Conditional Mutual Information and Petz Recovery in a $\mathbb{Z}_2$ Lattice Gauge Ground State

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Abstract

We study approximate quantum Markov structure in a $\mathbb{Z}_2$ lattice gauge ground state using the conditional mutual information (CMI) $I(A : C \mid B(w))$ and the performance of Petz recovery across a family of tripartitions $(A, B(w), C)$ parameterized by a buffer width w. We consider a 2×4 plaquette lattice with open boundaries and qubits on links, restricted to a gauge-invariant (Gauss-law) physical sector, at coupling $g = 1.0$. For each w we compute reduced density matrices, the entropies entering the CMI, and a Petz-recovered state $\sigma_{ABC} = (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{Petz}}_{B\to BC})(\rho_{AB})$, reporting fidelity $F(\rho_{ABC}, \sigma_{ABC})$ via the recovery error $E_{\mathrm{rec}}(w) = -\log F$. The Overleaf project includes the plot, a formatted table, raw CSV outputs, and a hash-based manifest; the appendix typesets raw artifacts and pipeline code when present. We also report numerical cross-checks (dense vs. low-rank method agreement and trace stability) to support validity.

1 Introduction

The conditional mutual information (CMI) quantifies residual correlations between A and C conditioned on B:

$$I(A : C \mid B) = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC}),$$

where $S(\rho) = -\mathrm{tr}(\rho \log \rho)$. Small CMI is characteristic of (approximate) quantum Markov chains and is closely tied to the possibility of reconstructing ρ_{ABC} from ρ_{AB} by a recovery channel acting only on B. A canonical candidate is the Petz map; more generally, several recovery constructions yield quantitative links between CMI and achievable reconstruction fidelity [1–3].

Lattice gauge models provide a concrete setting to test these ideas under gauge constraints. In this short reproducible note we compute CMI and Petz recovery across a controlled buffer geometry in a $\mathbb{Z}_2$ gauge model, and provide all numerical artifacts for verification.

2 Model and gauge-invariant sector

We place qubits on links ℓ of a square lattice with open boundary conditions. Let X_ℓ and Z_ℓ denote Pauli operators on link ℓ. We use the Hamiltonian convention implemented in the accompanying code:

$$H(g) = -g \sum_{\ell} Z_\ell - \frac{1}{g} \sum_{p} X_p, \qquad X_p = \prod_{\ell\in\partial p} X_\ell,$$

where p runs over plaquettes and ∂p denotes the boundary links of plaquette p. We work at $g = 1.0$ on a 2×4 plaquette lattice.

Gauge invariance is enforced by restricting to the physical (Gauss-law) subspace generated by star operators

$$G_s = \prod_{\ell \in \mathrm{star}(s)} X_\ell, \qquad G_s \psi = \psi \;\text{ for all vertices } s.$$

In this convention one has $[H(g), G_s] = 0$ for all vertices s, so the dynamics preserves the $G_s = +1$ sector. We compute the ground state ψ_0 in this physical sector and set $\rho = \psi_0 \psi_0$.

3 Regions A, $B(w)$, C and buffer construction

Subsystems are defined as subsets of links; reduced states are obtained by tracing out complement links. We fix disjoint link sets A and C ("terminals") and define a separating wall W between them, chosen as a brute-force minimum vertex cut in the vertex-adjacency graph used by the pipeline.

The pipeline uses two link-adjacency graphs:

- **Vertex-adjacency** (links adjacent if they share a vertex), used to construct and verify a separating wall W (e.g. via a minimum cut / separator routine).
- **Plaquette-adjacency** (links adjacent if they belong to a common plaquette), used to grow the buffer.

Let $d_\square(\ell, W)$ be the graph distance from link ℓ to the wall W in the plaquette-adjacency graph. For integer $w \geq 0$ we define the buffer region

$$B(w) = \{\ell : \; d_\square(\ell, W) \leq w\} \setminus (A \cup C),$$

ensuring A, $B(w)$, and C are disjoint by construction. The exact link indexing and the sets for each w are recorded in `repro_manifest.json` and `z2_petz_results.csv`.

4 Methods: CMI and Petz recovery

For each w we form reduced density matrices ρ_{AB}, ρ_{BC}, ρ_B, and ρ_{ABC} by partial traces, and compute entropies from eigenvalues using a small eigenvalue cutoff to avoid numerical $\log(0)$ instabilities (the cutoff value is recorded in the manifest). The implementation is hybrid: we use dense linear algebra when $|ABC|$ is small, and a low-rank (LR+OB) pipeline for larger buffers; both methods are cross-checked in the overlap regime. We evaluate Petz recovery only for instances where the constructed wall separates A and C (recorded as `SEP` in the CSV).

