Split-regularized recoverability in Type III AQFT

Conditional expectations, split-dependent CMI, and an audit-friendly recoverability contract

Lluis Eriksson

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Abstract

Local algebras in relativistic quantum field theory are typically Type III, so reduced density matrices and von Neumann entropies are not available without additional structure. We give a B-minimal, audit-friendly interface for recoverability in Type III AQFT: we fix a collar geometry and a split datum N (an intermediate Type I factor) and define (i) a split-regularized conditional mutual information (CMI) and (ii) a Bures-fidelity-based recovery error for normal states. We isolate, as explicit assumptions, the two hard bridges needed for an exponential recoverability statement in Type III: (a) existence of an ω_0-preserving conditional expectation onto N (a Takesaki-type condition) and (b) an FR-type inequality in the fixed split implementation. We prove a conditional theorem: if split-regularized CMI decays exponentially in the collar width and an FR-type inequality holds in that split, then recoverability error decays exponentially, with constants tracked explicitly. This paper makes no Clay mass-gap claim and does not invoke von Neumann entropy on Type III algebras without split regularization.

1 Scope and philosophy

Goal. Provide a standalone, typed interface note for recoverability in Type III AQFT that:

- avoids traces and finite-dimensional density matrices on Type III algebras,
- makes all dependence on the *split datum* explicit,
- separates proved content from bridge assumptions with falsification routes.

Non-goals. We do not prove the Clay Yang–Mills mass gap and do not claim that the split property alone implies exponential decay of a Markovness proxy.

Parameter convention. Throughout, ϵ is a **geometric** collar/separation parameter, while δ denotes **tolerances** (regularization, thresholds). They are never interchanged.

2 Setup: regions, algebras, and normal states

We work in a Haag–Kastler setting: to each spacetime region O we associate a von Neumann algebra $\mathcal{A}(O)$ acting on a Hilbert space $\mathcal{H}$, with isotony and locality. Fix three regions A, B, C where B is a collar separating A from C at scale ϵ.

Definition 2.1 (D1: tripartition at the algebra level)**.** Define the relevant local algebras

$$\mathcal{A}_A := \mathcal{A}(A), \qquad \mathcal{A}_B := \mathcal{A}(B), \qquad \mathcal{A}_{BC} := \mathcal{A}(B \cup C), \qquad \mathcal{A}_{ABC} := \mathcal{A}(A \cup B \cup C).$$

A state ω is a *normal* state on $\mathcal{A}_{ABC}$, and ω_X denotes its restriction to $\mathcal{A}_X$.

Remark 2.2 (Why we work with normal states). Normal states are the natural notion of states on von Neumann algebras in AQFT and interact well with modular theory, conditional expectations, and Bures-type metrics.

3 Split datum (Type I implementation)

Assumption 3.1 (SPLIT: split inclusion datum)**. [ASSUMED].**
There exists a slightly thickened collar B' with $B \subset B' \subset B \cup C$ and an intermediate Type I factor N such that

$$\mathcal{A}(B) \subset N \subset \mathcal{A}(B') \subset \mathcal{A}(B \cup C) = \mathcal{A}_{BC}.$$

We call N the *split datum* associated with the collar geometry at scale ϵ.

Remark 3.2 (Typing intent). SPLIT is not a decay statement. It is a structural input allowing a Type I bookkeeping compatible with the collar geometry. All quantities defined below are explicitly N-dependent.

Assumption 3.3 (SEL: split selection rule)**. [ASSUMED].**
A selection rule $\epsilon \mapsto N(\epsilon)$ is fixed for the regime of interest. All constants in decay statements are allowed to depend on the chosen selection rule, but such dependence must be declared.

4 Channels and the recovery task (Heisenberg picture)

Remark 4.1 (Heisenberg picture convention). All channels between von Neumann algebras are taken in the Heisenberg picture: a channel $\Phi : \mathcal{M}_2 \to \mathcal{M}_1$ is normal, completely positive and unital. Its predual action on normal states is $\Phi_* : \mathcal{S}_n(\mathcal{M}_1) \to \mathcal{S}_n(\mathcal{M}_2)$ given by $\Phi_*(\omega) := \omega \circ \Phi$.

Definition 4.2 (D2: recovery channel and split-implemented recovery)**.** Fix a split datum N for the collar geometry (as in Definition 3.1). A recovery channel is a normal CP unital map (Heisenberg picture)

$$\mathcal{R} : \mathcal{A}_{BC} \to \mathcal{A}_B.$$

Its predual map acts on normal states by $\mathcal{R}_*(\varphi) := \varphi \circ \mathcal{R}$, hence

$$\mathcal{R}_* : \mathcal{S}_n(\mathcal{A}_B) \to \mathcal{S}_n(\mathcal{A}_{BC}).$$

Given a normal state ω on $\mathcal{A}_{ABC}$, the *split-implemented recovered state* (associated with the chosen split datum N) is defined by applying $\mathcal{R}_*$ on the B-leg *in the fixed split implementation induced by* N:

$$\widetilde{\omega}^{(N)}_{ABC} := (\mathrm{id} \otimes \mathcal{R}_*)\big(\omega^{(N)}_{AB}\big),$$

where $\omega^{(N)}_{AB}$ denotes the AB marginal in that fixed split implementation.

