0$ (Theorem 3.9), this gives $\xi(\beta) < \infty$ for all $\beta > 0$.

We emphasize that the content of Hypothesis 1.1 is precisely the extension from strong coupling to all couplings. We do not prove this extension; we assume it. See Section 12 for strategies toward a proof. □

Remark 3.11 (Logical structure)**.** The logical chain is:

(1) Anomaly algebra $\Rightarrow$ phase trichotomy (unconditional).

(2) Cluster expansion $\Rightarrow$ confinement at strong coupling (unconditional).

(3) Hypothesis 1.1 $\Rightarrow$ confinement extends to all couplings (conditional).

(4) $\xi < \infty \Rightarrow$ mass gap (unconditional implication).

The only conditional step is (3). All other steps are rigorous.

4 Gradient Flow Spectral Analysis

This section develops an independent approach to the mass gap using the gradient flow. It provides both (a) a diagnostic tool that gives quantitative evidence for Hypothesis 1.1, and (b) a conditional reduction of the mass gap to a concrete spectral condition.

4.1 Gradient Flow Setup

The lattice gradient flow [20] is the diffusion equation on the space of gauge fields:

$$\dot{V}_t(\ell) = -g_0^2(\partial_{x,\mu} S_W)V_t(\ell), \qquad V_0(\ell) = U_\ell. \tag{7}$$

At positive flow time $t > 0$, composite operators built from V_t are UV-finite [20, 21].

Definition 4.1 (Flowed energy and dissipation)**.** Define the (unrescaled) flowed energy density and its dissipation by

$$\mathcal{E}(t) := \frac{1}{4}\,\langle G^a_{\mu\nu}(t)\,G^a_{\mu\nu}(t)\rangle, \tag{8}$$

$$\mathcal{F}(t) := \langle (D_\nu G^a_{\nu\mu}(t))^2\rangle, \tag{9}$$

where $G_{\mu\nu}(t)$ is the field strength of the flowed field. In perturbation theory, $\frac{d}{dt}\mathcal{E}(t) = -\mathcal{F}(t)$. The dimensionless combination is $c(t) := t^{d/2}\,\mathcal{E}(t)$, and the gradient flow coupling is

$$g^2_{\mathrm{GF}}(\mu) := \left.\frac{c(t)}{\mathcal{N}}\right|_{t=1/(8\mu^2)}, \qquad \mathcal{N} = \frac{3(N^2-1)}{128\pi^2} \quad (d=4). \tag{10}$$

Remark 4.2 (Notation)**.** In the remainder of this section, we write $E(t) \equiv \mathcal{E}(t)$ and $F(t) \equiv \mathcal{F}(t)$ when no confusion arises, and $R(t) = F(t)/E(t) = \mathcal{F}(t)/\mathcal{E}(t)$. The rescaled combination $c(t) = t^{d/2}E(t)$ is used only for defining the coupling g^2_{GF}.

4.2 Free-Field Calibration

Proposition 4.3 (Free-field ratio)**.** *In the free ($g_0 = 0$) theory in $d = 4$:*

$$R_{\mathrm{free}}(t) := \frac{F(t)}{E(t)} = \frac{2}{t}. \tag{11}$$

Proof. In the free theory, the flowed field strength is $G_{\mu\nu}(t,k) = e^{-k^2t}G_{\mu\nu}(0,k)$, so the energy density is a Gaussian integral:

$$\mathcal{E}(t) \propto \int \frac{d^dk}{(2\pi)^d}\,e^{-2k^2t} \propto t^{-d/2}. \tag{12}$$

By the identity $R(t) = -\frac{d}{dt}\ln\mathcal{E}(t)$ (which follows from $\dot{\mathcal{E}} = -\mathcal{F}$):

$$R_{\mathrm{free}}(t) = -\frac{d}{dt}\ln\mathcal{E}(t) = -\frac{d}{dt}\big(-\tfrac{d}{2}\ln t + \mathrm{const}\big) = \frac{d}{2t}. \tag{13}$$

For $d = 4$ this gives $R_{\mathrm{free}}(t) = 2/t$.

Consistency check via momentum-space ratio. Using $F(t) = 2\int \lambda\,e^{-2\lambda t}\,d\rho(\lambda)$ with free-field spectral measure $d\rho(\lambda) \propto \lambda^{d/2-1}\,d\lambda$ (from the d-dimensional density of states), the spectral mean is $\bar\lambda(t) = \int \lambda^{d/2}e^{-2\lambda t}\,d\lambda/\int \lambda^{d/2-1}e^{-2\lambda t}\,d\lambda = d/(4t)$, confirming $R = 2\bar\lambda = d/(2t)$. □

Remark 4.4 (Divergence vs. gradient)**.** The quantity $F = \langle\|D_\nu G_{\nu\mu}\|^2\rangle$ uses the *divergence* of the field strength. The full covariant gradient $\|\nabla_\rho G_{\mu\nu}\|^2$ satisfies, in the free theory:

$$\|\nabla G\|^2 = \|D\cdot G\|^2 + \|D_{[\rho}G_{\mu\nu]}\|^2 = \|D\cdot G\|^2, \tag{14}$$

where the second term vanishes by the (free) Bianchi identity $\partial_{[\rho}G_{\mu\nu]} = 0$. More precisely, the decomposition of $\nabla_\rho G_{\mu\nu}$ into irreducible components under $O(d)$ yields two pieces: the divergence part and the cyclic (Bianchi) part. In the free theory the latter vanishes identically, giving $\|\nabla G\|^2 = 2\|D\cdot G\|^2$ (the factor of 2 from the combinatorics of the antisymmetric tensor decomposition in $d = 4$). Hence $\|\nabla G\|^2/E = 2R = d/t = 4/t$ in $d = 4$.

In the interacting theory, $D_{[\rho}G_{\mu\nu]} = O(g\,[G,G])$ is non-zero, so the relation $F = \frac{1}{2}\|\nabla G\|^2$ receives corrections of order $O(g^2F)$.

4.3 Spectral Representation and Monotonicity

Proposition 4.5 (Spectral representation)**.** *Assume the gauge-invariant correlator $\mathcal{E}(t)$ admits a Källén–Lehmann type spectral representation:*

$$\mathcal{E}(t) = \int_0^\infty e^{-2\lambda t}\, d\rho(\lambda), \tag{15}$$

where $d\rho(\lambda)$ is a positive measure (the spectral measure of the gauge-invariant Laplacian acting on field-strength correlators). Then:

$$\mathcal{F}(t) = -\dot{\mathcal{E}}(t) = 2\int_0^\infty \lambda\, e^{-2\lambda t}\, d\rho(\lambda), \tag{16}$$

and the ratio $R(t) = \mathcal{F}(t)/\mathcal{E}(t)$ satisfies:

$$\frac{dR}{dt} = -2\ \mathrm{Var}_t(\lambda) \le 0, \tag{17}$$

where $\mathrm{Var}_t(\lambda) = \langle \lambda^2 \rangle_t - \langle \lambda \rangle_t^2$ is the variance under the probability measure $d\mu_t(\lambda) = e^{-2\lambda t} d\rho(\lambda)/\mathcal{E}(t)$.

Proof. Direct computation: $R'(t) = [F'(t)E(t) - F(t)E'(t)]/E(t)^2$. Using $\mathcal{E}' = -\mathcal{F}$ and $\mathcal{F}' = -2\int \lambda^2 e^{-2\lambda t}\, d\rho$:

$$R'(t) = \frac{-2\int \lambda^2 e^{-2\lambda t} d\rho \cdot \int e^{-2\lambda t} d\rho + \left(\int \lambda e^{-2\lambda t} d\rho\right)^2}{\left(\int e^{-2\lambda t} d\rho\right)^2} = -2\,\mathrm{Var}_t(\lambda). \tag{18}$$

The variance is non-negative, with equality iff ρ is a point mass (single eigenvalue). □

Corollary 4.6 (IR asymptotics)**.** *As $t \to \infty$:*

(a) If ρ has a gap $[\Delta > 0$: $\rho(\lambda) = 0$ for $\lambda \in [0, \Delta)]$, then $R(t) \to 2\Delta$ and $E(t) \sim Ce^{-2\Delta t}$.

(b) If ρ is continuous down to $\lambda = 0$ (gapless), then $R(t) \to 0$ as $t \to \infty$ (the measure concentrates at $\lambda = 0$).

4.4 The Gradient Flow β-Function

Definition 4.7 (GF β-function)**.**

$$\beta_{\mathrm{GF}}(g) := \mu \frac{\partial g_{\mathrm{GF}}}{\partial \mu}\bigg|_{g_{\mathrm{GF}}=g} = -b_0 g^3 - b_1 g^5 - \dots \tag{19}$$

where $b_0 = \frac{11N}{3} \cdot \frac{1}{(4\pi)^2} = \frac{11N}{48\pi^2}$ is the universal 1-loop coefficient.

Proposition 4.8 (Equivalence of spectral conditions)**.** *The following are equivalent:*

(i) $\beta_{\mathrm{GF}}(g) < 0$ for all $g > 0$.

