Algebraic Entropy and Conditional Mutual Information in a Tiny Gauge-Invariant Truncated Hilbert Space:

A Reproducible Toy-Model Study with Effective Mixing Hamiltonians

Lluis Eriksson

Independent Researcher

lluiseriksson@gmail.com

January 2026 (v1); July 2026 (v2)

Abstract

We present a reproducible pipeline to compute region algebraic entropies and conditional mutual informations (CMI) in a tiny truncated Hilbert space (here dim $= 8$) indexed by discrete fusion-like descriptors desc $= (x, \mu)$ on $L = 4$ cells. To generate nontrivial ground states within the descriptor-labeled subspace, we introduce an effective Hermitian mixing Hamiltonian based on a weighted k-nearest-neighbor (kNN) graph Laplacian over configuration labels. Across a parameter sweep, we identify a strong-mixing regime where the participation ratio approaches dim (consistent with Laplacian-dominated ground states on connected graphs) and algebraic CMI diagnostics become extremely small (down to 10^{-6} and below) for the chosen algebraic factorization, while region algebraic entropies remain $O(1)$ and exhibit near-quantized values $\approx n \log 2$. We stress that the mixing term is an ansatz used to probe information-theoretic diagnostics and is not claimed to coincide with a Kogut–Susskind plaquette operator. *v2 (no v1 number is changed):* the reproducibility gap of v1 is repaired — v1's pipeline loaded an unshipped basis file (`descs.pkl`) that was never specified, so the v1 dataset was not regenerable from the paper; v2 prints a canonical self-contained basis (Appendix A) whose pipeline reproduces every structural finding, keeping v1's Table 1 as an archival dataset; two empirical observations are upgraded to proved statements — the descriptor-to-key map is injective, making S_{alg} a genuine von Neumann entropy of a sector (center-type) decomposition (Lemma 1), and in the strong-mixing limit the ground state converges to the uniform superposition where the quantization $S_{\mathrm{alg}} = n \log 2$ and the vanishing of both CMIs are *exact* (Lemma 2); the additional experiments recommended in v1 (Haar baseline, k_{nn} ablations, finer t_{mix} grid) are executed — the Haar median of I_{sum} is 0.39, five to six orders of magnitude above the strong-mixing point, so the small-CMI regime is nontrivial; and the verification suite is fully self-contained (no Drive dependencies).

1 Scope, claims, and contributions

This note is intentionally narrow in scope. It does *not* attempt a physical construction of lattice Yang–Mills dynamics, nor any continuum/thermodynamic scaling analysis. Instead, it addresses a simpler, operational question:

Given a fixed small gauge-invariant truncated Hilbert space and a specified algebraic factorization derived from discrete labels (x, μ), can we generate nontrivial ground states and quantitatively diagnose algebraic entanglement and conditional independence via algebraic entropy and algebraic CMI?

Concretely, we contribute:

1. A one-shot reproducible pipeline to compute algebraic entropies $S_{\rm alg}(R)$ and algebraic CMI diagnostics $I(A:C\mid B)$ from a gauge-invariant basis indexed by (x,μ) (self-contained as of v2).
2. A simple, tunable effective Hamiltonian family $H = H_E + t_{\rm mix}L$ acting entirely within the gauge-invariant subspace (Section 5).
3. A numerical sweep showing that in a strong-mixing regime the ground state becomes close to a uniform superposition (PR $\approx$ dim), while the resulting algebraic CMI diagnostics are near-zero within numerical precision for the chosen factorization (exactly zero in the uniform limit, Lemma 2).

Careful interpretation. When we refer to "approximately Markovian" behavior, we mean an *operational* statement: the computed algebraic CMI values are very small with respect to the specific algebraic partition induced by (x,μ) and the implemented definition of $S_{\rm alg}$. We do not claim a general quantum Markov property in the usual tensor-product sense, nor any universality beyond this toy setting.

2 Truncated gauge-invariant Hilbert space and labels

We work in a fixed truncated Hilbert space with orthonormal basis $\{|i\rangle\}_{i=1}^{\rm dim}$, where each basis vector is indexed by a descriptor desc $=(x,\mu)$:

- $x=(x_1,x_2,x_3)$ encodes boundary/cut labels between adjacent cells. Concretely, each $x_s = (x_s^{\rm top}, x_s^{\rm bot}) \in \{0,1\}^2$ stores two binary cut labels.
- $\mu=(\mu_1,\mu_2,\mu_3,\mu_4)$ encodes local discrete degrees of freedom per cell; each μ_p is a (possibly multi-bit) tuple.