To quantify local reconstructibility we apply the Petz recovery map:

$$\mathcal{R}^{\mathrm{Petz}}_{B \to BC}(X) = \rho_{BC}^{1/2} \, \rho_B^{-1/2} \, X \, \rho_B^{-1/2} \, \rho_{BC}^{1/2}.$$

Numerically, we stabilize the inverse square root by a fixed regularization

$$\rho_B^{-1/2} \approx (\rho_B + \delta I)^{-1/2}, \qquad \delta = 10^{-6},$$

as recorded in the manifest (parameter `DELTA_PETZ`). The recovered state is

$$\sigma_{ABC} = (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{Petz}}_{B \to BC})(\rho_{AB}),$$

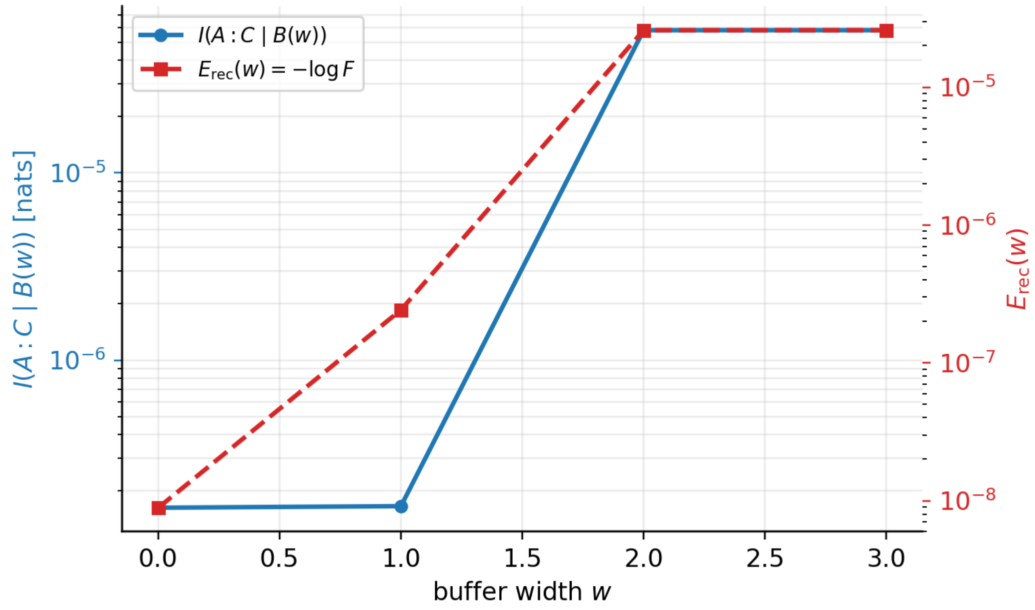

Figure 1: $\mathbb{Z}_2$ model on a 2×4 plaquette lattice at $g = 1.0$. Conditional mutual information $I(A : C \mid B(w))$ and Petz recovery error $E_{\mathrm{rec}}(w) = -\log F$ versus buffer width w.

and we explicitly renormalize $\sigma_{ABC} \leftarrow \sigma_{ABC}/\mathrm{tr}(\sigma_{ABC})$ before computing fidelity. We additionally symmetrize $\sigma_{ABC} \leftarrow (\sigma_{ABC} + \sigma_{ABC}^{\dagger})/2$ to suppress tiny non-Hermitian numerical noise. We verify $\sigma_{ABC} \succeq 0$ up to numerical tolerance (equivalently, we truncate tiny negative eigenvalues to zero) before evaluating the Uhlmann fidelity.

We report the Uhlmann fidelity

$$F(\rho_{ABC}, \sigma_{ABC}) = \left(\mathrm{tr}\sqrt{\sqrt{\rho_{ABC}}\sigma_{ABC}\sqrt{\rho_{ABC}}}\right)^2$$

and the recovery error $E_{\mathrm{rec}}(w) = -\log F$, where log denotes the natural logarithm (so E_{rec} is reported in nats).

5 Results

Figure 1 shows $I(A : C \mid B(w))$ together with $E_{\mathrm{rec}}(w)$. The CSV records whether each point was computed using the dense method or the LR+OB method (field `petz_method`). Table 1 reports the numerical values, including $1 - F$. The main qualitative observation in this instance is that Petz recovery remains high-fidelity (with recovery error of order 10^{-5}) over the tested buffer widths $w \in \{0, 1, 2, 3\}$, consistent with an approximately Markovian structure for this geometry at $g = 1.0$.

Table 1: $\mathbb{Z}_2$ model on a 2×4 plaquette lattice at $g = 1.0$. Region sizes and values of $I(A : C \mid B(w))$ and $1 - F$.

w	$\lvert A\rvert$	$\lvert B\rvert$	$I(A : C \mid B)$	$1 - F$
0	4	3	1.627985e-07	8.898586e-09
1	4	6	1.659899e-07	2.422318e-07
2	4	12	5.782748e-05	2.589104e-05
3	4	14	5.782759e-05	2.588882e-05

6 Validation checks

We include lightweight numerical checks to support reproducibility:

- **Method cross-check.** At the smallest buffer width where both implementations are feasible, dense evaluation versus the LR+OB pipeline agrees at the level $|\Delta E_{\text{rec}}| \approx 5.394 \times 10^{-8}$, and the trace agreement satisfies $|\Delta \text{tr}| \approx 5.6 \times 10^{-16}$ (see CSV for the corresponding rows).

