RIP-U and the $\omega = 0$ obstruction in Davies dynamics

Upper envelopes, witness floors, and falsification protocols for separation-dependent decoherence rates

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Abstract

We isolate the dynamic hinge in typed separation→rate→power pipelines within the Davies (weak-coupling, Markovian) setting in finite dimension. First, we formulate an upper-envelope statement (RIP-U): under an explicit factorized bath-correlation envelope with separation-dependent amplitude $f(\epsilon)$ and integrable time profile, Fourier-transformed Davies rates inherit an $O(f(\epsilon))$ envelope. Under additional regularity assumptions preventing trivial degeneracies, this yields a worst-case bound $\kappa^{\uparrow}(\epsilon) \leq Cf(\epsilon)$ for the instantaneous relative loss rate of a coherence-like functional. Second, we isolate a structural obstruction to lower-envelope statements: we prove an exact identity for the $\omega = 0$ contribution to the Davies Dirichlet form,

$$\mathcal{E}_{\sigma}^{(0)}(O) = \frac{\gamma(0)}{2} \, \|[S(0), O]\|_{2,\sigma}^2 \, ,$$

and derive a witness mechanism showing how $\omega = 0$ channels can enforce a non-vanishing dissipation contribution for suitable observable families. We emphasize directionality: RIP-U (upper) does not imply a positive lower envelope $\kappa^{\downarrow}(\epsilon)$ without additional family-qualified input. All assumptions are explicit and accompanied by falsification routes. *v2 (no v1 number is changed):* two v1 assumptions are upgraded to proved statements in their canonical instances — Δ-MONO holds unconditionally for the energy pinching (Davies covariance plus data processing, Lemma 2.7), and a sufficient bridge with an explicit constant is proved (Proposition 4.5); the paper's cross-reference labels are repaired (in v1 every assumption and theorem was typeset as "Definition $x.y$"); Table 1 is re-typeset (overlapping text in v1); the $\omega = 0$ identity is placed within the corrected Bohr-channel decomposition of the companion 2601.0023 (it is exactly the $\omega = 0$ sector, and the plain-commutator form provably fails at $\omega \neq 0$); and a verification suite reproduces every proved item numerically, including the identity at machine precision and an explicit family with $\kappa^{\downarrow} \ll \kappa^{\uparrow}$ under the same envelope.

1 Scope and conventions

What this paper does.

- Proves a Fourier-envelope inheritance lemma for rate coefficients under a factorized correlator bound.
- States a clean upper-envelope implication (RIP-U) under explicit additional bridge assumptions, and (v2) proves a sufficient bridge with an explicit constant on a regularized family.
- Proves an exact $\omega = 0$ Dirichlet identity in standard Davies form and a witness corollary.

What this paper does not do. We do not claim a Hamiltonian spectral gap and we do not claim RIP-U implies a family-wide true floor $\kappa^{\downarrow}(\epsilon) \geq \kappa_0$.

Parameter convention. ϵ is geometric separation/buffer width; δ is tolerance/regularization. They are never interchanged.

2 Davies setting and typed envelopes

We work in finite dimension: system Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = d < \infty$.

Assumption 2.1 (DAVIES: weak-coupling Markovian dynamics). ***[ASSUMED]****. The system dynamics is governed by a CPTP semigroup $\mathcal{T}_t = e^{t\mathcal{L}}$ with a Davies-type GKLS generator $\mathcal{L}$ and a faithful stationary state σ (typically $\sigma \propto e^{-\beta H}$). We assume the standard Bohr-frequency decomposition and GKLS form used in Davies theory [1, 2, 3].*

2.1 A coherence-like functional and instantaneous loss

Definition 2.2 (D1: Δ-track coherence)**.** Fix a dephasing (pinching) CPTP map Δ on $\mathcal{B}(\mathcal{H})$. Define

$$C_\Delta(\rho) := S(\rho \| \Delta[\rho]),$$

where $S(\rho\|\tau) := \mathrm{Tr}(\rho(\log \rho - \log \tau))$ when well-defined.

Definition 2.3 (D2: instantaneous loss)**.** Define the instantaneous change of C_Δ under $\mathcal{T}_t$ by

$$\dot{C}_{\Delta,\mathrm{loss}}(\rho) := - \left.\frac{d}{dt}\right|_{t=0} C_\Delta(\mathcal{T}_t(\rho)),$$

whenever the derivative exists.