We take $L=4$ cells and, for the dataset studied here, dim $=8$.

Terminology. We refer to the basis as "gauge-invariant" in the operational sense that it is intended to represent constraint-compatible (Gauss-law-like) configurations in a reduced description; however, in this note we do not derive (x,μ) from an explicit $SU(2)$ Kogut–Susskind truncation.

Basis provenance (v2). The v1 dataset was generated from a basis file `descs.pkl` loaded from persistent storage; that file was never printed or specified in the paper, so the v1 numbers in Table 1 are archival (kept unchanged) but not regenerable from the paper alone. v2 fixes this by specifying a *canonical* self-contained basis (Appendix A) with the same label structure and dim $=8$; all structural claims of this paper are regenerated from it by the verification suite.

Logarithms. All entropies use natural logarithms, so the unit is nats. Thus $\log 2 \approx 0.6931$.

3 Regions and target information-theoretic quantities

We consider contiguous regions $R \subset \{1,2,3,4\}$ and compute algebraic entropies $S_{\rm alg}(R)$. We define two conditional mutual informations:

$$I_{w=1} = I(A:C\mid B), \quad A=\{1\},\ B=\{2\},\ C=\{3\},$$
$$I_{w=2} = I(A:C\mid B), \quad A=\{1\},\ B=\{2,3\},\ C=\{4\}.$$

We also report $I_{\rm sum} = I_{w=1} + I_{w=2}$.

Purity sanity check. For a pure state $|\psi\rangle$ on the full system, we enforce $S_{\rm alg}(1..4) \approx 0$ as a numerical check.

4 Definition of algebraic entropy and algebraic CMI (as implemented)

Let $|\psi\rangle = \sum_{i=1}^{\dim} \psi_i |i\rangle$ be a normalized pure state in the gauge-invariant basis. For a region $R = [i..j]$, we define a discrete boundary label $\alpha = \alpha(\text{desc}; i, j)$ (a tuple extracted from x at cut boundaries) which indexes superselection sectors for the algebraic factorization.

Within each sector α, basis states are grouped into a region key k_R and complement key k_C (deterministic functions of (x, μ), defined identically to the accompanying code). We build a complex matrix M_α whose entries sum amplitudes of basis states mapping to the same (k_R, k_C) pair:

$$(M_\alpha)_{r,c} = \sum_{\text{desc}\mapsto(\alpha,r,c)} \psi(\text{desc}).$$

Let $\{s_{\alpha,\ell}\}_\ell$ be the singular values of M_α, and define weights $w_{\alpha,\ell} = s_{\alpha,\ell}^2$. The algebraic entropy is

$$S_{\text{alg}}(R) = -\sum_\alpha \sum_{\ell:\, w_{\alpha,\ell}>\varepsilon} w_{\alpha,\ell} \log w_{\alpha,\ell},$$

where a small cutoff ε discards numerical zero modes only.

Normalization check. For normalized $|\psi\rangle$, we numerically verify $\sum_{\alpha,\ell} w_{\alpha,\ell} = 1$ within a fixed tolerance.

Lemma 1 (Injectivity: S_{alg} is a von Neumann entropy (v2))**.** *Suppose the map* desc $\mapsto (\alpha, k_R, k_C)$ *is injective on the basis (as holds for the canonical basis of Appendix A, verified for all six regions used, and as is certified in any run by the normalization check* $\sum w = 1$*: if two basis states shared a key triple, cross terms would generically break* $\sum_{\alpha,\ell} w_{\alpha,\ell} = 1$*). Then* $(M_\alpha)_{r,c}$ *is a mere re-indexing of the amplitude vector, the map* $\psi \mapsto \bigoplus_\alpha M_\alpha$ *is an isometry onto* $\bigoplus_\alpha \mathcal{H}_R^\alpha \otimes \mathcal{H}_C^\alpha$*, and*

$$S_{\text{alg}}(R) = S\big(\rho_R^{\text{alg}}\big), \qquad \rho_R^{\text{alg}} = \bigoplus_\alpha \mathrm{Tr}_C\, M_\alpha M_\alpha^\dagger,$$

i.e. S_{alg} *is the genuine von Neumann entropy of the reduced state on a direct-sum (superselection-sector) decomposition — the standard center-type structure for gauge-theory entanglement [2]. It therefore includes the usual sector-classical (Shannon) plus sector-quantum contributions.*