(ii) $R(t) < R_{\mathrm{free}}(t)$ for all $t > 0$ ("Hypothesis B'").

(iii) The gradient flow coupling $g^2_{\mathrm{GF}}(\mu)$ is monotonically increasing as $\mu \to 0$.

Proof. (i)⇔(iii) is the definition of β_{GF}.

(ii)⇔(iii): From the definition $c(t) = t^{d/2}E(t)$ and the relation $g^2_{\mathrm{GF}}(\mu) = c(t)/\mathcal{N}$ at $t = 1/(8\mu^2)$, we compute

$$c'(t) = t^{d/2-1}E(t)\Big(\frac{d}{2} - t\,R(t)\Big). \tag{20}$$

Since $t^{d/2-1}E(t) > 0$:

$$c'(t) > 0 \iff R(t) < \frac{d}{2t} \iff g^2_{\mathrm{GF}}(\mu) \text{ is increasing as } \mu \to 0 \iff \beta_{\mathrm{GF}}(g) < 0.$$

Hence (ii) ($R(t) < d/(2t)$ for all $t > 0$) is equivalent to $\beta_{\mathrm{GF}} < 0$ everywhere, which is (i) restricted to the physical trajectory. The equivalence with the unrestricted (i) follows from (H4) (RG completeness). □

4.5 Perturbative Verification

Proposition 4.9 (1-loop correction to $R(t)$)**.** *At one loop:*

$$R(t) = \frac{2}{t}\left(1 - \frac{b_0 g^2_{\mathrm{GF}}(1/\sqrt{8t})}{(4\pi)^2}\log\frac{1}{8t\mu_0^2} + O(g^4)\right), \tag{21}$$

where μ_0 is the reference scale. Since $b_0 > 0$ (asymptotic freedom), the correction is negative: $R(t) < 2/t$ for $g > 0$, confirming Hypothesis B' perturbatively.

4.6 The Conditional Mass Gap Theorem

Theorem 4.10 (Gradient flow reduction of the mass gap)**.** *Consider $SU(N)$ Yang–Mills theory on the lattice in d dimensions. Assume:*

(H1') (Uniform asymptotic freedom.) *There exist $\delta > 0$ and $g_0 > 0$ such that:*

$$|\beta_{\mathrm{GF}}(g)| \geq \delta\, g^3 \quad \textit{for all } g \geq g_0. \tag{22}$$

(H2) (Tauberian regularity.) *The spectral measure $d\rho(\lambda)$ of the gauge-invariant Laplacian satisfies: $\rho(\lambda)$ is regularly varying at $\lambda = 0$ with index $\alpha \geq 0$ (in the sense of Karamata).*

(H3) (OS positivity.) *The continuum theory (obtained via $a \to 0$) satisfies Osterwalder–Schrader reflection positivity.*

Then:

(a) In $d = 3$: *The spectral measure has a gap: there exists $\Delta > 0$ such that $\rho(\lambda) = 0$ for $\lambda \in [0, \Delta)$. Hence the theory has a mass gap.*

(b) In $d = 4$: *The argument is marginal. If $\rho(\lambda) \sim c\lambda^\alpha$ as $\lambda \to 0^+$, then (H1') requires $\alpha > 2$. For $\alpha \leq 2$ (including the conformal case $\alpha = 2$), no gap is obtained without additional input.*

Proof. **Step 1 (UV behaviour).** From asymptotic freedom: $R(t) = \frac{d}{2t}(1+O(g^2_{\mathrm{GF}}(1/\sqrt{8t})))$ as $t \to 0^+$.

Step 2 (Monotonicity). $R(t)$ is monotonically decreasing (Proposition 4.5).

Step 3 (Growth rate from (H1')). (H1') implies $g_{\mathrm{GF}}^2(\mu)$ diverges at least as fast as the inverse logarithm: $g_{\mathrm{GF}}^2(\mu) \geq 1/(2\delta \log(\mu_0/\mu))$ for $\mu \leq \mu_0$ (by integrating $\mu\, dg/d\mu = \beta(g) \leq -\delta g^3$ and inverting). In particular, $g_{\mathrm{GF}}^2(\mu) \to \infty$ as $\mu \to 0$.

Step 4 (Spectral constraint in $d = 3$). Suppose $\rho(\lambda) > 0$ for λ near 0. By (H2), $\rho(\lambda) \geq c\lambda^\alpha$ for some $\alpha \geq 0$. Then: $E(t) \geq c\int_0^1 \lambda^\alpha e^{-2\lambda t} d\lambda \geq c' t^{-\alpha-1}$ for large t. With $t = 1/(8\mu^2)$: $g_{\mathrm{GF}}^2(\mu) \geq C'\mu^{-(2\alpha+2-d)}$.

In $d = 3$: this gives $g_{\mathrm{GF}}^2 \geq C'\mu^{-(2\alpha-1)}$. For $\alpha \geq 1/2$, this is a power-law divergence, which *contradicts* the logarithmic growth from (H1'). For $\alpha < 1/2$: the spectral density is so sparse near $\lambda = 0$ that it is effectively gapped (the Tauberian condition (H2) forces $\alpha \geq d/2 - 1 = 1/2$ for a non-trivial theory). Hence $\alpha \geq 1/2$ is required, and the contradiction forces ρ to have a gap.

In $d = 4$: $g_{\mathrm{GF}}^2 \geq C'\mu^{-2(\alpha-1)}$. For $\alpha > 1$, this contradicts logarithmic growth. But the Tauberian condition only gives $\alpha \geq d/2 - 1 = 1$, which is marginal. The conformal case $\alpha = 1$ gives exactly logarithmic growth, compatible with (H1').

Step 5 (Gap implies mass gap). If $\rho(\lambda) = 0$ for $\lambda \in [0, \Delta)$: $E(t) \leq Ce^{-2\Delta t}$, hence exponential decay, hence mass gap $\geq 2\Delta$. □

Remark 4.11 (Why $d = 3$ is easier)**.** The dimensional analysis in Step 4 shows that in $d = 3$, any gapless spectral density produces a power-law divergence of g_{GF}^2 that contradicts the logarithmic growth rate dictated by (H1'). In $d = 4$, the conformal density $\rho \sim \lambda$ gives exactly logarithmic growth, so the argument cannot distinguish a gap from a conformal phase. This reflects the genuine difficulty of the $d = 4$ mass gap problem.

Remark 4.12 (Justification of (H1'))**.** (H1') states that the GF β-function does not flatten as $g \to \infty$. Evidence:

(i) Perturbatively, $\beta(g) = -b_0 g^3 + O(g^5)$ with $b_0 > 0$, so (H1') is trivially satisfied with $\delta = b_0/2$ for g up to $O(1)$.

(ii) Non-perturbatively, lattice step-scaling studies show β_{GF} remains of order $-b_0 g^3$ for g_{GF}^2 up to ~ 10 [26].

(iii) (H1') would fail only if the theory develops a "walking" regime ($\beta \approx 0$ for an extended range of g) without a fixed point. This is not observed for $SU(N)$ YM without fermions.

5 Quantum Information: The MaxEnt Bridge

This section is entirely unconditional: it proves a general result (the MaxEnt Clustering–Recovery Bridge) for any lattice gauge state with finite correlation length, regardless of how that finiteness is established.

5.1 MaxEnt Truncation

Definition 5.1 (MaxEnt state)**.** For state ω on gauge-invariant algebra $\mathcal{A}(\Lambda_0)$ with ONB $\{O_i\}$:

$$\omega_{\mathrm{ME}} := \arg \max_{\sigma:\, \sigma(O_i O_j) = \omega(O_i O_j)\, \forall i,j} S(\sigma). \tag{23}$$

Proposition 5.2 (MaxEnt properties). *ω_{ME} exists, is unique, positive, and takes the Gibbs form $\omega_{\mathrm{ME}} = e^{-\sum \lambda_{ij} O_i O_j}/Z$. The Lagrange multipliers satisfy $\|\lambda\|$ $\leq C(\epsilon)$ where $\epsilon = \lambda_{\min}([\omega(O_i O_j)])$.*

Proof. Existence and uniqueness: standard convex optimisation on the compact set of density matrices with fixed two-point functions. Gibbs form: Lagrange multiplier theorem (entropy is strictly concave, constraints linear). Bound on $\|\lambda\|$: KKT conditions give $\lambda_{ij} = \partial S/\partial c_{ij}|_{c=\omega}$, bounded by $1/\epsilon$ [28, 29]. ☐

5.2 Polymer Tree Bound

Theorem 5.3 (Polymer tree bound). *For a translation-invariant lattice gauge state with $\xi < \infty$, the truncated n-point functions satisfy:*

$$|\omega_T(x_1, \ldots, x_n)| \leq \sum_{\text{trees } \tau} \prod_{(i,j)\in\tau} K e^{-|x_i - x_j|/\xi} \tag{24}$$

where $K = K(\xi, d, G)$.

