Gradient Flow Monotonicity and the Yang–Mills Mass Gap

A Conditional Reduction via Spectral Methods

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Abstract

We establish a conditional reduction of the Yang–Mills mass gap problem to a concrete spectral inequality involving the gradient flow. Our main result is:

Theorem (Informal). For pure $SU(N)$ Yang–Mills theory, if the gradient flow β-function satisfies a uniform strict asymptotic freedom condition $|\beta_{\mathrm{GF}}(g)| \geq \delta g^3$ for large g, and a Tauberian regularity condition holds for the spectral density, then: in $d = 3$ the theory has a mass gap $\Delta > 0$; in $d = 4$ the infrared trace anomaly vanishes, $a_{\mathrm{IR}} = 0$, ruling out a conformal infrared fixed point — combined with the phase exclusion of the companion paper [23], this reduces the mass gap to explicit spectral conditions. However, the spectral argument is marginal in $d = 4$ and requires additional non-perturbative input.

The proof uses three ingredients: (1) a spectral representation of the gradient flow energy $E(t)$ and a monotonicity identity $R'(t) = -2\,\mathrm{Var}_t(\lambda) \leq 0$ for the ratio $R(t) = F(t)/E(t)$; (2) the Komargodski–Schwimmer a-theorem constraining the IR behaviour; and (3) a gradient flow Poincaré inequality connecting functional inequalities to exponential clustering of correlators.

We verify all perturbative inputs: the free-field calibration gives $R_{\mathrm{free}}(t) = 2/t$ in $d = 4$, and the one-loop correction has the correct sign ($R(t) < 2/t$ for $g > 0$), confirming the reduction at weak coupling. We identify the non-perturbative obstruction (the indefiniteness of the Weitzenböck curvature term) as the precise technical barrier to closing the argument in $d = 4$.

v2 (no v1 numbered statement is changed): the status of the spectral representation is made honest — for the interacting (nonlinear) flow, the fixed-measure Laplace representation is *equivalent to complete monotonicity of* $E(t)$ and is now an explicit hypothesis (H0), proven only for the linearized flow (Table 1 retagged; Hypothesis 2.2 and Remark 2.3); an unresolved citation in Proposition 4.4 is repaired; the claim that the Holley–Stroock route gives the Poincaré inequality "unconditionally at finite β" is corrected — Holley–Stroock yields a constant exponential in the volume, so the L-uniform statement remains conditional outside the two controlled regimes (Remark 4.2); the proof of Corollary 2.15 is retagged as physical input (imported classification of $a_{\mathrm{IR}} = 0$ phases); the Karamata attribution in Theorem 2.11 is replaced by the elementary direct bound actually used; and a new verification suite adds numerical evidence: the exact spectral machinery, the free-field calibration on the lattice, the quantified $d = 3$ vs $d = 4$ dichotomy on synthetic spectral densities, and — new — a first in-framework Monte Carlo test of Hypothesis B′: SU(2) Wilson-action heat-bath configurations on 8^4 at $\beta = 2.4$, Wilson-flowed to $t = 2.4$, give $c(t) = t^2\langle E(t)\rangle$ increasing by 42% across the measured window and $R(t) < 2/t$ pointwise, i.e. $\beta_{\mathrm{GF}} < 0$ throughout — first in-framework numerical support for B′ in a controlled lattice window (evidence, not proof).

This paper is a companion to [23], which establishes the mass gap conditionally via anomaly algebra and quantum information methods. The two approaches are complementary: the present paper provides a quantitative spectral diagnostic, while [23] provides the algebraic phase exclusion.

1 Introduction

The existence of a mass gap in non-abelian Yang–Mills theory is one of the most profound open problems in mathematical physics, codified as one of the seven Millennium Prize Problems by the Clay Mathematics Institute [1]. The problem asks for a rigorous construction of a quantum field theory on $\mathbb{R}^4$ with gauge group $SU(N)$ satisfying the Wightman axioms, and a proof that the spectrum of the Hamiltonian possesses a strictly positive lower bound $\Delta > 0$ above the vacuum state.

Physically, this gap is associated with the phenomenon of colour confinement and the mass of the lightest glueball. While there is overwhelming numerical evidence from lattice Monte Carlo simulations supporting the existence of a mass gap [11, 12], a rigorous analytical proof remains elusive. The difficulty lies in controlling the non-perturbative behaviour of the theory in the infrared (IR) regime, where the running coupling constant becomes large, invalidating standard perturbative techniques.

Recent progress has come from several directions. Douglas [2] provides a comprehensive review, emphasising both the constructive approach and the role of computational methods. Faddeev [3] offers a complementary perspective focused on renormalisation. On the lattice, Lüscher's gradient flow formalism [5, 7] has provided powerful non-perturbative tools for defining running couplings and studying the approach to the continuum limit.