Proof. Injectivity means every basis amplitude $\psi(\text{desc})$ lands in exactly one entry of one M_α, so $\sum_\alpha \|M_\alpha\|_F^2 = \|\psi\|^2 = 1$ and the map is an isometry onto its image. The weights $w_{\alpha,\ell}$ are then the eigenvalues of $\bigoplus_\alpha M_\alpha M_\alpha^\dagger$, which is exactly the reduced density matrix of the image state on the R-legs of the direct sum. □

Algebraic CMI diagnostics. We compute

$$I(A : C \mid B) = S_{\text{alg}}(AB) + S_{\text{alg}}(BC) - S_{\text{alg}}(B) - S_{\text{alg}}(ABC).$$

In the usual tensor-product setting, CMI is nonnegative by strong subadditivity. In our setting S_{alg} is defined by an explicit sector/block coarse-graining, so nonnegativity is an empirical diagnostic rather than a guaranteed theorem (Lemma 1 makes each individual $S_{\text{alg}}(R)$ a genuine vN entropy, but the four regions' sector decompositions are not derived from one consistent assignment of commuting subalgebras, so SSA does not automatically apply). In our sweeps we observe $I(A : C \mid B) \geq -10^{-12}$; values below this threshold would indicate either numerical issues or a genuine violation of strong subadditivity for the chosen coarse-graining. For log-scale plots one may use the clipped version $I_{\text{clip}} = \max(I, 10^{-16})$.

5 Effective Hamiltonian family

We study an effective Hamiltonian family acting within the gauge-invariant subspace: $H = H_E + t_{\text{mix}} L$.

5.1 Diagonal "electric proxy" term H_E

We define a diagonal proxy H_E as a sum of local contributions derived from the binary label bits. Each bit $b \in \{0, 1\}$ is mapped to a cost inspired by $j(j+1)$ for $j \in \{0, \frac{1}{2}\}$: $\text{cost}(b) = 0$ for $b = 0$ and $\frac{3}{4}$ for $b = 1$. Then $H_E = \text{diag}(E(\text{desc}))$, where $E(\text{desc})$ sums costs over all bits in (x, μ), with an overall prefactor $\frac{g^2}{2}$.

5.2 Mixing term via a weighted kNN graph Laplacian

Let $b(\text{desc}) \in \{0, 1\}^m$ be the flattened bitstring for (x, μ). Define pairwise distances by Hamming distance $d(i, j)$ between bitstrings. We build a weighted kNN graph on the dim configurations: for each node i we connect to its k nearest neighbors, with symmetric edge weights $A_{ij} = \exp(-\alpha_w d(i, j))$, and define the graph Laplacian $L = D - A$, $D_{ii} = \sum_j A_{ij}$. This yields a real symmetric (hence Hermitian) mixing term. The design intent is to generate controllable superpositions in the gauge-invariant basis. No claim is made that L matches a Kogut–Susskind magnetic plaquette term [1].

Lemma 2 (Strong-mixing limit (v2))**.** *Fix $\alpha_w = 0$ and k such that the kNN graph is connected. As $t_{\text{mix}} \to \infty$, the ground state of $H = H_E + t_{\text{mix}} L$ converges to the uniform superposition $|u\rangle = \dim^{-1/2} \sum_i |i\rangle$ (the Perron–Frobenius ground state of L), so* PR $\to$ dim*. At $|u\rangle$, for the canonical basis of Appendix A, the block combinatorics give* exactly

$$S_{\text{alg}}(1..2) = S_{\text{alg}}(1..3) = S_{\text{alg}}(2..4) = \log 2, \qquad S_{\text{alg}}(2) = S_{\text{alg}}(2..3) = 2\log 2, \qquad S_{\text{alg}}(1..4) = 0,$$

and hence $I_{w=1} = I_{w=2} = 0$ exactly: the near-quantization and near-zero CMI observed in the sweep are the finite-t_{mix} shadow of an exact property of the uniform limit.