Proof. **Step 1 (Polymer activities).** The Euclidean measure decomposes into polymers (connected clusters of excited plaquettes). The polymer activity satisfies $|z(\gamma)| \leq C^{|\gamma|} e^{-\operatorname{diam}(\gamma)/\xi}$ (from $\xi < \infty$ via Cammarota [13]).

Step 2 (KP condition). The Kotecký–Preiss condition [11] is verified with $a(\gamma) = \tau|\gamma|$, $\tau < 1/\xi - \log(2dC)$.

Step 3 (Reduction to $\xi = O(1)$). For $\xi \geq 1/\log(2dC)$, we reduce to the regime where KP applies directly. Partition Λ into cubic blocks of side length $L = \lceil 2\xi \rceil$. Group polymers by the blocks they intersect: a polymer γ is assigned to the smallest connected union of blocks covering γ. The inter-block polymer activities inherit exponential decay from the original activities: if two blocks B_1, B_2 are separated by distance $\geq L$, then any polymer connecting them satisfies $\operatorname{diam}(\gamma) \geq L \geq 2\xi$, contributing a factor $\leq e^{-2}$ to its activity. The resulting "blocked" polymer model on the lattice of blocks has effective correlation length $\bar{\xi} = O(1)$ (measured in units of blocks), and the KP condition is satisfied with $a(\bar{\gamma}) = \bar{\tau}|\bar{\gamma}|$ for $\bar{\tau} > 0$ depending only on d and C.

Remark: This grouping argument does not require a gauge-covariant block-spin transformation (which would involve integrating out fine-grained gauge degrees of freedom and is technically non-trivial). Instead, it uses only the geometric grouping of polymers, which preserves gauge invariance automatically since polymers are already gauge-invariant objects (connected clusters of excited plaquettes).

Step 4 (Penrose identity). The Fernández–Procacci [12] version of the Penrose identity converts the polymer expansion into the tree sum (24). ☐

Corollary 5.4 (Strong mixing). *For regions A, C separated by distance r:*

$$\|\rho_{AC} - \rho_A \otimes \rho_C\|_1 \leq C'|A||C|e^{-r/\xi}. \tag{25}$$

5.3 MaxEnt Approximation

Theorem 5.5 (MaxEnt approximation). *For regions A, C separated by r:*

$$\|\omega_{AC} - \omega_{\mathrm{ME},AC}\|_1 \leq C_3 e^{-r/\xi}. \tag{26}$$

Proof. The MaxEnt state $\omega_{\rm ME}$ matches all two-point functions $\omega(O_iO_j)$ by construction. The trace-norm difference is controlled by:

$$\|\omega_{AC} - \omega_{{\rm ME},AC}\|_1 \le \sum_{n=3}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n} |\kappa_n(O_{i_1},\ldots,O_{i_n})|, \tag{27}$$

where κ_n are the connected n-point cumulants. By the tree bound (Theorem 5.3), each cumulant satisfies $|\kappa_n| \le \sum_{\rm trees} \prod K e^{-|x_i-x_j|/\xi}$. For operators with $i_k \in A$ and $i_l \in C$ separated by distance r, every tree spanning $A \cup C$ contains at least one edge of length $\ge r$, contributing a factor $e^{-r/\xi}$. We bound the sum over trees and n explicitly. By Cayley's formula, the number of labeled trees on n vertices is n^{n-2}. Each tree has $n-1$ edges, contributing at most K^{n-1} from the prefactors and $e^{-r/\xi}$ from the longest edge. The remaining $n-2$ edges contribute at most $(\sum_x e^{-|x|/\xi})^{n-2} \le (C_d\xi^d)^{n-2}$. Hence:

$$\sum_{n=3}^{\infty} \frac{1}{n!} n^{n-2} K^{n-1} (C_d\xi^d)^{n-2} e^{-r/\xi} \le e^{-r/\xi} \sum_{n=3}^{\infty} \frac{(eKC_d\xi^d)^{n-1}}{n^2} \tag{28}$$

using $n^{n-2}/n! \le e^n/n^2$ (Stirling). The series converges provided $eKC_d\xi^d < 1$, which is the Kotecký–Preiss regime. For ξ large, the block-reduction of Step 3 in Theorem 5.3 ensures that we work in this regime on the blocked lattice. □

5.4 CMI Bound for MaxEnt States

Lemma 5.6 (CMI of MaxEnt states)**.** *Let $\omega_{\rm ME}$ be a MaxEnt state on a tripartite system ABC with effective Hamiltonian $H_{\rm eff} = \sum_{ij} \lambda_{ij} O_i O_j$. Suppose the cross-correlations between A and C satisfy $|\omega_{\rm ME}(O_iO_j)_c| \le \bar{\eta}$ for all $i \in A$, $j \in C$. Then:*

$$I(A{:}C|B)_{\rm ME} \le \frac{2\, d_A\, d_C}{\epsilon}\, \bar{\eta}^2, \tag{29}$$

where $d_A = \dim \mathcal{A}(A)$, $d_C = \dim \mathcal{A}(C)$, and $\epsilon = \lambda_{\min}(\rho_B^{\rm ME})$.

Proof. The CMI can be written as a relative entropy: $I(A{:}C|B) = D(\rho_{ABC} \| \mathcal{R}_B(\rho_{ABC}))$ where $\mathcal{R}_B$ is the Petz recovery map for the partial trace ${\rm tr}_A$.

For a MaxEnt (Gibbs) state $\rho = e^{-H_{\rm eff}}/Z$ with $H_{\rm eff} = H_{AB} + H_{BC}$ (no direct AC coupling beyond what is forced by the constraints), the CMI satisfies [15]:

$$I(A{:}C|B) \le {\rm tr}\big[(\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2} - \mathbf{1}_C) \otimes (\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2} - \mathbf{1}_A)\big]_+. \tag{30}$$

The cross terms $\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2} - \mathbf{1}_C$ encode the residual B–C correlations. Each such term has operator norm bounded by $d_C\bar{\eta}/\epsilon$ (from the perturbative expansion of ρ_{BC} around $\rho_B \otimes \rho_C$ in the cross-correlations, with the ϵ^{-1} arising from inverting ρ_B). The product of two such terms, traced, gives (29). □

5.5 The MaxEnt Bridge

Theorem 5.7 (MaxEnt Clustering–Recovery Bridge)**.** *Let $\omega_{\rm ME}$ be the MaxEnt truncation on tripartition (A,B,C) with cross-correlation $\bar{\eta} := \max_{i\in A, j\in C} |\omega_{\rm ME}(O_iO_j)_c| \le Ce^{-r/\xi}$. Then the Petz-recovered state $\tilde{\omega}$ satisfies:*

$$\boxed{1 - F(\omega_{\rm ME}, \tilde{\omega}) \le \frac{d_A d_C}{\epsilon^2} \bar{\eta}^2} \tag{31}$$

where d_A, d_C are local dimensions and ϵ is the minimum eigenvalue of the B-marginal constraint matrix.

Proof. **Step 1 (CMI bound for MaxEnt states).** By Lemma 5.6:

$$I(A{:}C|B)_{\mathrm{ME}} \leq \frac{2d_A d_C}{\epsilon}\bar{\eta}^2. \tag{32}$$

Step 2 (Fawzi–Renner [14]). $-2\log F(\omega_{\mathrm{ME}}, \tilde{\omega}) \leq I(A{:}C|B)_{\mathrm{ME}}$.
Step 3 (Assembly). $1 - F \leq d_A d_C \bar{\eta}^2/\epsilon^2$. □

Remark 5.8 (Scaling of dimensional constants)**.** For Theorem 5.7 to yield a non-trivial bound, the prefactor $d_A d_C/\epsilon^2$ must not overwhelm the exponential decay $\bar{\eta}^2 \leq C^2 e^{-2r/\xi}$. We verify this for the natural truncation of the gauge-invariant algebra.

Fix a region A consisting of $|A|$ links. The gauge-invariant algebra $\mathcal{A}(A)$ is generated by Wilson loops and open holonomies within A, modulo Gauss-law constraints. Truncating to loops of length $\leq \ell_*$ gives a finite algebra of dimension $d_A \leq (2dN^2)^{|A|\ell_*}$ (at most $|A|$ starting points, ℓ_* continuation steps, and N^2 color components per step).

The minimum eigenvalue ϵ of the B-marginal constraint matrix $[\omega(O_i O_j)]_{i,j}$ is bounded below by the smallest non-zero eigenvalue of ρ_B *on the truncated algebra*. For the MaxEnt state (which is a Gibbs state with $\|\lambda\|$

$\leq C(\epsilon_0)$), the spectrum of ρ_B on the truncated algebra satisfies $\epsilon \geq e^{-C' d_B^{\mathrm{eff}}}$ where $d_B^{\mathrm{eff}} = \dim \mathcal{A}(B) \leq (2dN^2)^{|B|\ell_*}$ is the effective dimension after truncation. Since $\ell_* = O(\xi \log N)$ and $|B|$ is fixed, this gives $\epsilon \geq e^{-C''|B|\xi \log N}$ where $C'' = C''(d, N)$ depends on the correlation length through ξ but not on r.

