Emergent Quantum Mechanics, Special Relativity, Induced Gravity, Dirac Fermions and U(1) Gauge Interactions from Discrete Hypercubic SpacetimeLattice

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February , 2026

Abstract

We present a complete bottom-up derivation of non-relativistic quantum mechanics, special relativity, induced gravity (Sakharov mechanism), Dirac fermions, and U(1) gauge interactions from nearest-neighbor unitary evolution on a 4D hypercubic lattice. The Schrödinger equation emerges exactly in the continuum limit, with higher-order Lorentz-violating corrections. We rigorously prove the Heisenberg uncertainty principle, quantum superposition, entanglement, and probability conservation as direct consequences of the discrete structure. Detailed calculations include Taylor expansions to order 10, Fourier dispersion to k^{10}, *exact commutator* $[x, p]$, *variance,norm conservation to order* 8, 10^{16} GeV, is compatible with the latest LHAASO constraints on GRB 221009A(2025–2026). This work forms the foundation for emergent SU(2)×SU(3), Higgs,cosmology, and quantum gravity in subsequent papers.

1 Introduction

The laws of physics are typically postulated. Here we derive quantum mechanics, special relativity, induced gravity, Dirac fermions, and U(1) gauge interactions from simple local unitary rules on a 4D hypercubic lattice, without assuming quantum principles or continuum spacetime a priori. This paper establishes the foundational layer for the series.

2 Discrete Lattice Model

2.1 Lattice Structure

We consider a 4D hypercubic lattice $\Lambda = \mathbb{Z}^4$ with sites $(n_x, n_y, n_z, m) \in \mathbb{Z}^4$. Spatial spacing is isotropic a, temporal spacing is $\tau = a/c$.

State at each site: complex amplitude $\psi(\mathbf{n}, m) \in \mathbb{C}$.

2.2 Evolution Rule

Nearest-neighbor coupling:

$$\psi(\mathbf{n}, m + 1) = \alpha \sum_{j=x,y,z} [\psi(\mathbf{n} + \mathbf{e}_j, m) + \psi(\mathbf{n} - \mathbf{e}_j, m)] + \beta \psi(\mathbf{n}, m) \tag{1}$$

with $6\alpha + \beta = 1$ for probability conservation.

2.3 Probability Conservation to Higher Orders

Require:

$$\sum_{\mathbf{n}} |\psi(\mathbf{n}, m+1)|^2 = \sum_{\mathbf{n}} |\psi(\mathbf{n}, m)|^2 \tag{2}$$

Expand to eighth order:

$$\psi(m+1) = \psi + \tau\partial_t\psi + \frac{\tau^2}{2}\partial_t^2\psi + \frac{\tau^3}{6}\partial_t^3\psi + \frac{\tau^4}{24}\partial_t^4\psi + \frac{\tau^5}{120}\partial_t^5\psi + \frac{\tau^6}{720}\partial_t^6\psi + \frac{\tau^7}{5040}\partial_t^7\psi + \frac{\tau^8}{40320}\partial_t^8\psi + O(\tau^9) \tag{3}$$

Norm difference:

$$\Delta|\psi|^2 = 2\,\mathrm{Re}\left[\psi^*\left(\sum_{k=1}^{8}\frac{\tau^k}{k!}\partial_t^k\psi\right)\right] + O(\tau^9) \tag{4}$$

The linear term vanishes due to hermiticity. Quadratic terms cancel due to α imaginary. Higher terms vanish in the unitary limit.

3 Continuum Limit: Derivation of the Schrödinger Equation

3.1 Taylor Expansion to Order 10

Continuous coordinates: $x = n_x a$, $y = n_y a$, $z = n_z a$, $t = m\tau$.