Assumption 2.4 (Δ-MONO: sign convention for "loss")**.** ***[ASSUMED]*** *(family restriction;* proved for the energy pinching in v2*, Lemma 2.7). For the chosen Δ and semigroup $\mathcal{T}_t$, we restrict attention to families $\mathcal{F}_\epsilon$ such that $\dot{C}_{\Delta,\mathrm{loss}}(\rho) \geq 0$ for all $\rho \in \mathcal{F}_\epsilon$.*

Definition 2.5 (D3: ratios and envelopes)**.** Whenever $C_\Delta(\rho) > 0$, define $r(\rho) := \dot{C}_{\Delta,\mathrm{loss}}(\rho)/C_\Delta(\rho)$. For a specified family $\mathcal{F}_\epsilon$ define

$$\kappa^{\uparrow}(\epsilon) := \sup_{\rho \in \mathcal{F}_\epsilon :\, C_\Delta(\rho)>0,\ \dot{C}_{\Delta,\mathrm{loss}}(\rho)\ \text{exists}} r(\rho), \qquad \kappa^{\downarrow}(\epsilon) := \inf_{\rho \in \mathcal{F}_\epsilon :\, C_\Delta(\rho)>0,\ \dot{C}_{\Delta,\mathrm{loss}}(\rho)\ \text{exists}} r(\rho).$$

Remark 2.6 (Directionality). RIP-U targets $\kappa^{\uparrow}(\epsilon)$ (worst-case upper envelope). Minimum-maintenance-power lower bounds require $\kappa^{\downarrow}(\epsilon)$; RIP-U alone does not provide them.

2.2 Δ-MONO is unconditional for the energy pinching (v2)

Lemma 2.7 (Energy pinching: covariance and monotone loss)**.** ***[PROVED]*** *(v2). Let Δ_H be the pinching over the spectral projectors $\{\Pi_E\}$ of H and let $\mathcal{L}$ be a Davies generator built from Bohr components of the coupling (Assumption 5.3). Then:* (i) *$\Delta_H \circ \mathcal{T}_t = \mathcal{T}_t \circ \Delta_H$ for all $t \geq 0$;* (ii) *for every state ρ in the domain, $t \mapsto C_{\Delta_H}(\mathcal{T}_t(\rho))$ is nonincreasing, hence $\dot{C}_{\Delta_H,\mathrm{loss}}(\rho) \geq 0$ with no family restriction.*

Proof. (i) The Davies generator commutes with the Hamiltonian dressing $e^{iHt}(\cdot)e^{-iHt}$ (covariance is built into the Bohr construction: each dissipator term maps the ω' Bohr sector of its argument into itself). Since Δ_H is the Cesàro average of the dressing, $\mathcal{L} \circ \Delta_H = \Delta_H \circ \mathcal{L}$, and the semigroup inherits the commutation. (ii) By (i) and monotonicity of relative entropy under the CPTP map $\mathcal{T}_s$,

$$C_{\Delta_H}(\mathcal{T}_{t+s}\rho) = S(\mathcal{T}_s\mathcal{T}_t\rho \,\|\, \mathcal{T}_s\Delta_H\mathcal{T}_t\rho) \le S(\mathcal{T}_t\rho \,\|\, \Delta_H\mathcal{T}_t\rho) = C_{\Delta_H}(\mathcal{T}_t\rho). \qquad \square$$

Lemma 2.8 (Derivative identity). ***[PROVED]*** *(v2). Let $\Delta = \Delta_H$ and let ρ be full rank. Then*

$$\dot{C}_{\Delta,\text{loss}}(\rho) = -\mathrm{Tr}\big(\mathcal{L}(\rho)\,(\log\rho - \log\Delta[\rho])\big).$$

Proof. $\frac{d}{dt}\mathrm{Tr}(\rho_t \log\rho_t) = \mathrm{Tr}(\mathcal{L}(\rho)\log\rho)$ by $\mathrm{Tr}\,\mathcal{L}(\rho) = 0$. For the second term, $\frac{d}{dt}\mathrm{Tr}(\rho_t \log\Delta[\rho_t]) = \mathrm{Tr}(\mathcal{L}(\rho)\log\Delta[\rho]) + \mathrm{Tr}(\rho\,\frac{d}{dt}\log\Delta[\rho_t])$; since $\frac{d}{dt}\log\Delta[\rho_t]$ lies in the pinched (commutative) algebra, self-duality of Δ gives $\mathrm{Tr}(\rho\,\frac{d}{dt}\log\Delta[\rho_t]) = \mathrm{Tr}(\Delta[\rho]\,\frac{d}{dt}\log\Delta[\rho_t]) = \frac{d}{dt}\mathrm{Tr}(\Delta[\rho_t]) = 0$. $\square$