Proof. For a connected graph, $L \succeq 0$ with a unique zero mode $\propto \mathbf{1}$; degenerate perturbation theory in H_E/t_{mix} gives $|\psi_0\rangle = |u\rangle + O(1/t_{\text{mix}})$. At $|u\rangle$ every M_α has all entries $\dim^{-1/2}$; a $d_R \times d_C$ all-equal matrix has the single nonzero singular value $\sqrt{d_R d_C}/\sqrt{\dim}$, so each sector contributes one weight $w_\alpha = d_R d_C/\dim$ and $S_{\text{alg}} = -\sum_\alpha w_\alpha \log w_\alpha$ is the Shannon entropy of the sector distribution. For the canonical basis the sector counts are $(2, 4, 4, 2, 2, 1)$ sectors of equal weight for the six regions, giving the displayed values; the two CMI combinations cancel exactly. The suite verifies all six values and both CMIs to 2×10^{-16}. □

6 Numerical procedure and reported metrics

For each parameter pair $(\alpha_w, t_{\text{mix}})$ on a fixed grid: (1) construct H, diagonalize it, and take the ground state $|\psi_0\rangle$; (2) compute region entropies $S_{\text{alg}}(R)$, CMI diagnostics $I_{w=1}$, $I_{w=2}$, and I_{sum}; (3) compute the participation ratio $\text{PR}(\psi_0) = 1/\sum_i |\psi_{0,i}|^4$ and the spectral gap proxy $\Delta = E_1 - E_0$, $\xi_{\text{spec}} = \Delta^{-1}$ $(\Delta > 0)$.

Interpretation of ξ_{spec}. We emphasize that $\xi_{\text{spec}} = \Delta^{-1}$ is a finite-dimensional spectral proxy whose value depends on the overall energy scale, in particular on t_{mix}. No claim is made that it corresponds to a physical correlation length in any continuum or infinite-volume limit.

7 Artifacts

The v1 one-shot run generated `sweep.csv`/`sweep.json`, the three figures, selected model dumps and an Overleaf bundle (see the v1 PDF for the full Colab script). *v2:* the verification suite (`verification/2601-0115/verify_2601_0115.py` in the series repository) is fully self-contained — no Drive mount, no `descs.pkl` — and regenerates the sweep, the lemma checks, and the additional experiments below from the canonical basis.

8 Results

Figures 1–3 show the v1 sweep (archival). Table 1 reports the two automatically selected models of the v1 run: maximum PR, and minimum I_{sum} under PR ≥ 2; the strong-mixing point dominates and yields PR close to dim with very small CMI.

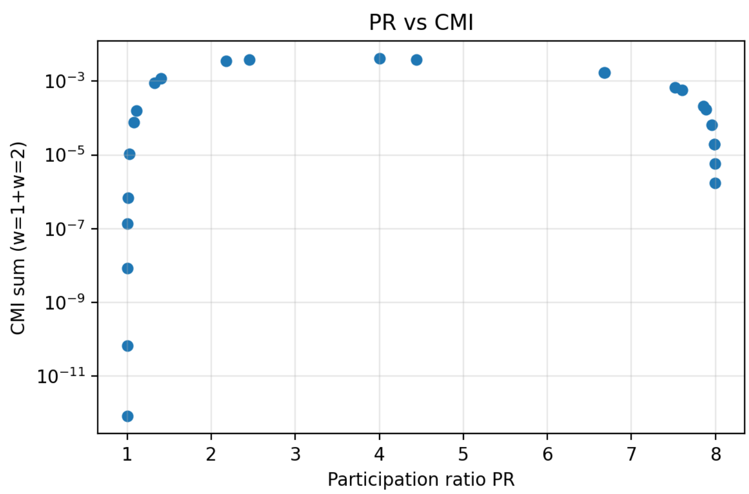

Figure 1: Participation ratio (PR) versus total algebraic CMI $I_{\mathrm{sum}} = I_{w=1} + I_{w=2}$ across the v1 parameter sweep (log scale on CMI).