1.1 The gradient flow approach

We adopt a perspective rooted in the gradient flow formalism. The gradient flow provides a gauge-invariant, non-perturbative definition of the running coupling $g^2_{\mathrm{GF}}(\mu)$ at an energy scale $\mu \sim 1/\sqrt{8t}$, where t is the flow time [5]. Our central insight is to link the spectral properties of the theory (the mass gap) directly to the monotonicity of the gradient flow β-function. We leverage the a-theorem of Komargodski and Schwimmer [4] and recent results on the irreversibility of RG flows to construct a conditional proof.

1.2 Main results

Theorem 1.1 (Main result — informal)**.** *Consider pure $SU(N)$ Yang–Mills theory in d dimensions. Assume that the gradient flow β-function satisfies a uniform strict asymptotic freedom condition, that the flowed energy admits the spectral representation (H0), and that standard axioms of unitarity and RG flow completeness hold. Then: (a) in $d = 3$: the theory has a mass gap $\Delta > 0$; (b) in $d = 4$: $a_{\mathrm{IR}} = 0$ (trivial IR), ruling out a conformal infrared fixed point and reducing the mass gap to explicit Poincaré-type conditions. However, the spectral argument is marginal.*

We also provide a bridge between the spectral gap of the Hamiltonian and functional inequalities (Poincaré and log-Sobolev) for the Euclidean measure, clarifying the necessary analytical conditions for the existence of the gap.

1.3 Status of results

#	Claim	Status	Key Input
1a	Spectral rep. ($d\rho \geq 0$), linearized flow	**PROVEN**	Finite volume
1b	Spectral rep. for interacting flow	**ASSUMED (H0)**	complete monotonicity of $E(t$
2	Monotonicity $R'(t) \leq 0$	**PROVEN given (H0)**	Spectral calculus
3	Free-field calibration $R = d/(2t)$	**PROVEN**	Gaussian integral
4	One-loop: $R < 2/t$ at weak coupling	**PROVEN**	Perturbation theory
5	Poincaré $\Rightarrow$ clustering	**PROVEN** (given L-uniform PI)	Heat kernel + spectral
6	Mass gap in $d = 3$	**CONDITIONAL**	(H0), (H1′)–(H4) + phase inpu
7	$a_{\mathrm{IR}} = 0$ in $d = 4$	**CONDITIONAL**	(H0), (H1′), (H2)
8	Mass gap in $d = 4$	**CONDITIONAL**	(H0), (H1′)–(H4)

Table 1: Classification of results (retagged in v2: claim 1 is split into the linearized statement 1a and the interacting-flow hypothesis 1b = (H0); claims 6–8 record their dependence on (H0) and, for claim 6, on the imported phase classification of Corollary 2.15).

2 The Conditional Theorem

2.1 Setup and notation

Let $\Lambda_L = (a\mathbb{Z}/La\mathbb{Z})^d$ be a d-dimensional periodic hypercubic lattice with spacing a and linear extent La, where $d \in \{3, 4\}$. To each oriented link $\ell = (x, \mu)$ we assign a variable $U_\ell \in SU(N)$. The Wilson plaquette action is

$$S_W[U] = \beta \sum_P \Big(1 - \tfrac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U_P\Big), \qquad \beta = \frac{2N}{g_0^2}, \tag{1}$$

where U_P is the ordered product around plaquette P.

Lattice gradient flow. Following Lüscher [5], the lattice gradient flow is

$$\dot{V}_t(\ell) = -g_0^2\big(\partial_{x,\mu} S_W[V_t]\big)V_t(\ell), \qquad V_0(\ell) = U_\ell, \tag{2}$$

where $\partial^a_{x,\mu} S_W = \frac{d}{ds}\big|_0 S_W[e^{sT^a}V_t(x,\mu)]$. Since $SU(N)$ is compact, the flow exists for all $t \geq 0$ on any finite lattice.

Flowed observables. The clover-leaf lattice field strength $G^a_{\mu\nu}(x;t)$ is constructed from the flowed links V_t. We define

$$E(t) := \frac{1}{4}\Big\langle \sum_x G^a_{\mu\nu}(x;t)G^a_{\mu\nu}(x;t)\Big\rangle, \qquad F(t) := \Big\langle \sum_x \big(D_\nu G^a_{\nu\mu}(x;t)\big)^2\Big\rangle, \qquad R(t) \equiv \frac{F(t)}{E(t)}. \tag{3}$$

The gradient flow coupling at scale $\mu = (8t)^{-1/2}$ is

$$g^2_{\mathrm{GF}}(\mu) := \frac{t^{d/2}}{\mathcal{N}} E(t)\Big|_{t=1/(8\mu^2)}, \qquad \mathcal{N} = \frac{d(d-1)(N^2-1)}{2^{d+2}\pi^{d/2}}. \tag{4}$$

Remark 2.1 (Convention: divergence vs. gradient). The quantity $F(t) = \|D \cdot G\|^2$ uses the *divergence* $D_\nu G_{\nu\mu}$, not the full gradient $D_\rho G_{\mu\nu}$. In the free theory, the Bianchi identity gives $F = \frac{1}{2}\|\nabla G\|^2$, and the resulting free-field ratio is $R_{\mathrm{free}} = d/(2t)$ (not d/t). See Section 3.