- **Trace stability.** The recovered state remains close to normalized; for example we observe $\text{tr}(\sigma) \approx 0.9999855$ in the reported run (see CSV).

7 Scope and outlook

This paper reports one system size (2×4) and one coupling ($g = 1.0$). Immediate extensions include scanning g across regimes, finite-size scaling, and comparing Petz recovery to alternative recovery maps and precision controls.

8 Reproducibility artifacts

This Overleaf project expects the following files in the project root:

- `fig_z2_petz_recovery.png`
- `tab_z2_petz_data.tex`
- `z2_petz_results.csv`
- `repro_manifest.json`

The manifest records run metadata (model and numerical parameters, cutoffs) together with file hashes to enable integrity checks of the artifacts. If any are missing, the PDF will still compile and will display a placeholder box for the missing artifact.

References

[1] D. Petz, Sufficient subalgebras and the relative entropy of states of a von Neumann algebra, Commun. Math. Phys. **105** (1986) 123–131.

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[3] D. Sutter, M. Berta, and M. Tomamichel, Multivariate trace inequalities, Commun. Math. Phys. **352** (2017) 37–58.

[4] F. J. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, J. Math. Phys. **12** (1971) 2259.

[5] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Rev. Mod. Phys. **51** (1979) 659.

[6] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. **303** (2003) 2–30.

A Reproducibility appendix

A.1 Raw data (CSV): z2_petz_results.csv

```
Nx,Ny,g,E0_phys,w,nA,nB,nC,sep,nABC,CMI,Erec,F,1-F,tr_sigma,wall_size,petz_method
2,4,1.0,-22.997180401986128,0,4,3,4,SEP,11,1.6279854714307262e-07,8.898585574692225e
    -09,0.9999999911014144,8.898585646122115e-09,0.9999960024027247,3,[DENSE]
2,4,1.0,-22.997180401986128,1,4,6,4,SEP,14,1.6598989827087962e-07,2.4223181230346393e
    -07,0.9999997577682169,2.4223178307636317e-07,0.9999853324155468,3,[LR+OB rABC=256 rB=32]
2,4,1.0,-22.997180401986128,2,4,12,4,SEP,20,5.782748071088417e-05,2.5891375884829603e
    -05,0.9999741089592938,2.5891040706160773e-05,0.9999855315493759,3,[LR+OB rABC=4 rB=48]
2,4,1.0,-22.997180401986128,3,4,14,4,SEP,22,5.782758913397412e-05,2.58891519139754e
    -05,0.9999741111832071,2.5888816792885017e-05,0.9999855315493761,3,[LR+OB rABC=1 rB=48]
```

A.2 Manifest (hashes + metadata): repro_manifest.json

```
{
  "argv": [
    "z2_overleaf_bundle.py"
  ],
  "created_utc": "2026-01-26T18:38:59.941986+00:00",
  "files": [
    {
      "bytes": 80280,
      "name": "fig_z2_petz_recovery.png",
      "sha256": "17d33a18621200aee2c378adf2dec0f3a28e48713150f5571fe3ba9188f8d398"
    },
    {
      "bytes": 293,
      "name": "tab_z2_petz_data.tex",
      "sha256": "d07e016d50e15769d34eb96030fe9a26dd27eb02e40ec4b132003c50f70c4abc"
    },
    {
      "bytes": 764,
      "name": "z2_petz_results.csv",
      "sha256": "710f9a48a4511d61dd5cfa0989f60e9f7b3b824a8eaac762f96d1475dad38176"
    }
  ],
  "meta": {
    "A_patch": [
      0,
      2,
      10,
      11
    ],
    "C_patch": [
      6,
      8,
      19,
      20
    ],
    "DELTA_PETZ": 1e-06,
    "DENSE_MAX_NABC": 12,
    "G_LIST": [
      1.0
    ],
    "MINCUT_KMAX": 10,
    "Nlinks": 22,
    "Np": 8,
    "Nv": 15,
    "Nx": 2,
    "Ny": 4,
    "PETZ_REQUIRE_SEP": true,
    "W_LIST": [
      0,
      1,
      2,
      3
    ],
    "created_utc": "2026-01-26T18:38:59.941986+00:00",
    "dim_phys": 256,
    "wall": [
      13,
      14,
      15
    ],
    "wall_size": 3
  },
  "platform": "Linux-6.6.105+-x86_64-with-glibc2.35",
  "python": "3.12.12 (main, Oct 10 2025, 08:52:57) [GCC 11.4.0]"
}
```