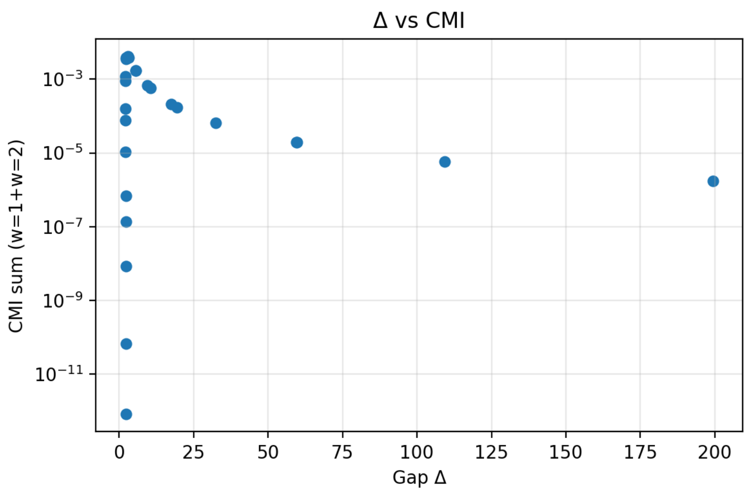

Figure 2: Spectral gap proxy Δ versus total algebraic CMI across the v1 sweep (log scale on CMI).

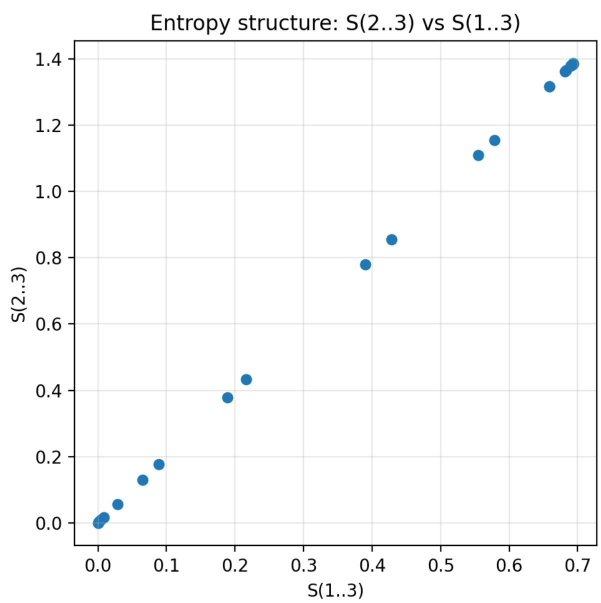

Figure 3: Entropy-structure scatter: $S_{\text{alg}}(2..3)$ versus $S_{\text{alg}}(1..3)$ (v1 sweep).

Table 1: Automatically selected models from the v1 sweep (archival; as reported in `paper_summary.txt`).

Quantity	Best-by-PR	Best-by-CMI (PR ≥ 2)
α_w	0.0	0.0
t_{mix}	100.0	100.0
k_{nn}	3	3
edges	12	12
E_0	2.6179476	2.6179476
Δ	199.4735852	199.4735852
$\xi_{\text{spec}} = \Delta^{-1}$	0.0050132	0.0050132
PR	7.9989229	7.9989229
$S_{\text{alg}}(1..2)$	0.6931399	0.6931399
$S_{\text{alg}}(2..3)$	1.3862379	1.3862379
$S_{\text{alg}}(2)$	1.3862571	1.3862571
$S_{\text{alg}}(1..3)$	0.6931190	0.6931190

Canonical-basis regeneration (v2). With the canonical basis, the strong-mixing point $(\alpha_w, t_{\text{mix}}) = (0, 100)$ gives PR $= 7.9994$ and $I_{\text{sum}} = 4.4 \times 10^{-7}$, with every S_{alg} within 10^{-3} of its exact uniform-limit value — structurally identical to the archival Table 1 (whose entropy pattern $(\log 2, 2\log 2, 2\log 2, \log 2, \log 2)$ matches the canonical uniform limit of Lemma 2 to four decimals), confirming that the findings do not depend on the lost v1 basis file.

