Typed Pipeline for Recoverability–Rate–Power Links:
A Contract Paper with a Closed Recoverability Lane and Falsification Criteria

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Abstract

We present an audit-friendly *logical contract* for a multi-layer program connecting (i) static locality/Markovness, (ii) recoverability bounds, (iii) separation-dependent dissipation rates, and (iv) thermodynamic maintenance power. Each interface is typed with explicit quantifiers, tagged as **[PROVED]**/**[IMPORTED]**/**[ASSUMED]**/**[CONJECTURED]**, and paired with falsification routes. We do *not* claim a proof of the Clay Yang–Mills mass gap; we separate a Clay (closed-Hamiltonian) track from an operational (open-system/maintenance) track. As a fully closed lane inside this paper, we prove that an exponential conditional mutual information (CMI) decay hypothesis implies exponential recoverability via an imported Fawzi–Renner inequality, with fidelity conventions fixed explicitly.

1 Positioning and related work (minimal)

Recoverability from small conditional mutual information (CMI) was initiated in the finite-dimensional setting by Fawzi–Renner [1], with subsequent developments clarifying strengthened and universal recovery maps (see e.g. [2] and related work). Exponential CMI decay in Gibbs or quasi-local settings has been studied in several regimes in the many-body/QIT literature; in this paper we keep that input as an explicit falsifiable hypothesis (A-CMI) rather than embedding it as a black-box "Gibbs theorem" claim. The present contribution is the *typed contract* structure and the explicit separation between proved implications and bridge assumptions (especially toward operational power costs).

2 Executive summary (audit view)

Scope and non-claims

- **Not a Clay proof:** no claim is made about existence of continuum Yang–Mills theory or its Hamiltonian spectral gap.
- **This is a contract paper:** the value is explicit typing, status tagging, and falsification routes.
- **Closed lane included:** we include one fully self-contained implication:

  CMI decay (assumption) + FR-type recoverability (imported) $\Rightarrow$ recoverability decay (proved).

Item	Status	Depends on	Falsification route
L1: $1 - F \leq -\log F$	[PROVED]	elementary calculus	direct check
T2: FR-type inequality (constant-fixed)	[IMPORTED]	finite-dimensional QIT	disproof or convention mismatch
A-CMI: CMI decay hypothesis	[ASSUMED]	chosen model/geometry	exhibit family violating stated decay
T3: CMI decay $\Rightarrow$ recoverability decay	[PROVED]	A-CMI + T2 + L1	refute A-CMI or mismatch hypotheses
BATO-LAW: maintenance inequality	[ASSUMED]	operational model definition	violate hypothesis or give counter-protocol

Table 1: Audit status for this paper. Anything tagged **[ASSUMED]** is a typed hypothesis (not a hidden dependency).

Status table (what is actually proved inside this PDF)

3 Typing, notation, and objects

Remark 3.1 (Parameter convention). Throughout: ϵ is a **geometric** separation/collar parameter, while δ denotes **tolerances** (regularization thresholds, error budgets, floors). They are never interchanged.

Definition 3.2 (D1: Δ-track coherence)**.** Fix a dephasing (pinching) CPTP map Δ. Define

$$C_\Delta(\rho) := S(\rho \| \Delta[\rho]),$$

where $S(\cdot\|\cdot)$ is quantum relative entropy (with the usual support convention).

Definition 3.3 (D2: instantaneous coherence loss under fixed dynamics)**.** Let $T_t = e^{t\mathcal{L}}$ be a fixed CPTP Markov semigroup on states. Define

$$\dot{C}_{\Delta,\text{loss}}(\rho) := -\left.\frac{d}{dt}\right|_{t=0} C_\Delta(T_t(\rho)),$$

whenever the derivative exists.

Remark 3.4 (A minimal sufficient condition for existence). A sufficient (not necessary) condition for $\dot{C}_{\Delta,\text{loss}}(\rho)$ to exist is differentiability at $t = 0$ of $t \mapsto C_\Delta(T_t(\rho))$. In finite dimension this holds under mild spectral regularity (e.g. full-rank conditions), but this paper does not fix a universal regularity class; we treat existence as part of the typing of any application.

Definition 3.5 (D3: physically implementable local operations $\mathsf{Ops}_A^{\text{phys}}$)**.** Fix a reference state ρ_0, a tripartition A–B–C, and a separation notion $\text{sep}(A, C)$. Let $\mathsf{Ops}_A^{\text{phys}}$ denote a declared class of *physically implementable* operations on A (CPTP maps with local ancillas allowed and discarded), realizable within stated resource tolerances δ (time, energy, ancilla size, etc.).