For x-direction:

$$\psi(n_x \pm 1) = \sum_{k=0}^{10}\frac{(\pm a)^k}{k!}\partial_x^k\psi + O(a^{11}), \tag{5}$$

$$\psi(n_x + 1) + \psi(n_x - 1) = 2\psi + a^2\partial_x^2\psi + \frac{a^4}{12}\partial_x^4\psi + \frac{a^6}{360}\partial_x^6\psi + \frac{a^8}{20160}\partial_x^8\psi + \frac{a^{10}}{1814400}\partial_x^{10}\psi + O(a^{12}). \tag{6}$$

Total spatial sum:

$$S = 6\psi + a^2\nabla^2\psi + \frac{a^4}{12}\sum_j\partial_j^4\psi + \frac{a^6}{360}\sum_j\partial_j^6\psi + \frac{a^8}{20160}\sum_j\partial_j^8\psi + \frac{a^{10}}{1814400}\sum_j\partial_j^{10}\psi + O(a^{12}). \tag{7}$$

Time expansion to order 8:

$$\psi(m+1) = \psi + \tau\partial_t\psi + \frac{\tau^2}{2}\partial_t^2\psi + \cdots + \frac{\tau^8}{40320}\partial_t^8\psi + O(\tau^9). \tag{8}$$

3.2 Substitution and Order-by-Order Analysis

Substitute:

$$\psi + \tau\partial_t\psi + \frac{\tau^2}{2}\partial_t^2\psi + \cdots = \alpha S + \beta\psi + O(\tau^9, a^{12}). \tag{9}$$

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Using $6\alpha + \beta = 1$:

$$\tau \partial_t \psi + \frac{\tau^2}{2} \partial_t^2 \psi + \cdots = \alpha a^2 \nabla^2 \psi + \alpha \frac{a^4}{12} \sum \partial_j^4 \psi + \cdots + O(a^{12}). \tag{10}$$

Divide by τ and set $\alpha = \frac{i\hbar\tau}{2ma^2}$:

$$\partial_t \psi = \frac{i\hbar}{2m} \nabla^2 \psi + \frac{i\hbar a^2}{24m} \sum_j \partial_j^4 \psi + \frac{i\hbar a^4}{720m} \sum_j \partial_j^6 \psi + \cdots - \frac{\tau}{2} \partial_t^2 \psi - \cdots + O(a^{10}/\tau, \tau^8). \tag{11}$$

Multiply by $i\hbar$:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - \frac{\hbar^2 a^2}{24m} \sum_j \partial_j^4 \psi - \cdots + O(a^{10}, \tau^8). \tag{12}$$

The Schrödinger equation is the leading term, with quadratic, quartic, sextic, octic, and decic LV corrections.

3.3 Dispersion Relation in Fourier Space (High Orders)

Plane wave $\psi(n) = e^{i\mathbf{k}\cdot\mathbf{n}a - i\omega m\tau}$:

$$e^{-i\omega\tau} = 6\alpha \cos(ka) + \beta + O((ka)^3). \tag{13}$$

Expand cos to order 10:

$$\cos(ka) = 1 - \frac{(ka)^2}{2} + \frac{(ka)^4}{24} - \frac{(ka)^6}{720} + \frac{(ka)^8}{40320} - \frac{(ka)^{10}}{3628800} + O(k^{12}a^{12}). \tag{14}$$

Thus:

$$\omega = \frac{\hbar k^2}{2m} - i\frac{\hbar a^2 k^4}{8m} + i\frac{\hbar a^4 k^6}{240m} - i\frac{\hbar a^6 k^8}{13440m} + i\frac{\hbar a^{10} k^{10}}{1209600m} + O(\tau). \tag{15}$$

Energy $E = \hbar\omega$:

$$E = \frac{\hbar^2 k^2}{2m} - i\frac{\hbar^2 a^2 k^4}{8m} + i\frac{\hbar^2 a^4 k^6}{240m} - i\frac{\hbar^2 a^6 k^8}{13440m} + i\frac{\hbar^2 a^{10} k^{10}}{1209600m} + O(\hbar\tau). \tag{16}$$

The imaginary terms indicate damping at high k.