2.2 Hypotheses

Hypothesis 2.2 (H0: spectral representation / complete monotonicity (v2, made explicit)). *The flowed energy admits the fixed-measure Laplace representation*

$$E(t) = \int_0^\infty e^{-2\lambda t}\, d\rho(\lambda), \qquad d\rho \geq 0. \tag{5}$$

Remark 2.3 (Status of (H0)). By Bernstein's theorem, (5) is *equivalent* to complete monotonicity of $E(t)$ ($(-1)^n E^{(n)}(t) \geq 0$ for all n). For the *linearized* flow this is proven on any finite lattice: each field-strength mode evolves with a heat kernel $e^{-\lambda t}$ and OS positivity gives $c_n > 0$, so $d\rho = \sum_n c_n \delta(\lambda - \lambda_n)\, d\lambda \geq 0$. For the full interacting flow, V_t is a *nonlinear* function of the initial configuration ($\partial_t G$ contains $[G, G]$ commutator terms), so $E(t)$ is not automatically a fixed-measure Laplace transform; v1's Table 1 overstated this point by tagging the representation PROVEN without qualification. We therefore record (H0) as an explicit hypothesis of the interacting theory. It is numerically testable (complete monotonicity of measured $\langle E(t)\rangle$; the dissipation identity $\dot{E} \leq 0$, its first consequence, is exact and is confirmed in the v2 suite), and it holds automatically at weak coupling where the linearization dominates. All results of this section are conditional on (H0) in the interacting case.

Hypothesis 2.4 (H1′: Uniform strict asymptotic freedom). *There exist $\delta > 0$ and $g_* > 0$ such that $|\beta_{\mathrm{GF}}(g)| \geq \delta g^3$ for all $g \geq g_*$.*

Justification. Perturbatively, $\beta_{\mathrm{GF}}(g) = -b_0 g^3 - b_1 g^5 - \cdots$ with $b_0 > 0$. Lattice step-scaling studies [8, 7] measure β_{GF} up to $g^2_{\mathrm{GF}} \sim 12$ for $SU(3)$ and find no flattening. *Failure mode.* If (H1′) fails, the coupling grows more slowly than any power of $\log(\mu_0/\mu)$ — a "walking" scenario inconsistent with confinement.

Remark 2.5 (Finite-volume corrections to β_{GF}). The coupling admits finite-volume corrections of order e^{-mL} where m is the mass gap (if it exists) [5]: if the transfer matrix has a spectral gap $\Delta > 0$ on Λ_L, then $|g^2_{\mathrm{GF},L} - g^2_{\mathrm{GF},\infty}| \leq Ce^{-\Delta L}$. All spectral arguments below are first performed at finite L and then taken to the thermodynamic limit; the finite-volume corrections are controlled by the uniform exponential clustering established in Section 4.

Hypothesis 2.6 (H2: Tauberian spectral regularity). *If the spectral measure $d\rho(\lambda)$ of the gauge-covariant Laplacian on adjoint-valued 2-forms has $\rho(\lambda) > 0$ for all $\lambda > 0$, then $\rho(\lambda) \geq c_\rho \lambda^{d/2}$ for $0 < \lambda \leq \lambda_0$, for some $c_\rho > 0$, $\lambda_0 > 0$.*

Justification. Here $\rho(\lambda)$ denotes the integrated spectral density. The exponent $d/2$ is the Weyl-law exponent, extended to the IR under the assumption that the spectrum is gapless, which is automatically satisfied by any conformal field theory. *When could (H2) fail?* A gapless theory with $\rho(\lambda) \sim \lambda^{d/2} |\log \lambda|^{-\alpha}$ would violate it; for pure $SU(N)$ this would require a walking regime, not observed in step-scaling studies [8], though not rigorously excluded.

Hypothesis 2.7 (H3: Osterwalder–Schrader positivity). *The Euclidean correlators obtained after the thermodynamic and continuum limits satisfy reflection positivity.*

Hypothesis 2.8 (H4: RG completeness). *$\beta_{\mathrm{GF}}(g)$ is continuous on $(0, \infty)$, and every RG trajectory with asymptotically free UV conditions extends to all $\mu > 0$, accumulating either at $g = \infty$ or at a fixed point.*

2.3 Spectral representation and monotonicity

Under (H0), from (5):

$$F(t) = -\dot{E}(t) = 2\int_0^\infty \lambda\, e^{-2\lambda t} d\rho(\lambda), \qquad \bar{\lambda}(t) \equiv \frac{\int \lambda\, e^{-2\lambda t} d\rho}{\int e^{-2\lambda t} d\rho}. \tag{6}$$

Proposition 2.9 (Monotonicity; given (H0))**.** $\frac{d}{dt}\bar{\lambda}(t) = -2\,\mathrm{Var}_t(\lambda) \leq 0$, *where the variance is taken under* $d\nu_t = e^{-2\lambda t} d\rho / E(t)$*. Equality holds iff* ρ *is a point mass.*

Corollary 2.10. $\lim_{t\to\infty} \bar{\lambda}(t) = \lambda_{\min} \equiv \inf \operatorname{supp} \rho$.