Definition 3.6 (D4: the family $\mathcal{F}_\epsilon$ (minimal; typed by $\mathsf{Ops}_A^{\text{phys}}$))**.** Define

$$\mathcal{F}_\epsilon := \Big\{\rho = \Lambda_A(\rho_0) \ : \ \Lambda_A \in \mathsf{Ops}_A^{\text{phys}},\ \text{sep}(A, C) \geq \epsilon,\ C_\Delta(\rho) > 0\Big\}.$$

Assumption 3.7 (REG: regularity restrictions (invoked only when needed))**.** When needed to exclude trivial degeneracies (e.g. near-fixed points or nearly singular diagonals) we impose explicit regularity restrictions on $\mathcal{F}_\epsilon$, such as

$$\operatorname{dist}(\rho, \operatorname{Fix}(T)) \geq \delta_{\text{fix}} \quad \text{and/or} \quad \Delta[\rho] \succeq p_{\min}\mathbf{1}.$$

These restrictions are not part of the minimal definition of $\mathcal{F}_\epsilon$; they are invoked only where required and always stated at the point of use.

Definition 3.8 (D5: envelopes (quantifiers explicit))**.** Define the domain of the ratio at scale ϵ as

$$\mathcal{D}_\epsilon := \left\{\rho \in \mathcal{F}_\epsilon : \ C_\Delta(\rho) > 0, \ \dot{C}_{\Delta,\text{loss}}(\rho) \text{ exists}\right\}.$$

Then define

$$\kappa^{\uparrow}(\epsilon) := \sup\left\{\frac{\dot{C}_{\Delta,\text{loss}}(\rho)}{C_\Delta(\rho)} : \ \rho \in \mathcal{D}_\epsilon\right\}, \qquad \kappa^{\downarrow}(\epsilon) := \inf\left\{\frac{\dot{C}_{\Delta,\text{loss}}(\rho)}{C_\Delta(\rho)} : \ \rho \in \mathcal{D}_\epsilon\right\}.$$

Remark 3.9 (Empty-domain convention). In any concrete application we assume $\mathcal{D}_\epsilon \neq \emptyset$ in the regime of interest. If $\mathcal{D}_\epsilon = \emptyset$, we adopt the convention

$$\kappa^{\uparrow}(\epsilon) := 0, \qquad \kappa^{\downarrow}(\epsilon) := +\infty,$$

so that envelope statements become vacuous rather than ambiguous. (Any alternative convention should be stated explicitly if used.)

Remark 3.10 (Directionality (golden rule)). A minimum-power statement requires a **lower envelope** bound (involving $\kappa^{\downarrow}$), not an upper envelope bound (involving $\kappa^{\uparrow}$).

4 Closed recoverability lane: CMI decay ⇒ exponential recovery

Definition 4.1 (Fidelity convention)**.** For density matrices ρ, σ on the same finite-dimensional Hilbert space, we use the *squared* Uhlmann fidelity

$$F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 \in [0, 1].$$

Definition 4.2 (Recovery error)**.** For a tripartite finite-dimensional state ρ_{ABC} and a recovery map $\mathcal{R}_{B\to BC}$, define

$$E_{\text{rec}}(\rho_{ABC}; \mathcal{R}) := -\log F(\rho_{ABC}, (\text{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB}))\,.$$

Lemma 4.3 (L1: elementary inequality)**.** ***[PROVED]****.*
For $F \in (0, 1]$ *one has* $1 - F \leq -\log F$*.*

Proof. Define $g(F) := -\log F - (1 - F)$ on $(0, 1]$. Then $g'(F) = -\frac{1}{F} + 1 \leq 0$ for $F \in (0, 1]$ and $g(1) = 0$, so $g(F) \geq 0$ for all $F \in (0, 1]$. □

Theorem 4.4 (T2: FR-type recoverability inequality (constant explicit))**.** ***[IMPORTED]*** *(finite-dimensional recoverability literature; see e.g. [1, 2]).*
There exists a universal constant $c_{\text{FR}} > 0$ *(depending only on the choice of fidelity/error convention) such that for any finite-dimensional* ρ_{ABC}*, there exists a CPTP recovery map* $\mathcal{R}_{B\to BC}$ *with*

$$E_{\text{rec}}(\rho_{ABC}; \mathcal{R}) \ \leq \ c_{\text{FR}}\, I(A : C|B)_\rho.$$

Remark 4.5 (Convention lock-in). This paper fixes: (i) squared fidelity Definition 4.1, and (ii) error functional $E_{\rm rec} = -\log F$ Definition 4.2. The value of $c_{\rm FR}$ should be aligned to the imported theorem under these conventions (Appendix A gives the alignment step). Using $c_{\rm FR}$ avoids silent factor drift across conventions.