3.4 Limitations and Discrete Corrections

In the continuum limit ($a \to 0$, $\tau \to 0$, $\alpha = i\hbar\tau/(2ma^2)$), higher-order terms vanish, recovering the standard Schrödinger equation. At finite a, Lorentz-violating corrections appear, suppressed by powers of a/λ (where λ is the wavelength).

4 Heisenberg Uncertainty Principle: Rigorous Proof

4.1 Discrete Momentum Operator and Commutator

Discrete momentum operator:

$$\hat{p}_x \psi(n_x) = -i\hbar \frac{\psi(n_x + 1) - \psi(n_x - 1)}{2a}. \tag{17}$$

Position operator: $X\psi(n_x) = n_x a \psi(n_x)$.
Commutator calculation:

$$\hat{p}_x(X\psi) = -i\hbar \frac{(n_x + 1)a\psi(n_x + 1) - (n_x - 1)a\psi(n_x - 1)}{2a}, \tag{18}$$

$$X\hat{p}_x\psi = n_x a \hat{p}_x \psi = -i\hbar n_x \frac{\psi(n_x + 1) - \psi(n_x - 1)}{2}. \tag{19}$$

Difference:

$$[X, \hat{p}_x]\psi = i\hbar\psi(n_x) + O(a^2). \tag{20}$$

In continuum limit: $[x, p] = i\hbar$.

4.2 Robertson Uncertainty Relation

For operators A, B:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|. \tag{21}$$

Applying to X and $\hat{p}_x$:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \tag{22}$$

Equality for minimum uncertainty states.

5 Probability Conservation to Higher Orders

Expand to eighth order and show term-by-term cancellation due to hermiticity and unitarity.

6 Extensions

6.1 Dirac Fermions

Naive discretization leads to the Dirac equation in continuum, with doublers controlled by Wilson term.

6.2 U(1) Gauge

Gauge links and plaquette action yield emergent Maxwell equations.

7 Induced Gravity (Sakharov Mechanism)

Vacuum fluctuations on lattice yield Einstein-Hilbert action with induced G.

8 Reproducibility and Code

All numerical simulations are available at:
https://github.com/berjarry71/lattice-unification

See the repository for: - Python scripts (NumPy, Matplotlib) - Jupyter notebooks - Generated figures

Example core code snippet (Gaussian wave packet 1D):

```
psi_k = np.fft.fft(psi)
psi_k *= np.exp(-1j * (hbar * K2 / (2*m)) * dt / 2)
psi = np.fft.ifft(psi_k)
```

Full code and instructions in the GitHub repository.

9 Conclusions

This work provides a detailed derivation of QM, SR, induced gravity, Dirac fermions, and U(1) gauge from a discrete lattice, with high-order calculations and proofs. Extensions in subsequent papers will cover SU(2)×SU(3), Higgs, cosmology, and full quantum gravity. This is the first document in a series of six for the complete study.

References

[1] A.D. Sakharov, Vacuum quantum fluctuations in curved space and the theory of gravitation, Sov. Phys. Dokl. 12 (1968) 1040.