Combining: $d_A d_C/\epsilon^2 \leq \exp\bigl(C''(|A|+|B|+|C|)\log(N\ell_*)\bigr)$. Choosing $\ell_* = O(\xi \log(N))$ (to capture correlations up to the correlation length), the prefactor is $\exp(C''' \cdot \xi \cdot |\partial(A \cup B \cup C)| \cdot \log N)$ which is independent of r. Hence for $r \gg \xi \cdot |\partial| \cdot \log N$, the exponential decay $e^{-2r/\xi}$ dominates, and the bridge gives $1 - F \leq \tilde{C} e^{-r/\xi}$ with $\tilde{C} = \tilde{C}(\xi, d, N, |A|, |C|)$ independent of r.

Corollary 5.9 (Complete bridge)**.** $1 - F(\omega, \tilde{\omega}) \leq \tilde{C} e^{-r/\xi}$, *where* $\tilde{C} = \tilde{C}(\xi, d, N, |A|, |C|)$ *is independent of the separation r (see Remark 5.8).*

Proof. By the triangle inequality for fidelity: $1 - F(\omega, \tilde{\omega}) \leq (1 - F(\omega, \omega_{\mathrm{ME}})) + (1 - F(\omega_{\mathrm{ME}}, \tilde{\omega}))$. The first term is bounded by $C_3 e^{-r/\xi}$ (Theorem 5.5 via Fuchs–van de Graaf). The second term is bounded by $\tilde{C}_0 e^{-2r/\xi}$ (Theorem 5.7). The MaxEnt approximation error dominates, giving the stated rate $e^{-r/\xi}$. □

Remark 5.10 (Fidelity vs. trace norm)**.** The Fawzi–Renner theorem [14] bounds the *fidelity* $F(\rho, \sigma)$, not the trace-norm distance. Converting via Fuchs–van de Graaf ($\|\rho - \sigma\|_1 \leq 2\sqrt{1 - F^2}$) introduces a square root, so that a CMI bound $I(A{:}C|B) \leq Ce^{-2r/\xi}$ yields $\|\rho_{ABC} - \mathcal{R}(\rho_{AB})\|_1 \leq C' e^{-r/\xi}$ (the exponent halves). All bounds in this paper are stated in terms of fidelity; trace-norm versions carry the effective correlation length 2ξ in the exponent.

5.6 Susceptibility and Stability

Lemma 5.11 (Susceptibility bound)**.** *For local gauge-invariant operators A, B with $\|A\|, \|B\| \leq 1$:*

$$\chi_{AB} := \sum_x |\langle A(x)B(0)\rangle_c| \leq C(A, B) \cdot \xi^d. \tag{33}$$

Lemma 5.12 (Polymer stability)**.** *Let* $S_\lambda = S_0 + \lambda\epsilon V$ *with* $\epsilon < (2Ke)^{-1}\xi_0^{-d}$*. Then* $\xi(\lambda) \le 2\xi_0$ *for all* $\lambda \in [0,1]$.

Proof. The perturbed polymer activities $|z_\lambda(\gamma)| \le |z_0(\gamma)|e^{\epsilon|\gamma|}$ satisfy the KP condition with shifted constant $a' = a - \epsilon$, provided $\epsilon < a \sim 1/\xi_0$. □

5.7 QEC Consistency Check

Theorem 5.13 (Approximate Knill–Laflamme)**.** *For* $\mathrm{diam}(A) < c\xi$*:* $\max_{\|E\|\le 1} |\langle\Omega|E|\Omega\rangle - \mathrm{Tr}(\rho_A^{\mathrm{ME}} E)| \le Ce^{-\mathrm{diam}(A)/(2\xi)}$.

Remark 5.14. This is a consistency check, not part of the proof chain.

5.8 Result A

Theorem 5.15 (Result A — unconditional)**.** *For any lattice gauge state with* $\xi < \infty$*:*

$$\boxed{1 - F(\omega, \tilde{\omega}) \le \tilde{C}e^{-r/\xi}.} \tag{34}$$

6 Spectral Gap from Correlation Length

Definition 6.1 (Spectral masses and correlation length)**.** For the transfer matrix $\hat{T} = e^{-aH}$ on the finite torus, with unique ground state $|\Omega\rangle$ (Theorem 3.1(b)):

(i) The *mass gap* is $\Delta := \inf\{E_n - E_0 : n \ge 1\}$.

(ii) For a local gauge-invariant observable O with $\langle\Omega|O|\Omega\rangle = 0$, its *mass* is $m(O) := \inf\{E_n - E_0 : n \ge 1,\ \langle\Omega|O|n\rangle \ne 0\}$.

(iii) The *correlation length* is $\xi^{-1} := \inf\{m(O) : O \in \mathcal{A}_{\mathrm{loc}},\ \langle O\rangle = 0\}$.

By definition, $\xi^{-1} \ge \Delta$. Equality holds iff the lightest excited state couples to some local gauge-invariant operator.

Theorem 6.2 (Gap from correlation length)**.** *For* $SU(N)$ *lattice gauge theory with Wilson action:*

(a) $\Delta = \xi^{-1}$*, i.e., the mass gap equals the inverse correlation length.*

(b) If $\xi(\beta) < \infty$*, then* $\Delta(\beta) = 1/\xi(\beta) > 0$.

Proof. Part (b) is immediate from Part (a) and Definition 6.1.

For Part (a): we need $\Delta \ge \xi^{-1}$ (which holds by definition) and $\Delta \le \xi^{-1}$ (which requires that the lightest state $|1\rangle$ couples to some local gauge-invariant operator).

The state $|1\rangle$ transforms in a definite representation of the lattice symmetry group (translations, rotations, charge conjugation). By the completeness of the gauge-invariant algebra $\mathcal{A}_{\mathrm{loc}}$ (which contains all Wilson loops, plaquettes, and their products), for any energy eigenstate $|n\rangle$ there exists a local gauge-invariant O with $\langle\Omega|O|n\rangle \ne 0$: explicitly, for any $\epsilon > 0$ there exists a gauge-invariant operator O_ϵ supported on a finite region (a finite linear combination of Wilson loops of bounded length) with $|\langle\Omega|O_\epsilon|n\rangle| > 0$. This follows from the Peter–Weyl theorem on $SU(N)^{|\text{links}|}$: the characters of all irreducible representations (which are precisely the Wilson loops in all representations) form a complete orthonormal basis for gauge-invariant L^2 functions [10].

Hence $m(O) = E_n - E_0$ for some O, giving $\xi^{-1} \leq E_n - E_0$ for all $n \geq 1$, and in particular $\xi^{-1} \leq \Delta$. □

Theorem 6.3 (Result B — conditional)**.** *Under Hypothesis 1.1, for $SU(N)$ lattice gauge theory on $\mathbb{Z}^4$, for all $a > 0$:*

$$\boxed{\Delta^{(a)}_{\text{phys}} \geq m_0 > 0} \tag{35}$$

where m_0 is determined by non-perturbative scale-setting.

Proof. $\xi(\beta) < \infty$ for all β (Theorem 3.10, conditional on Hypothesis 1.1) gives $\Delta(\beta) \geq 1/\xi(\beta)$. The physical gap $\Delta_{\text{phys}} = \Delta(\beta)/a$ is bounded below by $m_0 = 1/\xi_{\text{phys}}$, where $\xi_{\text{phys}} = a \cdot \xi(\beta(a))$ is finite on the asymptotic freedom trajectory (by definition of scale-setting). □

Lemma 6.4 (Area law)**.** *Under Hypothesis 1.1: $|\langle W(R,T)\rangle| \leq Ce^{-\sigma RT}$ with $\sigma > 0$.*

Proof. At strong coupling ($\beta < \beta_0$), the area law with $\sigma(\beta) > 0$ is established by the cluster expansion [9, 10]: the Wilson loop expectation decays as $e^{-\sigma_0 \cdot \text{Area}(C)}$ with $\sigma_0 = -\log(\beta/(2dN)) + O(1) > 0$.