3 Correlator envelope $\Rightarrow$ Fourier rate envelope

Assumption 3.1 (CORR: factorized correlator envelope). ***[ASSUMED]****. For each ϵ, the relevant bath correlators satisfy*

$$|G_{\alpha\beta}(t;\epsilon)| \le g_{\alpha\beta}(t)\, f(\epsilon),$$

where $g_{\alpha\beta} \in L^1(\mathbb{R})$ and $f(\epsilon) \ge 0$ captures separation dependence.

Definition 3.2 (D4: Fourier rate coefficients). Define the (formal) Fourier rate coefficients by $\gamma_{\alpha\beta}(\omega;\epsilon) := \int_{\mathbb{R}} dt\, e^{i\omega t}\, G_{\alpha\beta}(t;\epsilon)$, whenever the integral exists (or under the regularity hypotheses used in Davies theory).

Lemma 3.3 (Fourier envelope inheritance). ***[PROVED]*** *(under Assumption 3.1 and integrability). Under Assumption 3.1, for all ω and ϵ for which Definition 3.2 is well-defined,*

$$|\gamma_{\alpha\beta}(\omega;\epsilon)| \le \|g_{\alpha\beta}\|_{L^1}\, f(\epsilon).$$

Proof. By the triangle inequality and Assumption 3.1, $|\gamma_{\alpha\beta}(\omega;\epsilon)| \le \int_{\mathbb{R}} dt\, |G_{\alpha\beta}(t;\epsilon)| \le \int_{\mathbb{R}} dt\, |g_{\alpha\beta}(t)|\, f(\epsilon) = \|g_{\alpha\beta}\|_{L^1} f(\epsilon)$. $\square$

4 RIP-U: an upper envelope on instantaneous loss

Assumption 4.1 (REG: non-degeneracy for relative-entropy ratios). ***[ASSUMED]*** *(family restriction). The family $\mathcal{F}_\epsilon$ is restricted so that: (i) $C_\Delta(\rho) \ge c_{\min}(\delta) > 0$ uniformly over $\rho \in \mathcal{F}_\epsilon$; (ii) $\Delta[\rho] \succeq p_{\min}(\delta)\mathbf{1}$ uniformly over $\rho \in \mathcal{F}_\epsilon$; (iii) when needed, ρ stays a fixed distance (within tolerance δ) away from* $\mathrm{Fix}(\mathcal{T}_t)$.

Assumption 4.2 (BRIDGE: rate envelope $\Rightarrow$ loss envelope). ***[ASSUMED]*** *(*a proved sufficient instance in v2: *Proposition 4.5). Assume Assumptions 2.1, 3.1 and 4.1. Suppose the Davies generator is built from Fourier rate coefficients $\gamma_{\alpha\beta}(\omega;\epsilon)$ in such a way that the envelope from Lemma 3.3 implies: there exists a constant $C_{\text{RIP}} > 0$ such that for all $\rho \in \mathcal{F}_\epsilon$,*

$$\dot{C}_{\Delta,\text{loss}}(\rho) \le C_{\text{RIP}}\, f(\epsilon)\, C_\Delta(\rho).$$

Theorem 4.3 (T1: RIP-U (upper envelope)). ***[PROVED]*** *(given Assumptions 2.1, 2.4, 3.1, 4.1 and 4.2). Under Assumptions 2.1, 2.4, 3.1, 4.1 and 4.2, for all* $\rho \in \mathcal{F}_\epsilon$*,*

$$r(\rho) \leq C_{\mathrm{RIP}}\, f(\epsilon), \qquad \textit{and hence} \qquad \kappa^{\uparrow}(\epsilon) \leq C_{\mathrm{RIP}}\, f(\epsilon).$$

Proof. By Assumption 4.2, $\dot{C}_{\Delta,\mathrm{loss}}(\rho) \leq C_{\mathrm{RIP}} f(\epsilon) C_\Delta(\rho)$ for all $\rho \in \mathcal{F}_\epsilon$ with $C_\Delta(\rho) > 0$, hence $r(\rho) \leq C_{\mathrm{RIP}} f(\epsilon)$. Taking the supremum yields the stated envelope. □

Remark 4.4 (RIP-U does not imply a floor). RIP-U controls worst-case *upper* loss rates. It is compatible with $\kappa^{\downarrow}(\epsilon)$ being arbitrarily small on a large family. Thus RIP-U alone does not imply minimum maintenance power lower bounds in pipelines requiring a positive lower envelope. (The v2 suite exhibits this concretely: a weakly coupled level pair yields $\kappa^{\downarrow} \sim 10^{-3}\,\kappa^{\uparrow}$ under the same correlator envelope.)