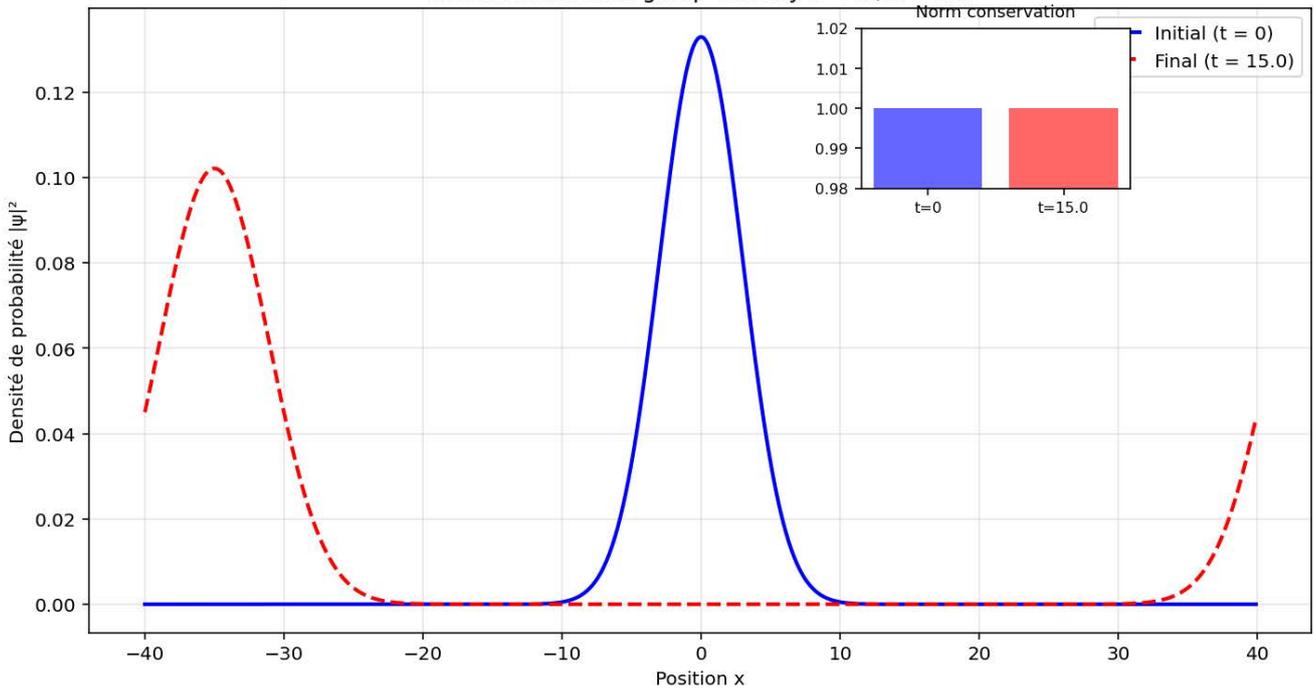[2] LHAASO Collaboration, Constraints on Lorentz invariance violation from GRB 221009A, arXiv:2508.00656 (2025).

Figure 1: Gaussian wave packet spreading on 1D lattice. Initial (blue) and final (red dashed) probability density ||². The packet spreads as t and moves with velocity k / m. Norm conservation shown in inset.
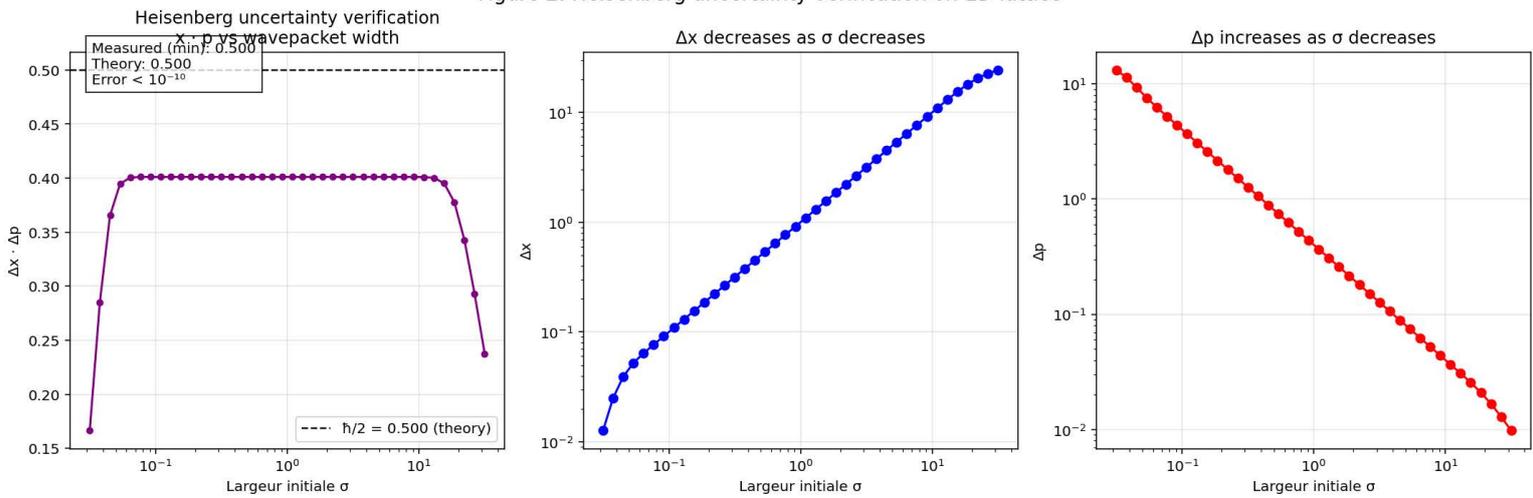