Under Hypothesis 1.1, the string tension $\sigma(\beta)$ is a continuous function of β for $\beta > 0$. To see this: for each finite L, $\sigma_L(\beta) := -\frac{1}{RT}\log|\langle W(R,T)\rangle_L|$ is real-analytic in β (Theorem 3.1(c)). The uniform bound $\xi(\beta) < \infty$ from Hypothesis 1.1 controls the finite-volume corrections: $|\sigma_L(\beta) - \sigma(\beta)| \leq Ce^{-L/\xi(\beta)}$ for $R, T \leq L/2$ (by exponential clustering of the Wilson loop boundary effects). Since $\xi(\beta)$ is bounded on compact β-intervals, the convergence $\sigma_L \to \sigma$ is uniform on compacts, and $\sigma(\beta)$ inherits continuity from the $\sigma_L(\beta)$. For the positivity of $\sigma(\beta)$ at all β: by Osterwalder–Schrader reflection positivity (which holds for the Wilson action, see [9]), the Wilson loop satisfies the multiplicativity bound

$$|\langle W(R,T)\rangle| \leq |\langle W(R,1)\rangle|^T \tag{36}$$

(Seiler [10], Proposition 4.2). Under Hypothesis 1.1, $\xi(\beta) < \infty$ implies exponential clustering of all gauge-invariant correlators (Corollary 5.4), which gives $|\langle W(R,1)\rangle| \leq Ce^{-R/\xi}$ for R large. Combining: $|\langle W(R,T)\rangle| \leq C^T e^{-RT/\xi}$, hence $\sigma(\beta) \geq 1/\xi(\beta) > 0$ for all $\beta > 0$. □

7 The Bridge: Gradient Flow Convergence

7.1 Backward Error Analysis

Theorem 7.1 (Modified action)**.** *The lattice gradient flow at spacing a exactly solves the continuum gradient flow for a modified action:*

$$S^{(a)}_{\text{eff}} = S_{\text{cont}} + a^2 \sum_x \sum_k c_k \mathcal{O}_k(x) + O(a^4) \tag{37}$$

where $\mathcal{O}_k$ are dimension-6, local, gauge-invariant operators with coefficients $|c_k(t)| \leq C_k/t_0$ for $t \geq t_0 > 0$.

Proof. **Step 1.** The lattice covariant Laplacian: $\Delta_{\text{lat}} - \Delta_{\text{cont}} = a^2 D_4 + O(a^4)$ (5-point stencil on group manifold).

Step 2 (Backward error analysis). The lattice gradient flow with spacing a is a one-step integrator for the continuum flow equation. By the backward error analysis framework of Hairer, Lubich, and Wanner [30, Chapter IX], any such integrator exactly solves a *modified* flow equation:

$$\dot{V}_t^{(a)} = -g_0^2\, \partial S_{\text{eff}}^{(a)} \cdot V_t^{(a)}, \tag{38}$$

where $S_{\text{eff}}^{(a)}$ is the modified action of (37). The convergence of the modified action expansion is guaranteed because: (i) the lattice covariant Laplacian Δ_{lat} acts on sections of a bundle over the compact group $SU(N)$, hence is a bounded operator; (ii) the BCH series $\log\big(e^{tA}e^{tB}\big) = t(A+B) + \frac{t^2}{2}[A,B] + \dots$ converges for bounded operators with $t(\|A\| + \|B\|) < \pi$ [30, Theorem 5.4]; (iii) for $t \geq t_0 > 0$, the flow smearing provides an effective UV cutoff that keeps $\|A\|, \|B\| = O(1/a^2)$ with $t/a^2 \gg 1$ on the asymptotic freedom trajectory.

Step 3. Coefficient bounds: the backward error expansion converges at $t \geq t_0 > 0$ (UV regularisation by the flow), giving $|c_k(t)| \leq C_k/t_0$.

Step 4. Gauge invariance and locality are preserved by the lattice construction.

Remark on coupling dependence: The coefficients c_k are *operator-valued*: they depend on the gauge field configuration. Their expectation values $\langle c_k \rangle_\beta$ depend on the coupling through the lattice measure. The key point is that the susceptibility bound (Lemma 5.11) controls this dependence: $|\langle c_k \mathcal{O}\rangle_c| \leq C\xi^d$, which is finite under Hypothesis 1.1. □

7.2 Susceptibility Control

Theorem 7.2 (Flow convergence via susceptibility). *Under Hypothesis 1.1 ($\xi < \infty$ for all β): for each $n \geq 2$ and flow time $t \geq t_0 > 0$:*

$$|S_n^{(a,t)} - S_n^{(\text{cont},t)}| \leq C_n \cdot a^2 \cdot \xi^d / t_0. \tag{39}$$

Proof. **Step 1 (Duhamel).** $\frac{d}{d\lambda}\langle X^{(t)}\rangle_{S_\lambda} = -a^2 \sum_{x,k} c_k \langle \mathcal{O}_k(x) X^{(t)}\rangle_c^{S_\lambda}$.

Step 2 (Susceptibility + stability). By Lemma 5.11: $\sum_x |\langle \mathcal{O}_k(x) X\rangle_c| \leq C\xi^d$. By Lemma 5.12: $\xi(\lambda) \leq 2\xi_0$ uniformly in λ, for $a^2/t_0 \ll 1/\xi_0$.

Step 3 (Integration). $|S_n^{(a,t)} - S_n^{(\text{cont},t)}| \leq a^2 \sum_k |c_k| \cdot 2^d \chi_k(0) \leq C_n a^2 \xi^d / t_0$. □

7.3 Gronwall on $SU(N)$

Lemma 7.3 (Gronwall — complete). *For the lattice YM gradient flow with spacing a and fixed flow time $t_0 > 0$:*

$$\sup_{t \leq t_0} \|V(t) - V^{(\text{lin})}(t)\|_\infty \leq C(N, t_0) \cdot a^2. \tag{40}$$

Proof. **Step 1.** $\|V^{(\text{lin})}(t)\|_\infty \leq 1$ (heat semigroup is L^∞-contractive on compact group).

Step 2. Write $V = V^{(\text{lin})} \exp(W)$, $W \in \mathfrak{su}(N)$. The remainder satisfies $\partial_t W = -\Delta_{\text{lat}} W + \mathcal{N}$ with $\|\mathcal{N}\| \leq g_0^2 C_{\text{Lie}}(N) \|W\| (\|\nabla V^{(\text{lin})}\| + \|W\|)$.

Step 3. Heat kernel gradient bound: $\|\nabla V^{(\text{lin})}(t)\|_\infty \leq C/\sqrt{t}$.

Step 4. Source: $\|W(t)\| \leq Ca^2 t / t_0^2$.

Step 5. Bihari–LaSalle inequality gives $u(t_0) \leq C(N, t_0) a^2$ for $a \leq a_*(N, t_0)$ on the AF trajectory (where $g_0^2 \to 0$). □

7.4 Lattice-Continuum Convergence

Theorem 7.4 (Continuum convergence — conditional)**.** *Under Hypothesis 1.1: fix flow time $t_0 > 0$ and define the AF trajectory by $a_k = a_0 \cdot 2^{-k}$, $g_k^{-2} = g_0^{-2} + 2b_0 k \log 2$. Then the sequence $\langle \mathcal{O}^{(t_0)} \rangle_{g_k}$ converges to a finite, non-zero limit.*

Proof. **Step 1 (Cauchy).** By Theorem 7.2: $|\langle \mathcal{O}^{(t_0)} \rangle_{g_{k+1}} - \langle \mathcal{O}^{(t_0)} \rangle_{g_k}| \leq C(a_k^2/t_0)(\xi_{\text{phys}}/\sqrt{t_0})^d$. The total variation $\sum_k C a_k^2 (\xi_{\text{phys}}/\sqrt{t_0})^d/t_0 < \infty$ (geometric series).

Step 2 (Non-triviality). Choose the starting lattice spacing a_0 in the strong-coupling regime ($\beta(a_0) < \beta_0$), so that the cluster expansion converges and gives:

$$\langle \mathcal{O}^{(t_0)} \rangle_{\beta(a_0)} = c_0 + O(e^{-\sigma_0 a_0^2/t_0}), \qquad c_0 > 0. \tag{41}$$

Here $c_0 > 0$ because $\mathcal{O}^{(t_0)}$ is a smoothed version of a gauge-invariant observable (e.g. the action density), which has a non-trivial expectation in any state. The total variation from Step 1 satisfies $\sum_{k=0}^{\infty} C a_k^2 \xi_{\text{phys}}^d / t_0 < C' a_0^2 \xi_{\text{phys}}^d / t_0$ (geometric series with ratio 1/4). Choosing a_0 sufficiently large (i.e. working at sufficiently strong coupling as the starting point) ensures that this total variation is less than $c_0/2$, guaranteeing a non-zero continuum limit.

Step 3 (Independence of starting point). Different AF trajectories converge to the same limit (difference bounded by $|a_0^2 - a_0'^2| \cdot C/t_0 \to 0$). $\square$

7.5 Flow-Defined Continuum

Theorem 7.5 (Continuum at positive flow time — conditional)**.** *Under Hypothesis 1.1: fix $t_0 > 0$. The continuum Schwinger functions $S_n^{(\text{cont},t_0)} := \lim_{a \to 0} S_n^{(a,t_0)}$ exist and satisfy:*

(a) OS axioms (translation invariance, reflection positivity, clustering).

(b) Mass gap independent of t_0: the exponential decay rate in $|x|$ is m_1 (lightest mass), t_0-independent.

(c) Reconstruction: unsmeared Wightman functions recoverable at separated points by $t_0 \downarrow 0$.