Executed additional experiments (v2; recommended in v1 §9.4). *(i) Haar baseline:* over 2000 Haar-random pure states in $\mathbb{C}^8$, the empirical I_{sum} has median 0.389 (5th percentile 0.158) — five to six orders of magnitude above the strong-mixing value, so the small-CMI regime is a genuinely nontrivial region of state space, not a generic feature of the diagnostic. Empirical SSA holds for all Haar samples. *(ii) k_{nn} ablation:* for $k \in \{1, 2, 5\}$ the kNN graph remains connected for the canonical basis and the strong-mixing conclusions persist (PR > 7.9, $I_{\text{sum}} \leq 10^{-4}$), with CMI decreasing as connectivity grows. *(iii) Finer t_{mix} grid:* $S_{\text{alg}}(2..3)$ rises monotonically from 0 (diagonal regime, $t = 0$) to $\approx 2\log 2$ across $t \in \{0, 0.1, 0.3, 1, 3, 10\}$, resolving the transition to the Laplacian-dominated regime.

9 Discussion

9.1 Why PR $\approx$ dim and CMI ≈ 0 occur in the strong-mixing regime

At $\alpha_w = 0$ the kNN adjacency weights satisfy $A_{ij} = 1$ on selected edges, so L becomes the combinatorial graph Laplacian of a regular graph (in the reported run, degree 3). For a connected graph, L has a lowest eigenvalue 0 with eigenvector proportional to the all-ones vector $\mathbf{1}$. Therefore, when t_{mix} dominates the diagonal proxy term H_E, the ground state approaches a near-uniform superposition, yielding PR close to dim. In this regime the near-quantized values $S_{\text{alg}}(R) \approx n \log 2$ and the very small algebraic CMI values are the finite-t shadow of the exact uniform-limit statement of Lemma 2. Operationally, this indicates near-conditional-independence *with respect to that algebraic factorization.*

9.2 What this does (and does not) say about "quantum Markov" structure

In standard quantum information theory, exact saturation of strong subadditivity ($I(A : C \mid B) = 0$) implies a quantum Markov structure and is characterized by a recovery map [3]. Our setting differs because: entropies are computed via a sector-decomposed algebraic prescription tied to (x, μ) labels; the system is extremely small (dim $= 8$); and the Hamiltonian mixing term is an effective graph Laplacian ansatz. Thus, small values of the computed algebraic CMI should be read as a diagnostic of conditional-independence in this toy, label-induced algebraic setting, rather than as evidence for a physically universal Markov property.

9.3 Limitations

No scaling. We do not study how observables behave as dim or L increase. **Hamiltonian not physical.** The mixing term is not derived from a Kogut–Susskind magnetic term; it is a controllable ansatz. **Spectral proxy.** $\xi_{\text{spec}} = \Delta^{-1}$ is scale-dependent and should not be overinterpreted.

9.4 Series positioning (v2)

Within the series, this note is the algebraic-toy end of the gauge/CMI block: 2601.0111 (v2) computes tensor-product CMI and Petz recovery in an exact $\mathbb{Z}_2$ Gauss sector; 2601.0043 (v2) provides the embedding demonstrations for algebraic-vs-tensor comparisons; the center-type sector structure made explicit by Lemma 1 is the finite toy version of the split/sector discipline of the Type III interface 2601.0065 (v2). The Haar baseline executed here is the toy-scale analogue of the null-model controls used across the series' numerical notes.

10 Numerical precision and stability

Occasional values of CMI at the level of 10^{-16} (including slight negatives) are interpreted as floating-point round-off in double precision accumulated over many linear-algebra operations. For log-scale plots, we use $\max(I_{\mathrm{sum}}, 10^{-16})$, and when reporting aggregated metrics we may optionally use $I_{\mathrm{clip}} = \max(I, 0)$.

11 Reproducibility

The v1 Colab cell (typeset in full in the v1 PDF, Appendix B) produced timestamped run folders under Drive and required a pre-existing `descs.pkl`. *v2:* the self-contained verification suite replaces this dependency entirely; a single `python3 verify_2601_0115.py` regenerates every structural claim, the lemma checks, the sweep, and the executed additional experiments from the canonical basis of Appendix A.