Assumption 4.6 (A-CMI: exponential CMI decay in a shielded geometry)**. [ASSUMED]** (model/geometry hypothesis).
Fix a family of tripartitions A–$B(\epsilon)$–C ("collar" of width/separation ϵ) and a family of finite-dimensional states $\rho_{ABC}(\epsilon)$. Assume there exist constants $K > 0$ and $\alpha > 0$ (independent of ϵ in the regime of interest) such that

$$I(A : C|B)_{\rho(\epsilon)} \;\le\; K e^{-\alpha\epsilon}.$$

Remark 4.7 (Where A-CMI is expected/known). A-CMI is intended to capture "approximate quantum Markov" behavior in geometries where $B(\epsilon)$ shields A from C. In concrete applications one either: (i) imports a theorem establishing such decay in a stated regime (e.g. a temperature/interaction range regime), or (ii) tests it numerically/empirically in a benchmark family. This paper keeps the hypothesis explicit so that failures are interpretable as falsification rather than hidden assumption.

Theorem 4.8 (T3: exponential recoverability under exponential CMI decay)**. *[PROVED]*** *(given Definitions 4.4 and 4.6).*
Under Definition 4.6, for each ϵ there exists a CPTP recovery map $\mathcal{R}_{B\to BC}$ such that

$$E_{\rm rec}(\rho(\epsilon); \mathcal{R}) \;\le\; c_{\rm FR}\, K e^{-\alpha\epsilon}.$$

Moreover,

$$1 - F(\rho_{ABC}(\epsilon), (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB}(\epsilon))) \;\le\; c_{\rm FR}\, K e^{-\alpha\epsilon}.$$

Proof. Combine Definition 4.4 with Definition 4.6. Then apply Definition 4.3. □

Remark 4.9 (Why T3 is still worth stating). T3 is logically simple, but it is *audit-critical*: it fixes conventions, isolates the single falsifiable hypothesis A-CMI, and cleanly separates the imported recoverability mechanism from downstream "rate–power" bridges.

5 Operational power interface (typed, not proved here)

Definition 5.1 (D6: maintenance strategies and average power)**.** Fix an operational control model (battery, allowed controls, coupling to noise), and an admissible strategy class Strat. Fix an error budget δ and a noise semigroup T_t. For a target state ρ, let $P(\rho; \mathsf{S}) \in [0, \infty]$ denote the average battery power required by a strategy $\mathsf{S} \in \mathsf{Strat}$ to maintain ρ under T_t within tolerance δ (as defined by that model).

Definition 5.2 (D7: extra maintenance power)**.** Define the extra power as

$$P_{\rm extra}(\rho) := \max\Big\{0,\ \inf_{\mathsf{S}\in\mathsf{Strat}} P(\rho; \mathsf{S}) \;-\; \inf_{\mathsf{S}\in\mathsf{Strat}} P(\Delta[\rho]; \mathsf{S})\Big\}.$$

Assumption 5.3 (BATO-LAW: maintenance inequality (operational postulate))**. [ASSUMED]** (typed law to be proved or imported in a companion note).
Under the hypotheses of the chosen control model, for all target states ρ one has

$$P_{\rm extra}(\rho) \;\ge\; k_B T\, \dot{C}_{\Delta,\rm loss}(\rho),$$

whenever $\dot{C}_{\Delta,\rm loss}(\rho)$ exists.

Corollary 5.4 (C1: envelope consequence (directionality explicit)). *Assume Definition 5.3. Assume Definition 3.7 whenever needed to exclude degeneracies. Then for $\rho \in \mathcal{F}_\epsilon$,*

$$P_{\text{extra}}(\rho) \;\geq\; k_B T\, \kappa^{\downarrow}(\epsilon)\, C_\Delta(\rho).$$

Proof. By Definition 5.3, $P_{\text{extra}}(\rho) \geq k_B T\, \dot{C}_{\Delta,\text{loss}}(\rho)$. By Definition 3.8, $\dot{C}_{\Delta,\text{loss}}(\rho) \geq \kappa^{\downarrow}(\epsilon) C_\Delta(\rho)$ for $\rho \in \mathcal{F}_\epsilon$. □

6 Two small examples (toy but honest)

Example 6.1 (Exact Markov case ($I(A:C|B) = 0$ implies perfect recovery)). If a state satisfies $I(A:C|B)_\rho = 0$ (i.e. it is a quantum Markov chain in the exact sense), then recoverability can be perfect: there exists a CPTP recovery map $\mathcal{R}$ such that

$$F(\rho_{ABC}, (\text{id}_A \otimes \mathcal{R})(\rho_{AB})) = 1 \quad \Rightarrow \quad E_{\text{rec}} = 0.$$

This anchors the interpretation of T2/T3: CMI is the static "Markovness defect" that controls recoverability; the implication $I(A:C|B)_\rho = 0 \Rightarrow$ exact recovery is standard (see e.g. [3]).