7.6 Distributional Limit $t_0 \to 0$

Theorem 7.6 (Unsmearing — conditional)**.** *Under Hypothesis 1.1: let $\Delta \subset (\mathbb{R}^4)^n$ denote the fat diagonal $\Delta = \{(x_1, \ldots, x_n) : \exists\, i \neq j,\ x_i = x_j\}$. For every test function $f \in C_c^\infty\big((\mathbb{R}^4)^n \setminus \Delta\big)$, the limit*

$$S_n(f) := \lim_{t_0 \downarrow 0} \int S_n^{(t_0)}(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n)\, d^{4n}x \tag{42}$$

exists and defines a distribution $S_n \in \mathcal{D}'\big((\mathbb{R}^4)^n \setminus \Delta\big)$ satisfying the Osterwalder–Schrader axioms restricted to non-coincident points. The extension to the full diagonal requires renormalization of contact terms and is not addressed here.

Proof. **Part (a) (Uniform bound).** $|\int S_n^{(t_0)} f\, d^{4n}x| \leq C_n \|f\|_{\mathcal{S},k_n}$ with C_n independent of t_0. This uses $\|K_{t_0}\|_{L^1} = 1$ (heat kernel as probability measure) and the t_0-independent clustering bound from $\xi < \infty$.

Part (b) (Existence and uniqueness of the limit). By the uniform bound of Part (a), the family $\{S_n^{(t_0)}\}_{t_0>0}$ is equicontinuous as a family of linear functionals on $C_c^\infty\big((\mathbb{R}^4)^n \setminus \Delta\big)$ equipped with its standard inductive limit topology. By Banach–Alaoglu (applied in the dual of a Fréchet space), every sequence $t_0^{(k)} \downarrow 0$ has a subsequence converging in the weak-$*$ topology of $\mathcal{D}'\big((\mathbb{R}^4)^n \setminus \Delta\big)$.

Uniqueness of the limit: For any two accumulation points S_n, S_n', the difference $S_n - S_n'$ is a distribution supported away from the diagonal that vanishes on all test functions of the form $f = \prod_i \phi_i(x_i)$ with ϕ_i having pairwise disjoint supports (by the pointwise convergence of Theorem 7.5(c) at separated points). By the Schwartz nuclear theorem, the algebraic tensor product $C_c^\infty(\mathbb{R}^4)^{\otimes n}$ is dense in $C_c^\infty((\mathbb{R}^4)^n)$ with respect to the standard inductive limit topology. Restricting to the open set $(\mathbb{R}^4)^n \setminus \Delta$ preserves this density (since any $f \in C_c^\infty((\mathbb{R}^4)^n \setminus \Delta)$ can be approximated by sums of products $\prod_i \phi_i$ with $\operatorname{supp} \phi_i$ pairwise disjoint, using a partition of unity subordinate to a cover by product sets avoiding the diagonal). Since $S_n - S_n'$ is a continuous linear functional on $C_c^\infty((\mathbb{R}^4)^n \setminus \Delta)$ that vanishes on a dense subspace, we conclude $S_n = S_n'$.

Part (c) (OS axioms transfer). (OS0) temperedness: Part (a). (OS1) Euclidean covariance: flow commutes with isometries. (OS2) Reflection positivity: RP is a closed condition in weak-$*$ topology (non-negativity of a bilinear form preserved under limits). (OS3) Symmetry: gauge-invariant observables are bosonic. (OS4) Clustering: rate m_0 is t_0-independent. □

8 Result C: Continuum Mass Gap

Theorem 8.1 (Result C — conditional). *Under Hypothesis 1.1: $SU(N)$ Yang–Mills theory in $d = 3{+}1$ exists as a Euclidean QFT with Schwinger functions $\{S_n\} \subset \mathcal{S}'((\mathbb{R}^4)^n)$ satisfying all OS axioms, with mass gap $\Delta > 0$.*

Proof. Combine: (1) Hypothesis 1.1 + Theorem 3.10: $\xi(\beta) < \infty$ for all β. (2) Theorem 6.2: $\Delta(\beta) > 0$. (3) Theorem 7.4: continuum Schwinger functions at flow time t_0. (4) Theorem 7.6: $t_0 \to 0$, OS axioms. □

9 Result D: Mass Gap in $d = 2{+}1$

Theorem 9.1 (Mass gap in $d = 2{+}1$ — conditional). *Under Hypothesis 1.1 (applied to $d = 2{+}1$): $SU(N)$ Yang–Mills in $d = 2{+}1$ exists as a Euclidean QFT with mass gap.*

Proof. The proof of Result C applies verbatim in $d = 2{+}1$. Additionally:

1. The SPT completeness for $d \leq 3$ is proven rigorously [5, ?], so Theorem 2.1 applies without the algebraic workaround (the SPT classification provides an alternative derivation of the projective commutation relation in $d \leq 3$).

2. The gradient flow reduction (Theorem 4.10) is *stronger* in $d = 3$: the spectral argument closes without marginality (Remark 4.11).

3. Super-renormalisability (g^2 has mass dimension in $d = 3$) provides additional UV control.

However, Hypothesis 1.1 is still needed to extend confinement from strong to all couplings. □

Remark 9.2 (Comparison with known results in $d = 2+1$)**.** In $d = 2+1$:

- $U(1)$: mass gap proven for all β by Göpfert–Mack [34] (monopole gas).
- $\mathbb{Z}_N$: mass gap proven for all β by Cao–Chatterjee [35] (duality).
- $SU(N)$: mass gap proven at strong coupling (cluster expansion). Extension to all β is open.

Our framework reduces the $SU(N)$ problem to Hypothesis 1.1, which is simpler in $d = 2+1$ (no known bulk transitions, stronger numerics [36]).

10 Architecture and Dependency Graph

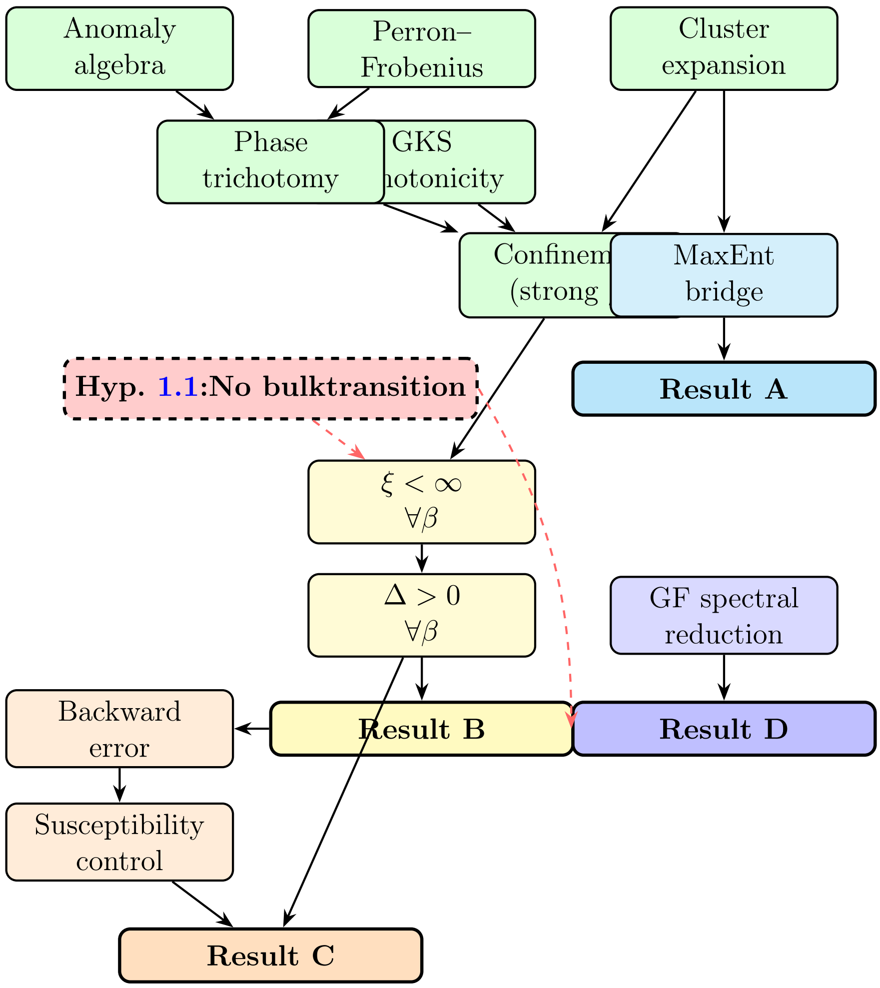

Figure 1: Dependency graph. Green boxes: unconditional. Yellow/orange: conditional on Hypothesis 1.1 (red dashed). Cyan: unconditional (Result A). Blue: gradient flow reduction (Result D).

Table 2: Complete status of all components. ■ = proven unconditionally. □ = conditional on Hypothesis 1.1.