4.1 A proved sufficient bridge (v2)

Proposition 4.5 (BRIDGE-P: explicit-constant bridge on a regularized family). ***[PROVED]*** *(v2). Let* $\Delta = \Delta_H$*, and strengthen Assumption 4.1 to REG*$^+$*: (i)* $C_\Delta(\rho) \geq c_{\min} > 0$ *and (ii′)* $\rho \succeq p_{\min}\mathbf{1}$ *(hence also* $\Delta[\rho] \succeq p_{\min}\mathbf{1}$*) over* $\mathcal{F}_\epsilon$*. Let the Davies dissipator be built from Bohr components* $S(\omega)$ *with rates* $\gamma(\omega;\epsilon)$ *obeying the envelope of Lemma 3.3 with a common profile,* $\gamma(\omega;\epsilon) \leq \bar{g} f(\epsilon)$*, and let* N_B *be the number of Bohr channels and* $M^2 := \max_\omega \|S(\omega)\|^2$*. Then Assumption 4.2 holds with the explicit constant*

$$C_{\mathrm{RIP}} = \frac{4\, N_B\, \bar{g}\, M^2\, \log(1/p_{\min})}{c_{\min}}.$$

Proof. By Lemma 2.8 and Hölder, $\dot{C}_{\Delta,\mathrm{loss}}(\rho) \leq \|\mathcal{L}_D(\rho)\|_1\, \|\log\rho - \log\Delta[\rho]\|_\infty$. Under REG$^+$, $\|\log\rho\|_\infty \leq \log(1/p_{\min})$ and likewise for $\Delta[\rho]$, so the second factor is at most $2\log(1/p_{\min})$. Each dissipator term is 1-norm bounded by $2\gamma(\omega;\epsilon)\|S(\omega)\|^2$, so $\|\mathcal{L}_D(\rho)\|_1 \leq 2N_B\bar{g}M^2 f(\epsilon)$. (The Hamiltonian part contributes zero loss: unitary conjugation commuting with Δ_H leaves C_Δ invariant.) Dividing by $C_\Delta(\rho) \geq c_{\min}$ gives the claim. □

Remark 4.6. BRIDGE-P demotes the bridge from an assumption to a checkable statement on the regularized family: the constant is explicit, dimension-aware through N_B and M, and blows up only as $p_{\min} \to 0$ or $c_{\min} \to 0$, which is the honest signature of the relative-entropy ratio degenerating. The v2 suite instantiates it numerically (worst observed $r/(C_{\mathrm{RIP}} f) \approx 2 \times 10^{-3}$, i.e. the bound is loose but never violated).

5 The $\omega = 0$ obstruction: an exact Dirichlet identity

Definition 5.1 (D5: σ-weighted 2-norm). For a faithful state σ, define $\|X\|_{2,\sigma} := \sqrt{\mathrm{Tr}\big(\sigma^{1/2}X^\dagger\sigma^{1/2}X\big)}$.

Definition 5.2 (D6: Dirichlet form (one convenient convention)). Let $\mathcal{L}$ be a GKLS generator with faithful stationary state σ. Define the Dirichlet form on observables by

$$\mathcal{E}_\sigma(O) := -\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}\,\mathcal{L}^\sharp(O)\Big),$$

where $\mathcal{L}^\sharp$ denotes the Heisenberg-picture action of the generator on observables.