2.4 Main argument

Step 1: UV behaviour. By asymptotic freedom, $\bar{\lambda}(t) = \frac{d}{4t}(1 + O(g^2_{\mathrm{GF}})) \to +\infty$ as $t \to 0^+$.

Step 2: Monotonicity $\Rightarrow$ IR bound. By Proposition 2.9, $\bar{\lambda}(t) \leq \bar{\lambda}(t_0) < \infty$ for all $t \geq t_0$.

Step 3: (H1′) + (H2) $\Rightarrow$ spectral gap in $d = 3$.

Theorem 2.11 (Spectral gap, $d = 3$)**.** *Under Hypotheses (H0)–(H4) with* $d = 3$*: the spectral measure has a gap* $\Delta > 0$*.*

Proof. Suppose for contradiction that $\lambda_{\min} = 0$. *Lower bound on* g^2_{GF} *from the spectrum:* for any $\lambda_* > 0$, $E(t) \geq e^{-2\lambda_* t}\rho(\lambda_*)$. With $t = 1/(8\mu^2)$ and $\lambda_* = c\mu^2$ $(0 < c < 4)$:

$$g^2_{\mathrm{GF}}(\mu) = \frac{t^{3/2}}{\mathcal{N}} E(t) \geq \frac{e^{-c/4}\,\rho(c\mu^2)}{8^{3/2}\mathcal{N}\mu^3}. \tag{7}$$

Upper bound from (H1′): integrating $\mu\, dg/d\mu \leq -\delta g^3$ gives $g^2_{\mathrm{GF}}(\mu) \leq \frac{1}{2\delta \log(\mu_0/\mu)}(1 + o(1))$ as $\mu \to 0$. *Combining:* $\rho(c\mu^2) \leq C\mu^3/\log(\mu_0/\mu)$, i.e. $\rho(\lambda) \leq C'\lambda^{3/2}|\log\lambda|^{-1}$ for small λ. (v2 note: this inversion is *elementary* — it follows directly from the pointwise bound (7); no Tauberian theorem is needed for this direction. The "Tauberian" label of (H2) refers to the regularity hypothesis itself, which plays the role such theorems require.) But (H2) gives $\rho(\lambda) \geq c_\rho \lambda^{3/2}$; dividing, $c_\rho \leq C'/|\log\lambda| \to 0$, contradicting $c_\rho > 0$. Hence $\lambda_{\min} > 0$. □

Remark 2.12 ($d = 4$: the marginal case). In $d = 4$ the same argument gives: upper bound $\rho(\lambda) \leq C'\lambda^2|\log\lambda|^{-1}$, lower bound $\rho(\lambda) \geq c_\rho\lambda^2$. The formal contradiction has the same structure, but C' depends on non-perturbative corrections of order $b_0 g^2_{\mathrm{GF}}/(4\pi)^2$, which become $O(1)$ at the confinement scale: the upper bound becomes $\rho(\lambda) \leq C'(1 + \delta_{\mathrm{NP}}(\lambda))\lambda^2|\log\lambda|^{-1}$ with $\delta_{\mathrm{NP}} = O(1)$ for $\lambda \lesssim \Lambda^2_{\mathrm{QCD}}$. Since the factor $(1 + O(1))$ can compensate the $|\log\lambda|^{-1}$ decay, the contradiction cannot be established without non-perturbative control of δ_{NP}. This marginality reflects the genuine difficulty of the $d = 4$ problem. We note that Hypothesis 2.2 of the companion [23] ($\xi < \infty$ for all β) would immediately resolve it.

Step 4: Spectral gap $\Rightarrow$ exponential decay.

Theorem 2.13. *If* $\rho(\lambda) = 0$ *for* $\lambda \in [0, \Delta)$ *with* $\Delta > 0$*, then* $E(t) \leq Ce^{-2\Delta t}$ *for all* $t \geq 0$*.*

Theorem 2.14 (Main theorem: trivial IR in $d = 3$)**.** *Under Hypotheses (H0)–(H4) with* $d = 3$*: (i) the spectral measure has a gap* $\Delta > 0$*; (ii)* $E(t) \leq Ce^{-2\Delta t}$*; (iii)* $a_{\mathrm{IR}} = 0$*.*

Corollary 2.15 (Mass gap for pure $SU(N)$ YM, $d = 3$)**.** *Under the hypotheses of Theorem 2.14: the theory has a mass gap* $\Delta_{\mathrm{phys}} > 0$*.*

Argument (v2: retagged as physical input). For pure $SU(N)$ without matter, the classification "the only phases with $a_{\mathrm{IR}} = 0$ are (1) trivially gapped (confining) or (2) TQFT" is *imported physical input* (it rests on the expected landscape of IR phases, not on a theorem proved here); both classes imply exponential decay of connected correlators of local gauge-invariant operators. v1 presented this as a proof; v2 records it as the imported classification step, consistent with the CONDITIONAL tag of Table 1. □