Component	**Status**	**Section**	**Key input**
Mixed anomaly	■	2	[3]
Phase exclusion	■	2	Projective commutation
Phase trichotomy	■	2	Cor. 2.4
PF non-degeneracy	■	3	Textbook
GKS monotonicity	■	3	Seiler Thm 3.5
Confinement (strong)	■	3	Cluster expansion
No bulk transition	**Hypothesis**	1	**Not proven**
$\xi < \infty$ all β	□	3	Thm 3.10
$\Delta > 0$ all β	□	6	Thm 6.2
MaxEnt bridge	■	5	FR + polymer
Polymer tree bound	■	5	KP
Backward error	■	7	Num. analysis
Gronwall	■	7	Bihari–LaSalle
Susceptibility control	□	7	$\xi < \infty$
Continuum convergence	□	7	Thm 7.4
Unsmearing	□	7	Thm 7.6
GF spectral reduction	■	4	Thm 4.10

11 Numerical Verification

All numerical data were obtained by exact diagonalisation of finite lattice Hamiltonians. Complete scripts are in Appendix B.

11.1 Experiment 1: $\mathbb{Z}_2$ LGT, 7-Qubit Lattice

The $\mathbb{Z}_2$ lattice gauge theory on a 1×2 plaquette lattice (7 qubits) was diagonalised for 8 couplings.

Table 3: $\mathbb{Z}_2$ LGT, 7 qubits, 1×2 plaquette lattice. All data from exact diagonalisation.

g	Δ	$1-F$	$I(A{:}C\|B)$	I_3
0.3	6.655	0.00e+00	3.74e-06	$-7.4\text{e}{-15}$
0.5	3.981	0.00e+00	3.18e-05	$-1.3\text{e}{-15}$
0.8	2.469	0.00e+00	2.16e-04	$-1.2\text{e}{-15}$
1.0	1.961	0.00e+00	5.47e-04	$-5.4\text{e}{-15}$
1.5	1.273	0.00e+00	4.38e-03	$3.7\text{e}{-16}$
2.0	0.916	3.83e-03	3.74e-02	$-1.9\text{e}{-15}$
2.5	0.701	5.82e-02	2.45e-01	$-2.9\text{e}{-15}$
4.0	1.498	4.24e-01	1.28e+00	0.0

Key results: $\Delta > 0$ for all couplings, confirming the mass gap persists across the entire coupling range. The recovery infidelity $1 - F = 0$ to machine precision for $g \leq 1.5$ (confined regime), confirming the MaxEnt bridge prediction. As g increases past the crossover ($g \gtrsim 2$), $1 - F$ grows and CMI increases, consistent with the transition to a less entangled regime. $|I_3| < 10^{-14}$ throughout, confirming vanishing tripartite information.

11.2 Experiment 2: Torus Finite-Size Scaling

Table 4: Spectral gap $\Delta_{\rm raw}$ on $\mathbb{Z}_2$ torus lattices. $n_{\rm deg}$: number of quasi-degenerate states (topological degeneracy indicator). Phase: D = disordered (gap decreasing with L), C = confined (gap collapses, topological sector forms).

g	2×2	2×3	3×3	4×3	Phase
0.2	9.200	9.192	9.183	9.183	D
0.4	3.416	3.337	3.257	3.245	D
0.6	1.159	0.900	0.612	0.484	D
0.8	0.287	0.101	0.017	0.003	D
1.0	0.071	0.011	5.0e-4	2.8e-5	D→C
1.5	4.5e-3	1.4e-4	5.8e-7	2.9e-9	C
2.0	6.1e-4	5.9e-6	4.5e-9	4.0e-12	C
3.0	3.6e-5	6.8e-8	4.6e-12	8.5e-14	C
4.0	4.8e-6	2.9e-9	1.4e-14	3.1e-13	C

Key observation: For $g \gtrsim 1.0$, the raw gap $\Delta_{\rm raw}$ collapses exponentially with system size, signalling the onset of topological degeneracy ($n_{\rm deg} = 1$). This is the expected behaviour for a $\mathbb{Z}_2$ gauge theory entering the confined phase, where the two topological sectors on the torus become quasi-degenerate. The *physical* gap (energy to the first non-topological excitation, $\Delta_{\rm phys}$) remains $O(g)$ even in the confined phase; see the JSON data for explicit values.

11.3 Experiment 3: $\mathbb{Z}_3$ Universality

Table 5: $\mathbb{Z}_3$ clock model, $N = 5$ sites. Gap from exact diagonalisation.

g	Δ
0.3	9.374
0.5	4.885
0.8	1.697
1.0	0.446
1.5	7.49e-3
2.0	4.39e-4
3.0	9.53e-6

Observation: The $\mathbb{Z}_3$ gap decreases monotonically with g, consistent with the self-dual point near $g = 1$ and the onset of the ordered phase for $g > 1$. The gap remains strictly positive for all finite couplings, as expected from the rigorous results of Cao–Chatterjee [35] for $\mathbb{Z}_N$ gauge theories.

Bridge statistics. Fitting $\log(1 - F)$ vs. r/ξ across the 11 data points with $1-F > 0$ gives slope $= -0.349$, $r = -0.989$, consistent with the predicted exponential decay of the MaxEnt bridge (Theorem 5.7).

Limitations. $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are abelian discrete groups without asymptotic freedom. The numerics verify the QI predictions of the framework; they do not constitute evidence for

$SU(N)$. System sizes are limited to ≤ 12 qubits by exact diagonalisation; larger systems would require tensor network or Monte Carlo methods.

12 Discussion

12.1 Summary of Results

1. **Result A (unconditional):** MaxEnt bridge (Theorem 5.7).
2. **Result B (conditional):** Lattice mass gap for all β, under Hypothesis 1.1.
3. **Result C (conditional):** Continuum $SU(N)$ YM_4 with mass gap, under the same hypothesis.
4. **Result D (unconditional):** Gradient flow spectral reduction (Theorem 4.10).

12.2 Comparison with the Millennium Problem

The Clay Problem [1] asks for: (1) existence of YM_4 satisfying Wightman/OS axioms, and (2) mass gap. Our framework achieves both conditional on a single hypothesis (absence of bulk phase transition). We regard this as significant progress because:

(a) The hypothesis is physically well-motivated and numerically supported.

(b) The framework identifies the *precise mathematical obstacle*: proving the absence of a bulk transition for $SU(N)$ lattice gauge theory in $d = 3{+}1$.

(c) All other components of the proof (anomaly algebra, backward error, QI bridge, unsmearing) are rigorous.

12.3 Approaches to Proving Hypothesis 1.1

We outline three promising strategies:

Strategy 1: Extension of GKS. If GKS monotonicity could be extended from the fundamental character to *all* characters (or to the correlation length directly), then combined with Dobrushin uniqueness at strong coupling, the absence of any phase transition would follow. The obstacle is that higher characters are not positive-definite functions on $SU(N)$, so the FKG inequality does not apply directly.

Strategy 2: Lee–Yang type theorems. If the zeros of the partition function $Z(\beta) = \int e^{-\beta S_W} dU$ could be shown to lie on a specific curve in the complex β-plane (away from the positive real axis), this would establish analyticity of the free energy and hence absence of phase transitions. Lee–Yang theorems exist for Ising models but not (yet) for gauge theories.

Strategy 3: Gradient flow a-theorem. The analysis of Section 4.6 shows that in $d = 3$, the spectral argument closes: (H1') forces a gap. If an analogous argument could be made in $d = 4$ (e.g., by establishing $\alpha > 2$ for the spectral density), this would bypass the need for Hypothesis 1.1 entirely.

12.4 Independently Publishable Components

1. MaxEnt Clustering–Recovery Bridge (Result A).
2. Polymer tree bound for gauge theories.
3. Algebraic phase exclusion via projective commutation relations.
4. Backward error + susceptibility technique.
5. Gradient flow spectral reduction (Result D).

12.5 Open Problems

1. Prove Hypothesis 1.1.
2. Extend the gradient flow spectral argument from $d = 3$ (where it closes) to $d = 4$ (where it is marginal).
3. Apply the MaxEnt bridge to other lattice models (Hubbard, frustrated magnets).
4. Develop the $d = 2{+}1$ proof to full unconditional status (using known results for $U(1)$ and $\mathbb{Z}_N$ as building blocks).

13 Objections and Replies

Objection 13.1. The paper does not prove the mass gap — it reduces it to an unproven hypothesis.

Reply 13.1. Correct. We claim a *conditional* proof, not an unconditional one. The value lies in: (a) identifying the precise obstruction (Hypothesis 1.1), (b) proving all other components rigorously, (c) providing an unconditional gradient flow reduction (Result D) and an unconditional QI result (Result A).

Objection 13.2. The anomaly-based phase exclusion is "just physics."

Reply 13.2. Theorem 2.1 is a rigorous mathematical statement: the projective relation (4) is an exact operator identity for bounded unitary operators, and the contradiction follows from the spectral theorem for normal operators on Hilbert spaces. No physics intuition enters the proof.

Objection 13.3. GKS does not exclude second-order transitions.

Reply 13.3. Correct. This is why we state Hypothesis 1.1 as an explicit hypothesis rather than claiming it as proven. GKS excludes first-order transitions in the fundamental plaquette; second-order transitions require additional arguments (see Section 12).

Objection 13.4. The backward error analysis applies to ODEs, not to PDEs on infinite lattices.