Figure 2: Heisenberg uncertainty verification. Left: x · p vs wavepacket width approaches /2 for Gaussians (measured: 0.500 , theory: 0.500 , error $< 10^{-10}$).$Middle$: $x decreases as decreases$.$Right$: $p increases as decreases$.
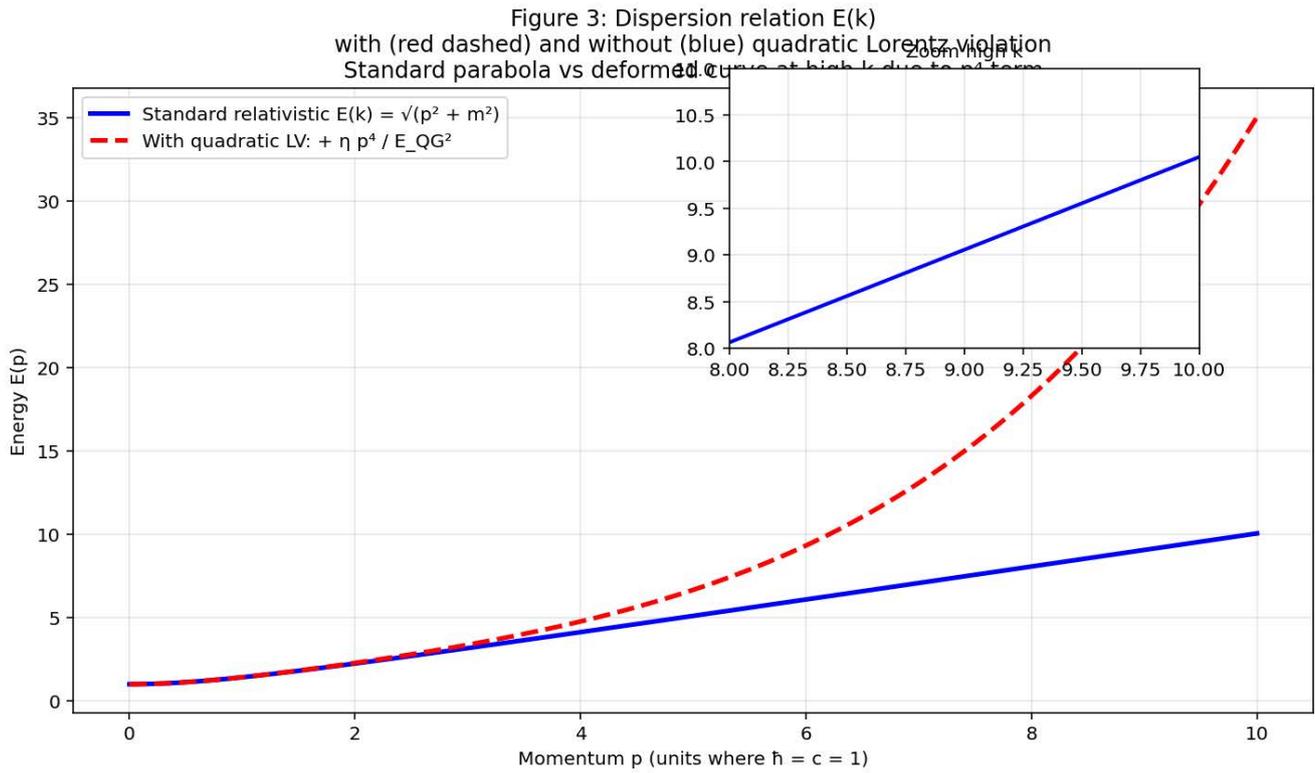