2.5 The dimensional dichotomy

	$d = 3$	$d = 4$
Weyl exponent (H2)	$\rho \gtrsim \lambda^{3/2}$	$\rho \gtrsim \lambda^2$
Coupling bound (H1′)	$g^2 \leq O(\log)$	$g^2 \leq O(\log)$
Spectral bound	$\rho \leq C\lambda^{3/2}\lvert\log\rvert^{-1}$	$\rho \leq C\lambda^2\lvert\log\rvert^{-1}$
Contradiction?	**Yes**	**Marginal**

In $d = 3$, YM is super-renormalisable and perturbative corrections are genuinely small at scales $\mu \gg g^2$: the logarithmic contradiction is robust. In $d = 4$, the coupling is marginal and corrections become $O(1)$ at the confinement scale: this is the genuine difficulty of the Millennium Problem.

3 Spectral Calibration

3.1 Definitions and conventions

We work in $d = 4$ Euclidean space with gauge group $SU(N)$ and pure Yang–Mills. The flow equation is $\partial_t B_\mu = D_\nu G_{\nu\mu}$, $B_\mu(0, x) = A_\mu(x)$. The energy-dissipation identity gives $\frac{d}{dt}\langle E(t)\rangle = -\langle F(t)\rangle$, hence $R(t) = -\frac{d}{dt}\ln\langle E(t)\rangle$.

3.2 Free-field calibration: $R_{\mathrm{free}}(t) = 2/t$ in $d = 4$

Proposition 3.1. *In the free theory in $d = 4$: $R_{\mathrm{free}}(t) = 2/t$.*

Proof. $\langle E(t)\rangle \propto \int d^4k\, e^{-2k^2t} \propto t^{-d/2}$ gives $R = -\frac{d}{dt}\ln(t^{-d/2}) = d/(2t) = 2/t$. □

3.3 Hypothesis B′ and equivalences

Define $c(t) := t^{d/2}\langle E(t)\rangle$. Then $c'(t) = t^{d/2-1}\langle E(t)\rangle(\frac{d}{2} - tR(t))$, so

$$c'(t) > 0 \iff R(t) < \frac{d}{2t} \iff \beta_{\mathrm{GF}}(g) < 0. \tag{8}$$

Thus Hypothesis B′ ($R(t) < d/(2t)$ for all $t > 0$) is precisely monotone running of the gradient flow coupling.

3.4 One-loop verification

Using Lüscher's NLO expansion [5] for pure $SU(N)$: $R(t) = \frac{2}{t}\big(1 - \frac{b_0}{2}g^2(\mu) + O(g^4)\big)$ with $b_0 = 11N/(48\pi^2) > 0$: the correction is negative, $R(t) < 2/t$ for $g > 0$, confirming B′ at one loop.

3.5 The non-perturbative obstruction

A natural strategy for proving $R(t) < d/(2t)$ non-perturbatively would bound $F(t)$ by $(\mathrm{const}/t)E(t)$ using functional inequalities. However, the relevant quadratic form involves a Weitzenböck decomposition $D^*D = -D^2 + \mathcal{R}(G)$, where the curvature term $\mathcal{R}(G)$ is *indefinite* in the non-abelian theory. This prevents a diamagnetic comparison with the free Laplacian and is the central technical barrier to a non-perturbative proof in $d = 4$.

3.6 Numerical evidence (v2)

The v2 verification suite (`verification/2602-0020/` in the programme repository) adds four layers of numerical evidence. *(i) Exact spectral machinery:* on random discrete positive measures, the identity $d\bar{\lambda}/dt = -2\mathrm{Var}_t$ holds to 10^{-8} against finite differences, $\bar{\lambda} \to \lambda_{\min}$, and Theorem 2.13 is confirmed. *(ii) Dichotomy quantified:* a gapless Weyl density $d\rho \sim \lambda^{d/2-1}d\lambda$ gives $g^2_{\mathrm{GF}}(\mu)$ *constant* in the IR (machine-precision flat over three decades) — a conformal fixed point, incompatible with (H1′), exactly as Step 3 requires; imposing the (H1′) bound, the contradiction margin $c_\rho \leq C'/|\log|$ decays logarithmically in $d = 3$, while in $d = 4$ an $O(1)$ factor $(1 + \delta_{\mathrm{NP}})$ can absorb it (Remark 2.12 reproduced). *(iii) Free calibration on the lattice:* $R(t) = d/(2t)$ to 1% in the window $a^2 \ll t \ll L^2$ for $d = 3$ $(L = 64)$ and $d = 4$ $(L = 32)$. *(iv) First in-framework Monte Carlo test of B':* SU(2) Wilson-action heat-bath configurations on 8^4 at $\beta = 2.4$ (Kennedy–Pendleton, mean plaquette 0.6294, matching the known value), Wilson-flowed to $t = 2.4$: $\langle E(t)\rangle$ is strictly decreasing (the dissipation identity, the exact first consequence of the flow), $c(t) = t^2\langle E(t)\rangle$ *increases by* 42% across $t \in [0.4, 2.4]$, and $R(t) < 2/t$ pointwise in the window — i.e. $\beta_{\mathrm{GF}} < 0$ throughout the window — numerical support for B′ in the regime where lattice artifacts $(a^2 \ll 8t)$ and finite volume $(\sqrt{8t} \ll L)$ are simultaneously controlled. This does not prove B′ (single group, single β, one volume), but it is the first direct check of the paper's central hypothesis inside its own observable.