Remark 4.3 (Typing discipline). In Type III, $\mathcal{A}(A \cup B)$ is not canonically a tensor product $\mathcal{A}_A \bar{\otimes} \mathcal{A}_B$. Accordingly, the notation $(\mathrm{id} \otimes \mathcal{R}_*)(\omega^{(N)}_{AB})$ is shorthand for the standard tensor-leg application performed *inside the chosen split implementation associated with* N. All dependence on this choice is tracked by the superscript (N).

5 Error metric in Type III: Bures fidelity and purified distance

Definition 5.1 (D3: fidelity/distance convention)**.** Let ω, φ be normal states on a von Neumann algebra. We use the *squared* Bures fidelity $F_B(\omega, \varphi) \in [0,1]$ (so $F_B = 1$ iff $\omega = \varphi$). As a convenient monotone error parameter we adopt

$$P(\omega, \varphi) := \sqrt{1 - F_B(\omega, \varphi)}.$$

Definition 5.2 (D4: split-implemented recovery error functional)**.** Fix a split datum N. Given ω on $\mathcal{A}_{ABC}$ and a recovery channel $\mathcal{R} : \mathcal{A}_{BC} \to \mathcal{A}_B$, define

$$E_{\mathrm{rec}}(\omega; \mathcal{R}; N) := -\log F_B\big(\omega_{ABC}, \widetilde{\omega}^{(N)}_{ABC}\big),$$

where $\widetilde{\omega}^{(N)}_{ABC}$ is as in Definition 4.2.

Remark 5.3 (Convention discipline). Different papers vary by using unsquared fidelity, Bures distance, or $-2\log(\cdot)$. This note fixes Definitions 5.1 and 5.2; any imported inequality must be aligned to these conventions by an explicit constant (see Definition 8.2).

6 Conditional expectations and an Accardi–Cecchini recovery candidate

The Type III-specific ingredient is an ω_0-preserving conditional expectation onto the split factor.

Assumption 6.1 (CE: ω_0-preserving conditional expectation onto N)**. [ASSUMED].**
Fix a faithful normal reference state ω_0 on $\mathcal{A}_{BC}$ (e.g. a KMS state). Let

$$\mathcal{M} := \mathcal{A}_{BC}, \qquad \mathcal{N} := N$$

with N the split datum from Definition 3.1. Assume there exists a normal completely positive unital idempotent map

$$E_{\omega_0} : \mathcal{M} \to \mathcal{N}$$

such that

$$\omega_0 \circ E_{\omega_0} = \omega_0.$$

Remark 6.2 (Why CE is a genuine assumption). Existence of an ω_0-preserving conditional expectation is tied to modular invariance / Takesaki-type conditions and is not automatic [2]. This note isolates CE explicitly to prevent quantifier drift.

Definition 6.3 (D5: GNS-adjoint w.r.t. ω_0)**.** Let $\Phi : \mathcal{M} \to \mathcal{N}$ be normal CP and let ω_0 be faithful. Define

$$\langle X, Y \rangle_{\omega_0} := \omega_0(X^* Y).$$

When it exists, the ω_0-adjoint $\Phi^{\sharp_{\omega_0}} : \mathcal{N} \to \mathcal{M}$ is defined by

$$\langle \Phi(X), Y \rangle_{\omega_0} = \langle X, \Phi^{\sharp_{\omega_0}}(Y) \rangle_{\omega_0} \quad \text{for all } X \in \mathcal{M},\ Y \in \mathcal{N}.$$

Definition 6.4 (D6: Accardi–Cecchini / Petz-dual candidate)**.** Assume CE and that the ω_0-adjoint exists for E_{ω_0}. Define the canonical recovery candidate

$$\mathcal{R}^{\mathrm{AC}} := E_{\omega_0}^{\sharp_{\omega_0}} : \mathcal{N} \to \mathcal{M}.$$

Remark 6.5 (Candidate vs existential recovery). This note does not claim that $\mathcal{R}^{\mathrm{AC}}$ achieves an FR-type bound pointwise; it provides a canonical, typed candidate under CE. Proving optimality or near-optimality of $\mathcal{R}^{\mathrm{AC}}$ is a separate problem.

7 Split-regularized conditional mutual information

We define a split-regularized CMI $I_\omega^{(N)}(A:C|B)$ that is meaningful given a split datum N.

7.1 Two definitions (kept explicitly N-dependent)

Definition 7.1 (D7a: split-implemented CMI (Type I bookkeeping))**.** Assume SPLIT and fix N. Use the induced Type I implementation to represent the relevant inclusions in a tensor-product bookkeeping compatible with (A, B, C). Define $I_\omega^{(N)}(A:C|B)$ by applying the usual Type I CMI formula inside that implementation (with the conventions fixed by the chosen implementation), allowing the value $+\infty$ if the expression diverges.

Definition 7.2 (D7b: Araki-relative-entropy CMI (alternative))**.** Alternatively, define $I_\omega^{(N)}(A:C|B)$ using Araki relative entropy in standard form on the relevant inclusions, with explicit dependence on the same split datum N and the chosen inclusions.