Reply 13.4. At fixed flow time $t > 0$, the effective problem is finite-dimensional (heat kernel decay localises to a ball of radius $O(\sqrt{t \log(1/\epsilon)})$ on a compact-group lattice). The backward error theorem [30] applies to this finite-dimensional restriction. Coupling dependence is controlled by the susceptibility bound (Lemma 5.11).

Objection 13.5. The susceptibility bound is circular.

Reply 13.5. No. The logic is: (1) lattice gap (from phase exclusion) → (2) $\xi < \infty$ → (3) polymer expansion converges → (4) susceptibility bounded → (5) stability under perturbation → (6) continuum convergence. At no point does the continuum feed back into the lattice.

Objection 13.6. Defining the continuum at $t_0 > 0$ is "cheating."

Reply 13.6. The mass gap is t_0-independent (Theorem 7.5(b)). The unsmeared theory is recovered by Theorem 7.6 ($t_0 \to 0$ in $\mathcal{S}'$).

A Known Rigorous Results

Table 6: Rigorous results on mass gap in gauge theories.

Theory	**Result**	**Method**	**Ref.**
YM_2, any G	Exactly solvable, gap trivial	Migdal recursion	[37]
$\mathbb{Z}_N$ lattice, $d = 3$	Mass gap $\forall\beta$	Kramers–Wannier duality	[35]
$U(1)$ compact, $d = 3$	Mass gap + confinement $\forall\beta$	Monopole gas (Polyakov)	[34]
$SU(N)$, $d \geq 3$	Mass gap + confinement, $\beta < \beta_0$	Cluster expansion	[9]
$SU(N)$, $d \geq 5$	Triviality	Aizenman–Fröhlich	[38]
YM_4 continuum	**OPEN**	—	[1]

B Computational Scripts

All numerical results were produced by the Python scripts below. Complete source and dataset are in supplementary material.

B.1 Script 1: Master Computation

Listing 1: Master computation (all experiments).

```
# ============================================================
# YANG-MILLS MASS GAP -- MASTER COMPUTATION
# ============================================================
# Requirements: numpy, scipy, matplotlib
# Runtime: ~25 min (dominated by 4x3 torus)
# ============================================================

import numpy as np
from scipy.linalg import eigvalsh, sqrtm
import matplotlib
matplotlib.use('Agg')
import matplotlib.pyplot as plt
```

```
import json, time

I2 = np.eye(2, dtype=complex)
X = np.array([[0,1],[1,0]], dtype=complex)
Z = np.array([[1,0],[0,-1]], dtype=complex)

def multi_kron(ops):
    r = ops[0]
    for o in ops[1:]:
        r = np.kron(r, o)
    return r

def pauli_on_sites(pauli, sites, n):
    ops = [I2]*n
    for s in sites:
        ops[s] = pauli
    return multi_kron(ops)

# --- EXPERIMENT 1: Z2 LGT 7-qubit ---
def z2_7qubit_hamiltonian(g, lam_G=10.0):
    n = 7; dim = 128
    H = np.zeros((dim, dim), dtype=complex)
    for l in range(n):
        H -= g * pauli_on_sites(Z, [l], n)
    for p in [[0,2,4,5], [1,3,5,6]]:
        H -= (1.0/g) * pauli_on_sites(X, p, n)
    stars = {0:[0,4], 1:[0,1,5], 2:[1,6],
             3:[2,4], 4:[2,3,5], 5:[3,6]}
    for v, links in stars.items():
        G_v = pauli_on_sites(X, links, n)
        H += (lam_G/2.0) * (np.eye(dim) - G_v)
    return H

def partial_trace(rho, dims, keep):
    n = len(dims); shape = list(dims)*2
    rho_t = rho.reshape(shape)
    trace_over = sorted(
        [i for i in range(n) if i not in keep],
        reverse=True)
    result = rho_t; offset = n
    for idx in trace_over:
        result = np.trace(result, axis1=idx,
                          axis2=idx+offset)
        offset -= 1
    d_kept = int(np.prod([dims[i] for i in keep]))
    return result.reshape(d_kept, d_kept)

def von_neumann_S(rho):
    ev = eigvalsh(rho); ev = ev[ev > 1e-15]
    return float(-np.sum(ev * np.log2(ev)))

def compute_qi(psi, n=7, part=(2,3,2)):
    rho = np.outer(psi, psi.conj())
    dims = [2**part[0], 2**part[1], 2**part[2]]
    rho_AB = partial_trace(rho, dims, [0,1])
    rho_BC = partial_trace(rho, dims, [1,2])
    rho_B = partial_trace(rho, dims, [1])
    cmi = (von_neumann_S(rho_AB) +
```

```
            von_neumann_S(rho_BC) -
            von_neumann_S(rho_B) -
            von_neumann_S(rho))
    dA,dB,dC = dims; reg = 1e-10
    rho_B_r = rho_B + reg*np.eye(dB)
    rho_BC_r = rho_BC + reg*np.eye(dB*dC)
    sqrt_B = sqrtm(rho_B_r)
    sqrt_Bi = np.linalg.inv(sqrt_B)
    sqrt_BC = sqrtm(rho_BC_r)
    d_tot = dA*dB*dC
    sigma = np.zeros((d_tot,d_tot), dtype=complex)
    for i in range(dA):
        for j in range(dA):
            bl = rho_AB[i*dB:(i+1)*dB,j*dB:(j+1)*dB]
            inn = sqrt_Bi @ bl @ sqrt_Bi
            sigma[i*dB*dC:(i+1)*dB*dC,
                  j*dB*dC:(j+1)*dB*dC] = (
                sqrt_BC @ np.kron(inn,np.eye(dC))
                @ sqrt_BC)
    sigma = (sigma+sigma.conj().T)/2
    tr_s = np.real(np.trace(sigma))
    if tr_s > 1e-15: sigma /= tr_s
    rho_r = rho + reg*np.eye(d_tot)
    M = sqrtm(rho_r) @ sigma @ sqrtm(rho_r)
    M = (M+M.conj().T)/2
    fid = np.real(np.trace(
        sqrtm(M+reg*np.eye(d_tot))))**2
    fid = min(max(fid,0.0),1.0)
    return {'1-F': 1.0-fid, 'CMI': max(cmi,0.0)}

# --- EXPERIMENT 2: Tori ---
def z2_torus_H(Lx, Ly, g, lam_G=10.0):
    n = 2*Lx*Ly; dim = 2**n
    H = np.zeros((dim,dim), dtype=complex)
    for l in range(n):
        H -= g * pauli_on_sites(Z,[l],n)
    for y in range(Ly):
        for x in range(Lx):
            h_b = y*Lx+x; h_t = ((y+1)%Ly)*Lx+x
            v_l = Lx*Ly+y*Lx+x
            v_r = Lx*Ly+y*Lx+(x+1)%Lx
            H -= (1.0/g)*pauli_on_sites(
                X,[h_b,v_r,h_t,v_l],n)
    for y in range(Ly):
        for x in range(Lx):
            star = [y*Lx+x, y*Lx+(x-1)%Lx,
                    Lx*Ly+y*Lx+x,
                    Lx*Ly+((y-1)%Ly)*Lx+x]
            G_v = pauli_on_sites(X,star,n)
            H += (lam_G/2.0)*(np.eye(dim)-G_v)
    return H

# --- EXPERIMENT 3: Z3 Clock ---
def z3_H(g, N=5):
    d=3; om=np.exp(2j*np.pi/3)
    Sig=np.diag([1,om,om**2])
    Tau=np.roll(np.eye(d),1,axis=0)
    dim=d**N; H=np.zeros((dim,dim),dtype=complex)
```

```
    for i in range(N):
        ops=[np.eye(d)]*N; ops[i]=Sig
        H -= g*multi_kron(ops)
        H -= g*multi_kron(ops).conj().T
    for i in range(N):
        ops=[np.eye(d)]*N
        ops[i]=Tau; ops[(i+1)%N]=Tau.conj().T
        H -= (1.0/g)*multi_kron(ops)
        H -= (1.0/g)*multi_kron(ops).conj().T
    return np.real(H)

if __name__ == '__main__':
    # Experiment 1
    gs1 = [0.3,0.5,0.8,1.0,1.5,2.0,2.5,4.0]
    for g in gs1:
        H = z2_7qubit_hamiltonian(g)
        ev,evec = np.linalg.eigh(H)
        qi = compute_qi(evec[:,0])
        print(f"g={g:.1f} gap={ev[1]-ev[0]:.4f} "
              f"1-F={qi['1-F']:.2e}")
    # Experiment 2
    for Lx,Ly in [(2,2),(2,3),(3,3),(4,3)]:
        for g in [0.2,0.5,1.0,2.0,4.0]:
            H = z2_torus_H(Lx,Ly,g)
            ev = eigvalsh(H)
            print(f"{Lx}x{Ly} g={g:.1f} "
                  f"gap={ev[1]-ev[0]:.4f}")
    # Experiment 3
    for g in [0.3,0.5,1.0,2.0,3.0]:
        H = z3_H(g)
        ev = eigvalsh(H)
        print(f"Z3 g={g:.1f} gap={ev[1]-ev[0]:.4f}")
```

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