4 From Poincaré Inequality to Mass Gap

4.1 The lattice Poincaré inequality

We work on Λ_L with Gibbs measure $d\mu_\beta = Z^{-1}e^{-S_W}\prod_\ell dU_\ell$. For a gauge-invariant observable $\mathcal{O}$, define the flowed observable $\mathcal{O}^{(t)}[U] = \mathcal{O}[\{V_t(\ell; U)\}]$.

Theorem 4.1 (Gradient flow Poincaré inequality)**.** *For $SU(N)$ lattice gauge theory with Wilson action at $\beta > 0$, and any gauge-invariant $\mathcal{O} \in L^2(\mu_\beta)$:*

$$\mathrm{Var}_{\mu_\beta}(\mathcal{O}) \leq 2\int_0^\infty \sum_\ell \langle \|\nabla_\ell \mathcal{O}^{(t)}\|^2\rangle_{\mu_\beta} dt, \tag{9}$$

where ∇_ℓ is the gradient on $SU(N)$ with respect to $V_t(\ell)$.

Proof. Step (a): variance dissipation. $\mathrm{Var}(\mathcal{O}) = -\int_0^\infty \frac{d}{dt}\mathrm{Var}(\mathcal{O}^{(t)})\,dt$, using $\mathrm{Var}(\mathcal{O}^{(\infty)}) = 0$: by the Łojasiewicz gradient inequality for real-analytic functions on compact manifolds [19], each trajectory converges to a single critical point of S_W with finite arc length; averaged convergence follows by dominated convergence. *Step (b):* by the chain rule and integration by parts on $SU(N)$, $\frac{d}{dt}\mathrm{Var}(\mathcal{O}^{(t)}) = -2\sum_\ell\langle\|\nabla_\ell\mathcal{O}^{(t)}\|^2\rangle + 2\sum_\ell\langle(\nabla_\ell\mathcal{O}^{(t)}) \cdot \mathcal{Q}_\ell^{(t)}\rangle$, where $\mathcal{Q}_\ell^{(t)}$ collects nonlinear terms. By Cauchy–Schwarz, $|\langle\nabla_\ell\mathcal{O}^{(t)} \cdot \mathcal{Q}_\ell^{(t)}\rangle| \leq \kappa(\beta)\langle\|\nabla_\ell\mathcal{O}^{(t)}\|^2\rangle$. The force

$g_0^2 \partial_\ell S_W$ is $O(1)$ uniformly in β, while the nonlinear remainder carries an additional $g_0^2 = 2N/\beta$: $\kappa(\beta) \le 2NC_{\mathrm{Lie}}(N)/\beta < 1$ for $\beta > 2NC_{\mathrm{Lie}}(N)$, giving (9) with $C_{\mathrm{PI}} = 1/(1-\kappa(\beta))$ in that regime. *Step (c):* integration. □

Remark 4.2 (Poincaré constant: regimes and the L-uniformity caveat (v2)). Step (b) establishes the PI with L-uniform constant at weak coupling ($\beta > 2NC_{\mathrm{Lie}}$). At strong coupling ($\beta < \beta_0$), the cluster expansion directly establishes exponential clustering (Theorem 3.8 of [23]), hence a transfer-matrix gap, hence the PI. *v2 correction:* v1 additionally claimed the PI "unconditionally at finite β" via Bakry–Émery plus the Holley–Stroock perturbation lemma [21]. That route is correct but its constant is multiplied by $e^{\mathrm{osc}(S_W)}$, and $\mathrm{osc}(S_W) = O(\beta L^d)$: the resulting constant is exponential in the *volume*, hence useless for the L-uniform statement that Theorem 4.5 needs. The honest summary is: L-uniform PI is proven at both coupling extremes; its extension to all β is conditional on the absence of a phase transition where the gap closes (Hypothesis 1 of [23]).

Remark 4.3 (Morse genericity). If S_W is Morse (generic β), the stable manifold theorem implies the set of initial conditions converging to non-minimum critical points has μ_β-measure zero; the averaged convergence in Step (a) suffices, so we do not use this.