Assumption 5.3 (BOHR: frequency decomposition). ***[ASSUMED]*** *(standard in Davies theory). Let H be the system Hamiltonian with spectral projectors $\{\Pi_E\}$. For a system coupling operator S, define its Bohr components $S(\omega) := \sum_{E'-E=\omega} \Pi_E S \Pi_{E'}$. Assume the Davies generator uses dissipators built from $S(\omega)$ with scalar rates $\gamma(\omega) \geq 0$.*

Lemma 5.4 (Commutation at zero frequency). ***[PROVED]****. If $\sigma \propto e^{-\beta H}$ and $S(0)$ is the $\omega = 0$ Bohr component, then $[S(0), H] = 0$, and hence $[S(0), \sigma] = 0$ and $[S(0), \sigma^{1/2}] = 0$.*

Proof. By construction, $S(0) = \sum_E \Pi_E S \Pi_E$ commutes with $H = \sum_E E\, \Pi_E$, hence with any function of H, in particular σ and $\sigma^{1/2}$. □

Theorem 5.5 (T2: $\omega = 0$ Dirichlet identity). ***[PROVED]*** *(under Assumptions 2.1 and 5.3 in standard Davies form). The zero-frequency contribution $\mathcal{E}_\sigma^{(0)}$ to the Dirichlet form satisfies*

$$\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2} \, \|[S(0), O]\|_{2,\sigma}^2.$$

Proof. We give a direct expansion in Appendix A. □

Remark 5.6 (The $\gamma(0) = 0$ case). If $\gamma(0) = 0$ (e.g. baths with vanishing zero-frequency spectral density), then $\mathcal{E}_\sigma^{(0)} \equiv 0$ and this particular $\omega = 0$ obstruction mechanism disappears. Any lower-envelope floor must then come from other frequency sectors or from structural restrictions defining the family.

Corollary 5.7 (C1: $\omega = 0$ witness mechanism). ***[PROVED]****. Assume $\gamma(0) > 0$. If there exists an observable family O_ϵ such that $\|[S(0), O_\epsilon]\|_{2,\sigma}^2 / \|O_\epsilon\|_{2,\sigma}^2 \geq c > 0$, then $\mathcal{E}_\sigma^{(0)}(O_\epsilon) \geq \frac{\gamma(0)}{2} \, c \, \|O_\epsilon\|_{2,\sigma}^2$.*

Proof. Immediate from Theorem 5.5. □

Remark 5.8 (C1 is not RIP-L). C1 is a mechanism for a chosen observable family. Converting it into a uniform lower-envelope bound $\kappa^{\downarrow}(\epsilon) \geq \kappa_0$ over a state family requires additional typed bridges.

Assumption 5.9 (RIP-L: true floor (family-qualified)). ***[CONJECTURED]****. There exist $\epsilon_* > 0$ and $\kappa_0 > 0$ such that $\kappa^{\downarrow}(\epsilon) \geq \kappa_0$ for all $\epsilon \geq \epsilon_*$, for a specified family $\mathcal{F}_\epsilon$ (with any regularity restrictions stated explicitly).*

Remark 5.10 (Position within the Bohr-channel decomposition of 2601.0023 (v2)). The companion 2601.0023 (v2) proves the exact channel-wise decomposition $\mathcal{E}_\sigma(O) = \sum_\omega \gamma(\omega)\, e^{\beta\omega/2}\, \mathrm{Re}\langle S(\omega)O, [S(\omega), O]\rangle_{\mathrm{KMS}}$ and shows that the naive plain-commutator form $\frac{\gamma(\omega)}{2} \|[S(\omega), O]\|_{2,\sigma}^2$ *fails* at $\omega \neq 0$ (deviations of order 20% there; up to 57% in the present suite's model). Theorem 5.5 is exactly the $\omega = 0$ sector of that decomposition: at $\omega = 0$ the KMS weight $e^{\beta\omega/2}$ is 1 and $S(0)$ commutes with $\sigma^{1/2}$ (Lemma 5.4), which is precisely what collapses the exact quadratic form to the plain commutator. The verification suite checks both facts side by side: machine-precision agreement at $\omega = 0$, systematic deviation at every $\omega \neq 0$.

6 Falsification matrix

Note added in v2

No v1 definition, assumption label, theorem statement or numbering is changed; new v2 material is appended at the end of the affected sections. v2: (1) repairs the cross-reference labels — in v1 every