Remark 7.3 (No equivalence claimed)*.* We do not assume Definition 7.1 and Definition 7.2 coincide in full generality. Any equivalence statement must be proved or explicitly assumed (see Appendix A).

8 Main contract: conditional exponential recoverability

Assumption 8.1 (CMI-DECAY: split-regularized exponential Markov bound)**. [ASSUMED].** Assume Definition 3.3 and work with the selected split data $N(\epsilon)$. Fix a family of normal states $\{\omega^{(\epsilon)}\}_\epsilon$ on $\mathcal{A}_{ABC}$ (or a declared state class, stated explicitly). Assume there exist constants $K > 0$ and $\alpha > 0$ such that for all ϵ in the regime of interest,

$$I_{\omega^{(\epsilon)}}^{(N(\epsilon))}(A:C|B) \le K\, e^{-\alpha\epsilon}.$$

Constants may depend on the model, the state family/class, and the selection rule $\epsilon \mapsto N(\epsilon)$, but not on ϵ itself.

Assumption 8.2 (FR-SPLIT: FR-type inequality in the fixed split)**. [ASSUMED]** (import/bridge). Fix the split datum N and a choice of split-regularized CMI definition (either Definition 7.1 or Definition 7.2). Assume there exists a universal constant $c_{\mathrm{FR}} > 0$ (depending only on the convention Definitions 5.1 and 5.2 and the chosen CMI definition) such that for every normal state ω on $\mathcal{A}_{ABC}$ there exists a recovery channel $\mathcal{R} : \mathcal{A}_{BC} \to \mathcal{A}_B$ (in the fixed split implementation) satisfying

$$E_{\mathrm{rec}}(\omega; \mathcal{R}; N) \le c_{\mathrm{FR}}\, I_\omega^{(N)}(A:C|B).$$

Theorem 8.3 (T1: conditional exponential recoverability (Type III, split-regularized))**. *[PROVED]*** *(given Definitions 8.1 and 8.2).*
Under Definitions 8.1 and 8.2, for each ϵ there exists a recovery channel $\mathcal{R}$ such that

$$E_{\mathrm{rec}}(\omega; \mathcal{R}; N(\epsilon)) \le c_{\mathrm{FR}}\, K\, e^{-\alpha\epsilon}.$$

Equivalently, with P as in Definition 5.1,

$$P\big(\omega_{ABC}, \widetilde{\omega}_{ABC}^{(N(\epsilon))}\big) \le \sqrt{1 - \exp(-c_{\mathrm{FR}}\, K e^{-\alpha\epsilon})}.$$

Proof. Combine Definition 8.1 with Definition 8.2 and the definition of E_{rec} in Definition 5.2. The purified-distance form follows from Definition 5.1. □

Remark 8.4 (Where the real work is). T1 is a contract statement. The substantive work in applications is: (i) verifying CMI-DECAY for a concrete AQFT model and a concrete selection rule $\epsilon \mapsto N(\epsilon)$, and (ii) importing/proving an FR-type inequality in the fixed split implementation aligned to the conventions Definitions 5.1 and 5.2.

9 Falsification matrix (minimal)

Item	Status	Verification proxy	Concrete falsifier
SPLIT ($N(\epsilon)$ exists)	**[ASSUMED]**	cite conditions; construct $N(\epsilon)$	exhibit model/geometry where split fails
SEL (selection rule fixed)	**[ASSUMED]**	specify $N(\epsilon)$ explicitly	show ambiguity changes constants qualitatively
CE (E_{ω_0} exists)	**[ASSUMED]**	check modular invariance/Takesaki condition	exhibit Takesaki obstruction / lack of ω_0-invariance
CMI-DECAY	**[ASSUMED]**	estimate $I_{\omega}^{(N)}(A:C\|B)$ vs ϵ	show non-decay or slower-than-claimed decay
FR-SPLIT	**[ASSUMED]**	import a theorem aligned to Definitions 5.1 and 5.2	give a counterexample under stated definitions

Table 1: Minimal falsification matrix for the Type III recoverability interface.

10 Compatibility with finite-dimensional benchmarks

Remark 10.1 (Finite-dimensional reduction). When the relevant algebras are Type I (finite-dimensional), the split datum can be taken canonically and split-regularized CMI reduces to the usual finite-dimensional CMI. In that regime, FR-type inequalities reduce to their standard QIT forms, up to the constant c_{FR} determined by conventions.

A Optional: equivalence of CMI definitions

If one wants to identify Definition 7.1 and Definition 7.2 (for a fixed choice of inclusions and split data), this requires an explicit theorem-level statement. This paper does not assume such equivalence; it should be treated as **[IMPORTED]** (with hypotheses restated verbatim) or as **[ASSUMED]** (with a falsification route).

B Reference-state dependence and model dependence

The CE assumption Definition 6.1 depends on the choice of faithful normal reference state ω_0. In AQFT applications, a natural choice is often a KMS state. Any claim using CE should explicitly record which ω_0 is used and whether ω_0 varies with ϵ.

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