4.2 Exponential clustering via finite propagation speed

Proposition 4.4 (Gaussian localisation)**.** *For a gauge-invariant observable supported on $A \subset \Lambda_L$: $\|\nabla_\ell \mathcal{O}^{(t)}\| \le C_1 \|\mathcal{O}\|_\infty e^{-\mathrm{dist}(\ell,A)^2/(C_2 t)}$ for all $t > 0$ and links ℓ.*

Proof. Linear part: for the linearised flow, Gaussian localisation follows from standard heat-kernel estimates on lattice graphs [18]. *Nonlinear correction:* write $V_t = V_t^{(\mathrm{lin})} \exp(W_t)$; the remainder satisfies a forced heat equation with source of norm $\le g_0^2 C_{\mathrm{Lie}} \|W_t\| (\|\nabla V_t^{(\mathrm{lin})}\| + \|W_t\|)$; a Gronwall–Bihari comparison argument (v2: the unresolved citation of v1 is replaced by this self-contained statement; see e.g. the standard Bihari inequality) gives $\|W_t\|_\infty \le C g_0^2 t$ for $t \le t_*$ on the asymptotic-freedom trajectory. The chain rule then preserves the Gaussian factor with $C_2 = 4(1+\epsilon)$. At strong coupling the cluster expansion [9] provides exponential clustering directly, bypassing the flow. □

Theorem 4.5 (Exponential clustering)**.** *If the Poincaré constant in (9) is uniform in L, then for gauge-invariant $\mathcal{O}_A, \mathcal{O}_B$ with $\mathrm{dist}(A,B) = r$: $|\langle \mathcal{O}_A \mathcal{O}_B \rangle_c| \le C \|\mathcal{O}_A\|_\infty \|\mathcal{O}_B\|_\infty e^{-r/\xi}$, with $\xi = \sqrt{C_2 C_{\mathrm{PI}}/2}$.*

Proof. $\langle \mathcal{O}_A \mathcal{O}_B \rangle_c = 2\int_0^\infty I(t)\, dt$ with $I(t) = \sum_\ell \langle \nabla_\ell \mathcal{O}_A^{(t)} \cdot \nabla_\ell \mathcal{O}_B^{(t)} \rangle_c$. By Proposition 4.4 and $\mathrm{dist}(\ell,A)^2 + \mathrm{dist}(\ell,B)^2 \ge r^2/2$: $|I(t)| \le C' \|\mathcal{O}_A\| \|\mathcal{O}_B\| e^{-r^2/(2C_2 t)}$. Splitting at $t_* = r^2/(4C_2)$: for $t < t_*$ the Gaussian factor decays; for $t \ge t_*$, the PI with L-uniform constant gives $\mathrm{Var}(\mathcal{O}^{(t)}) \le \mathrm{Var}(\mathcal{O}) e^{-2t/C_{\mathrm{PI}}}$ (Gronwall on the variance dissipation), yielding $e^{-r^2/(2C_2 C_{\mathrm{PI}})}$. □

4.3 Connection to the physical mass gap

Proposition 4.6. *If all gauge-invariant local observables satisfy exponential clustering with rate $M > 0$, then the transfer matrix has spectral gap $\Delta_{\mathrm{phys}} \ge M$; conversely $\Delta_{\mathrm{phys}} > 0$ implies clustering with rate $M = \Delta_{\mathrm{phys}}$.*

Remark 4.7 (Combining Sections 2 and 4). The logical chain is: (H0) + (H1′)–(H4) $\Rightarrow$ spectral gap $\Delta > 0$ (§2, $d = 3$) $\Rightarrow$ [Poincaré $\Rightarrow$ clustering] (§4, dimension-independent) $\Rightarrow$ mass gap $\Delta_{\mathrm{phys}} > 0$.

5 Discussion and Open Problems

5.1 Comparison with other approaches

Our framework offers a complementary route to the constructive QFT programme: instead of constructing the measure directly, we reduce the problem to a global property of the β-function — a statement about a single scalar function of one variable. This reduction is analogous to how the a-theorem simplifies RG analysis. The conformal bootstrap [13] provides complementary constraints, consistent with our conclusion $a_{\mathrm{IR}} = 0$. (The typed, audit-first presentation — explicit hypotheses, status table, failure modes — follows the contract discipline used across the author's companion series, e.g. ai.viXra:2601.0066.)

5.2 The $d = 3$ vs. $d = 4$ dichotomy

In $d = 3$, super-renormalisability makes the logarithmic contradiction robust; this mirrors the state of rigorous results ($U(1)$: Göpfert–Mack [14]; $\mathbb{Z}_N$: Cao–Chatterjee [15]; $SU(N)$ open). In $d = 4$, even compact $U(1)$ has no rigorous mass gap proof.

5.3 Evidence for (H1′)

(i) Perturbation theory: $b_0 > 0$. (ii) Lattice step-scaling [8]: no flattening up to $g^2_{\mathrm{GF}} \sim 12$. (iii) Anomaly constraints: 't Hooft matching of the $\mathbb{Z}_N^{(1)}$ 1-form symmetry prohibits a trivially gapped symmetric phase [23]. (iv) *v2:* the SU(2) Monte Carlo test of Section 3.6: $\beta_{\mathrm{GF}} < 0$ throughout the measured window.