Item	Status	Verification proxy	Concrete falsifier
CORR	[ASSUMED]	compute $G_{\alpha\beta}(t;\epsilon)$	separation dependence not factorizable as $g(t)f(\epsilon)$
Fourier envelope lemma	[PROVED]	compute $\|g\|_{L^1}$	violate integrability/envelope inequality
Δ-MONO	[ASSUMED] general; [PROVED] for Δ_H (v2)	evaluate $\dot{C}_{\Delta,\mathrm{loss}}(\rho)$	find ρ with $\dot{C}_{\Delta,\mathrm{loss}}(\rho) < 0$ in family (for $\Delta \neq \Delta_H$)
REG	[ASSUMED]	lower-bound spectrum of $\Delta[\rho]$	sequence with $\Delta[\rho_n]$ nearly singular
BRIDGE	[ASSUMED] general; sufficient instance [PROVED] (v2)	compute $r(\rho)$ on the family	show $r(\rho)$ violates $O(f(\epsilon))$ despite CORR
RIP-U	[PROVED]	apply Theorem 4.3	refute any upstream assumption
$\omega = 0$ identity	[PROVED]	direct Dirichlet-form expansion	refute Davies form/hypotheses
RIP-L	[CONJECTURED]	estimate $\kappa^{\downarrow}(\epsilon)$	construct ρ_ϵ with $r(\rho_\epsilon) \to 0$

Table 1: Falsification matrix for the dynamic layer (re-typeset in v2; v1's table had overlapping column text).

reference was typeset as "Definition $x.y$" regardless of environment, so e.g. "By Definition 4.2" in v1 denotes Assumption 4.2 and "Immediate from Definition 5.5" denotes Theorem 5.5; (2) re-typesets Table 1, which in v1 printed overlapping, partially illegible status/proxy entries; (3) proves Δ-MONO unconditionally for the canonical energy pinching via Davies covariance plus data processing (Lemma 2.7), together with the derivative identity used downstream (Lemma 2.8); (4) proves a sufficient bridge with an explicit constant on a regularized family (Proposition 4.5), demoting BRIDGE from assumption to checkable statement in that instance; (5) places Theorem 5.5 as the $\omega = 0$ sector of the corrected Bohr-channel decomposition of 2601.0023 v2 (Remark 5.10); (6) adds series positioning (v1 cited no companion papers) and a verification suite (`verification/2601-0064/` in the series repository) reproducing every proved item: the identity at machine precision (2.9×10^{-16} over random O), the envelope lemma with a concrete correlator, covariance and nonnegative loss for Δ_H, BRIDGE-P, an explicit weak-pair family with $\kappa^{\downarrow} \approx 10^{-3}\kappa^{\uparrow}$ (Remark 4.4), the witness C1, and the $\omega \neq 0$ failure of the plain-commutator form.

References

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A Line-by-line proof of the $\omega = 0$ identity

We give a direct expansion showing $\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2}\|[S(0), O]\|_{2,\sigma}^2$. For clarity we present the Hermitian case $S(0)^\dagger = S(0)$; the general case replaces $S(0)^2$ by $S(0)^\dagger S(0)$ and yields the same commutator structure.

The $\omega = 0$ Heisenberg dissipator has the standard GKLS form

$$\mathcal{L}^{\sharp,(0)}(O) = \gamma(0)\Big(S(0)\,O\,S(0) - \tfrac{1}{2}\{S(0)^2, O\}\Big).$$

Using the Dirichlet form convention from Definition 5.2,

$$\mathcal{E}_\sigma^{(0)}(O) = -\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}\,\mathcal{L}^{\sharp,(0)}(O)\Big).$$

Expand:

$$\begin{aligned}\mathcal{E}_\sigma^{(0)}(O) = &-\gamma(0)\,\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}S(0)\,O\,S(0)\Big) + \frac{\gamma(0)}{2}\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}S(0)^2O\Big)\\ &+ \frac{\gamma(0)}{2}\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}O\,S(0)^2\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|[S(0), O]\|_{2,\sigma}^2 &= \mathrm{Tr}\Big(\sigma^{1/2}\big(S(0)O - OS(0)\big)^\dagger\sigma^{1/2}\big(S(0)O - OS(0)\big)\Big)\\ &= \mathrm{Tr}\Big(\sigma^{1/2}(O^\dagger S(0) - S(0)O^\dagger)\sigma^{1/2}(S(0)O - OS(0))\Big).\end{aligned}$$

Using Lemma 5.4 to commute $S(0)$ through $\sigma^{1/2}$ and expanding the product yields

$$\|[S(0), O]\|_{2,\sigma}^2 = \mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}S(0)^2O\Big) - 2\,\mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}S(0)\,O\,S(0)\Big) + \mathrm{Tr}\Big(\sigma^{1/2}O^\dagger\sigma^{1/2}O\,S(0)^2\Big).$$

Comparing with the expression for $\mathcal{E}_\sigma^{(0)}(O)$ gives $\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2}\|[S(0), O]\|_{2,\sigma}^2$, as claimed.