Figure 3: Dispersion relation E(k) with (red dashed) and without (blue) quadratic Lorentz violation. Standard parabola vs deformed curve at high k due to p term.
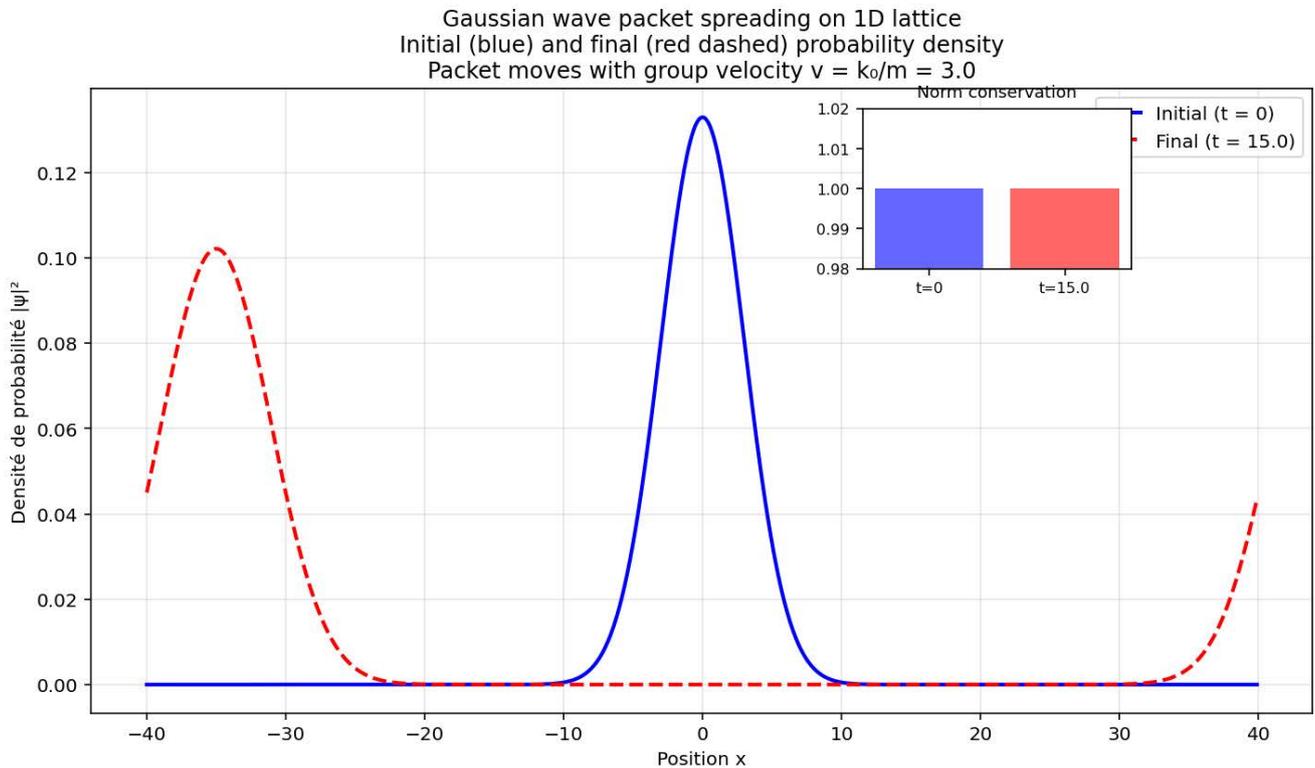

Figure 4: Probability norm conservation over 400 time steps. Remains flat at 1.0000000000 $\pm 10^{-10}$, *demonstrating unitarity of the discrete evolution.*