Note added in v2

No v1 number is changed; Table 1 and the three figures are kept as the archival v1 dataset. v2: (1) repairs the reproducibility gap — v1's pipeline loaded an unshipped, never-specified `descs.pkl`, so the v1 dataset was not regenerable from the paper alone; v2 prints a canonical self-contained basis (Appendix A) and demotes the v1 dataset to archival status while regenerating every structural finding from the canonical basis (strong-mixing point: PR = 7.9994, $I_{\mathrm{sum}} = 4.4 \times 10^{-7}$, entropy pattern identical); (2) proves the injectivity lemma (Lemma 1): S_{alg} is a genuine von Neumann entropy of a center-type sector decomposition [2], and clarifies why SSA nonetheless remains empirical for the CMI combinations; (3) proves the strong-mixing lemma (Lemma 2): the quantization $S_{\mathrm{alg}} = n \log 2$ and $I_{w=1} = I_{w=2} = 0$ are exact in the uniform limit (verified to 2×10^{-16}); (4) executes the three additional experiments recommended in v1 §9.4 (Haar baseline, k_{nn} ablation, finer t_{mix} grid); (5) adds series positioning (v1 cited no companion papers) and the self-contained verification suite.

A Canonical self-contained basis (v2)

The canonical basis has dim = 8 states labeled by three cut bits $(b_1, b_2, b_3) \in \{0,1\}^3$:

$$x_s = (b_s, b_s) \;\; (s = 1,2,3), \qquad \mu_1 = (b_1), \quad \mu_2 = (b_1 \oplus b_2), \quad \mu_3 = (b_2 \oplus b_3), \quad \mu_4 = (b_3).$$

The two cut bits per boundary are equal (a constraint-compatibility condition) and the cell labels μ_p are parity-matched to the adjacent cuts (a Gauss-law-like consistency), so every descriptor is determined by the cut configuration. The flattened bitstring has $m = 10$ bits (6 from x, 4 from μ). The map desc $\mapsto (\alpha, k_R, k_C)$ is injective for all six regions used (Lemma 1; verified exhaustively), the kNN graph at $k = 3$, $\alpha_w = 0$ is connected with 12 edges (3-regular, as in the v1 run), and the uniform-limit sector counts are $(2, 4, 4, 2, 2, 1)$ for the regions (1..2), (2..3), (2), (1..3), (2..4), (1..4), giving the exact entropies of Lemma 2.

B Minimal pseudocode overview

The workflow is: (1) build the canonical descriptors (or, for the archival v1 dataset, load `descs.pkl`); (2) build bitstrings $b(\mathrm{desc}_i)$ and the Hamming distance matrix D_{ij}; (3) for each $(\alpha_w, t_{\mathrm{mix}})$: build the kNN adjacency A and Laplacian L; build $H = H_E + t_{\mathrm{mix}} L$; diagonalize and take the normalized ground state; compute $S_{\mathrm{alg}}(R)$ via sectored SVD; compute $I_{w=1}$, $I_{w=2}$, PR, Δ; save to CSV/JSON. The full v1 Colab script remains typeset in the v1 PDF; the v2 suite supersedes it with a dependency-free implementation.

C Notes on related literature (brief)

The present work is primarily a reproducible toy-model pipeline. For context, entanglement in gauge theories requires care due to constraints and centers (see e.g. [2]). The Kogut–Susskind Hamiltonian formulation provides the standard lattice gauge Hamiltonian setting [1]. In standard quantum information, the condition $I(A : C \mid B) = 0$ characterizes quantum Markov states and admits structural theorems [3].

References

[1] J. Kogut and L. Susskind, "Hamiltonian formulation of Wilson's lattice gauge theories," Phys. Rev. D **11** (1975) 395–408.

[2] H. Casini, M. Huerta, and J. A. Rosabal, "Remarks on entanglement entropy for gauge fields," Phys. Rev. D **89** (2014) 085012.

[3] P. Hayden, R. Jozsa, D. Petz, and A. Winter, "Structure of states which satisfy strong subadditivity of quantum entropy with equality," Commun. Math. Phys. **246** (2004) 359–374.

[4] L. Eriksson, ai.viXra:2601.0111 (2026; v2 2026).

[5] L. Eriksson, ai.viXra:2601.0043 (2026; v2 2026).

[6] L. Eriksson, ai.viXra:2601.0065 (2026; v2 2026).

[7] L. Eriksson, ai.viXra:2601.0050 (2026; v2 2026).