5.4 Relation to the companion paper

The companion [23] establishes the gap via anomaly algebra ⇒ phase exclusion ⇒ confinement, conditional on the absence of a bulk phase transition. The two reductions are complementary: this paper reduces to $\beta_{\mathrm{GF}} < 0$; the companion to a condition on the lattice phase diagram. Neither implies the other; both are supported by the same lattice data; a proof of either would yield the mass gap. The independent value of the present work lies in: (a) the $d = 3$ spectral reduction; (b) the identification of the Weitzenböck obstruction; (c) the gradient flow tools of independent interest.

5.5 The No-CFT Conjecture

Conjecture 1 (Trivial IR — Conjecture A)**.** *For pure $SU(N)$ Yang–Mills in $d = 4$, $N \geq 2$: $a_{\mathrm{IR}} = 0$; the theory does not flow to any non-trivial conformal fixed point.*

Conjecture 2 (No intermediate fixed points — Conjecture B)**.** *The RG flow of pure $SU(N)$ YM does not pass through any interacting conformal fixed point at intermediate scales.*

Conjecture 1, combined with the phase exclusion of [23] and the spectral framework of Section 2, would yield the mass gap without assuming (H1′). It is not implied by anomaly matching alone (a conformal phase can carry the anomaly through line operators). Evidence: no interacting IR fixed point of pure YM has been observed; $\beta < 0$ perturbatively; for $SU(2)$, $a_{\mathrm{UV}} = 31/60$ and no known CFT populates $(0, 31/60)$ with the required 1-form symmetries; bootstrap exclusion may become feasible.

5.6 Open problems

(1) Non-perturbative control of Weitzenböck terms. (2) Proving (H1′) non-perturbatively. (3) YM_3 as testbed: the $d = 3$ argument is closest to unconditional. (4) Combining the anomaly-algebraic and gradient-flow approaches. (5) *v2:* proving (H0) (complete monotonicity of $E(t)$) beyond the linearized regime, or replacing the spectral representation by a monotonicity structure intrinsic to the nonlinear flow.

Note added in v2

No v1 numbered statement, hypothesis or conclusion is changed; v2 is a status-honesty and evidence release. (1) The spectral representation (5) is made an explicit hypothesis (H0) for the interacting flow (it is equivalent to complete monotonicity of $E(t)$ and proven only for the linearized flow); Table 1 is retagged accordingly (v1 tagged it PROVEN without qualification). (2) The unresolved citation "[?, Lemma 6.4]" in Proposition 4.4 of v1 is replaced by a self-contained Gronwall–Bihari statement. (3) Remark 4.2: the Holley–Stroock route claimed in v1 to give the PI "unconditionally at finite β" yields a volume-exponential constant; the L-uniform claim is now correctly restricted to the two controlled regimes. (4) Corollary 2.15's proof step is retagged as imported physical input (phase classification at $a_{\mathrm{IR}} = 0$). (5) The Karamata attribution in Theorem 2.11 is replaced by the elementary direct bound actually used. (6) A verification suite is added (Section 3.6): exact spectral machinery, quantified dichotomy, lattice free-field calibration, and a first in-framework SU(2) Monte Carlo test of Hypothesis B′ (8^4, $\beta = 2.4$: $c(t)$ rises 42%, $R(t) < 2/t$ pointwise; plaquette 0.6294 validated against the literature).

A Conventions and Detailed Calculations

A.1 Gradient flow conventions

We follow Lüscher [5] throughout. The normalisation $\mathcal{N}$ in (4) is chosen so that $g^2_{\mathrm{GF}} = g^2_{\overline{\mathrm{MS}}} + O(g^4)$ at tree level. For $d = 4$, $N_f = 0$: $\mathcal{N} = 3(N^2 - 1)/(128\pi^2)$.

A.2 Derivation of $R_{\mathrm{free}} = d/(2t)$

In the free theory with propagator $\delta^{ab}\delta_{\mu\nu}/k^2$ (Feynman gauge), $|G_{\mu\nu}(k)|^2 = 2(k^2|A|^2 - |k\cdot A|^2)$ and $|k_\nu G_{\nu\mu}(k)|^2 = k^4|A_\mu|^2 - k^2|k\cdot A|^2$. After contracting and integrating with e^{-2k^2t}, the cleanest derivation uses $\langle E\rangle \propto t^{-d/2}$, hence $R = d/(2t)$.

A.3 One-loop β-function coefficient

$b_0 = \frac{11N}{3}\cdot\frac{1}{(4\pi)^2} = \frac{11N}{48\pi^2}$; in the gradient flow scheme $R(t) = \frac{d}{2t}\big(1 - \frac{b_0}{2}g^2(\mu) + O(g^4)\big)$.

A.4 Known rigorous results

Theory	Result	Method	Ref.
YM_2, any G	Exactly solvable	Migdal recursion	[16]
$\mathbb{Z}_N$, $d = 3$	Gap $\forall\beta$	Duality	[15]
$U(1)$ compact, $d = 3$	Gap + confinement	Monopoles	[14]
$SU(N)$, $d \geq 3$	Gap, $\beta < \beta_0$	Cluster exp.	[9]
$SU(N)$, $d \geq 5$	Triviality	Aizenman–Fröhlich	[17]
YM_4 continuum	**OPEN**	—	[1]

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