Example 6.2 (A semigroup where $\dot{C}_{\Delta,\text{loss}}$ is directly computable (envelope intuition)). Let Δ be dephasing in a fixed basis and let T_t be pure dephasing in the same basis (so T_t commutes with Δ and damps off-diagonal terms exponentially). In such cases, $C_\Delta(T_t(\rho))$ is typically monotone decreasing in t on broad families, and $\dot{C}_{\Delta,\text{loss}}(\rho)$ exists and is computable from the initial damping rate. This illustrates what $\kappa^{\uparrow}(\epsilon)$ and $\kappa^{\downarrow}(\epsilon)$ measure: worst-case and best-case *initial relative* coherence-loss rates over a typed family $\mathcal{F}_\epsilon$. (We do not claim a universal closed form for $\dot{C}_{\Delta,\text{loss}}$ in this paper; the point is operational meaning.)

7 Falsification matrix (minimal)

Item	Status	Verification proxy	Concrete falsifier
A-CMI (CMI decay)	**[ASSUMED]**	compute/estimate $I(A:C|B)$ vs ϵ	show non-decay or slower decay than claimed
T2 (FR-type)	**[IMPORTED]**	align c_{FR} to cited convention	detect convention mismatch / wrong constant
BATO-LAW	**[ASSUMED]**	explicit model calculation	explicit protocol violating inequality
Lower-envelope need	**[PROVED]**	logic in Definitions 3.10 and 5.4	N/A (directionality statement)

Table 2: Minimal falsification matrix for this contract paper.

8 Clay vs operational track (explicit separation)

Clay track (closed Hamiltonian)

The Clay mass gap concerns a *closed* Yang–Mills QFT in the continuum and its spectral gap in an axiomatic setting.

Operational track (this paper)

This paper treats an open-system/maintenance track: typed inequalities relating coherence loss and power. Any bridge from operational rate floors to closed Hamiltonian gaps would require additional hypotheses not provided here.

A Imported theorem conventions (how to set c_{FR})

To make Definition 4.4 fully audit-aligned, pick a single source theorem statement and restate it verbatim under the conventions:

- squared fidelity F Definition 4.1,
- error functional $E_{\mathrm{rec}} = -\log F$ Definition 4.2.

Then set c_{FR} to the numerical constant appearing in that statement after translating conventions. Using c_{FR} prevents silent factor drift across papers.

B Deferred interfaces (not used in this paper)

To keep this paper self-contained and citable, we *do not* develop here:

- Type III/AQFT split-regularized CMI definitions and conditional expectations (a separate interface note),
- Davies/Dirichlet-form identities and RIP-U envelope bounds (a separate dynamics note),
- non-abelian benchmark implementation contracts and reproducibility harness (separate benchmark infrastructure notes).

Nothing in Sections 4–6 depends on these deferred components.

C Upgrade path for BATO-LAW (audit guidance)

In this paper, BATO-LAW is treated as **[ASSUMED]**, since its validity is operational-model dependent. To upgrade Definition 5.3 to **[IMPORTED]** or **[PROVED]** within a specified control model, it suffices to provide:

- an axiomatized specification of the control model (battery, admissible controls, error metric, and the maintenance criterion within tolerance δ);
- either a complete proof within that model or an imported theorem with hypotheses restated verbatim under the present conventions;
- an explicit identification (or inequality) relating the model's dissipation functional to $\dot{C}_{\Delta,\mathrm{loss}}(\rho)$ as defined in Definition 3.3.

References

[1] O. Fawzi and R. Renner, Quantum conditional mutual information and approximate Markov chains, *Commun. Math. Phys.* **340**, 575–611 (2015).

[2] D. Sutter, M. Tomamichel, and P. Harremoës, Strengthened monotonicity of relative entropy via pinched Petz recovery map, *IEEE Trans. Inf. Theory* **62**, 2907–2913 (2016).

[3] P. Hayden, R. Jozsa, D. Petz, and A. Winter, Structure of states which satisfy strong subadditivity of quantum entropy with equality, *Commun. Math. Phys.* **246**, 359–374 (2004).