CMI-Based Recoverability versus Wilson-Loop Diagnostics in $\mathbb{Z}_2$ Lattice Gauge Theory (2+1D)

Exact diagonalization benchmark on small open lattices

Version 2

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Abstract

We provide a finite-size benchmark testing whether a CMI-based recoverability proxy correlates with Wilson-loop confinement diagnostics in $\mathbb{Z}_2$ lattice gauge theory in 2+1 dimensions, computing ground states by sparse exact diagonalization on 2×2 and 2×3 open lattices (link qubits, Gauss-law penalty verified by $\langle G_v \rangle \simeq 1$) and evaluating entropic quantities by pure-state Schmidt/SVD. We state a conditional bridge from a spatial area law to an operational "information horizon" via explicit hypotheses (H1)–(H4), with two recovery-length conventions (absolute, and normalized by the boundary prefactor $\Sigma_B(w) = |\partial B(w)| \log 2$). *v2 (no v1 number is changed) adds a structural lemma that v1's own Table 1 was exhibiting unnoticed:* for a pure global state, when the collar saturates ($B = (A \cup C)^c$) purity forces $I(A{:}C|B) = I(A{:}C|\emptyset)$ – this is exactly why v1's Table 1 shows $I(w{=}2) = I(w{=}0) = 0.4992999$ to seven digits; the saturated row carries no buffer information, and the informative range of that benchmark is $w \in \{0, 1\}$. v2 also anchors the non-monotonicity of CMI under collar enlargement as geometry-dependent (v1's wall geometry shows growth at $w{=}0 \to 1$; a BFS-patch geometry on the same states decays monotonically – there is no data-processing theorem in that direction); cross-links the CMI benchmark to the Petz-error twin on the same lattices (densified $n = 8$ sweep: Spearman rank-trend +1.00, permutation $p = 10^{-4}$, against both $1/\sigma_{\text{eff}}$ *and* the inverse spectral gap – so, as in the twin, confinement specificity is unresolved at these sizes); harmonizes the fidelity convention (v1 correctly uses root fidelity with $I \geq -2 \log f$, equivalent to the companions' squared-convention $I \geq -\log F$); removes an internal phase label leaked into v1's Section 1 title and a duplicated heading; and adds series positioning and a regenerable verification suite. The conditional proposition and its hypotheses are unchanged.

1 Conditional bridge: confinement as an information horizon

As in v1 (unchanged; the internal phase label in the v1 heading is removed and the duplicated subsection title merged): transfer-matrix state ρ_{β_T} (or ground-state limit); spatial Wilson loops and spatial string tension σ_{str} with finite-size Creutz proxies on small lattices; tripartition with collar $B(w)$ and boundary prefactor $\Sigma_B(w) = |\partial B(w)| \log 2$; CMI in nats; absolute and normalized recovery lengths $\xi_{\text{rec}}^{\text{abs}}(\varepsilon)$, $\xi_{\text{rec}}^{\text{norm}}(\varepsilon)$; hypotheses (H1) spatial area law, (H2) wall geometry forces $\text{Area}_{\min}(w) \geq c_0 w$, (H3$_g$) flux-tube-dominated exponential suppression of loop-restricted connected leakage $\Delta_{\text{loop}}(w)$, (H4$_{\text{QI}}$) structural leakage-to-CMI input; and the conditional proposition

$I(A{:}C|B(w)) \le K\,\Sigma_B(w)\,e^{-\alpha w}$ with $\alpha = p\kappa\sigma_{\rm str}c_0$, $K = K_I C_f K_\Delta^p$, plus the recovery corollary. *Convention note (added in v2):* v1 states the Fawzi–Renner corollary with root fidelity $F = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ and $I \ge -2\log F$ – which is correct; the companions use the squared convention with $I \ge -\log F^2$-equivalently. The identity $-2\log f = -\log(f^2)$ makes both the same statement (suite-checked); we flag it because this factor has historically been the most error-prone convention point of the series.

2 Saturation lemma and the informative range (added in v2)

Lemma 2.1 (Saturation identity). *Let ρ_Λ be pure on a finite slice and let A, C be disjoint with $B := (A \cup C)^c$. Then*

$$I(A{:}C\,|\,B) \;=\; I(A{:}C\,|\,\emptyset) \;=\; S(A) + S(C) - S(AC).$$

Proof. Purity gives $S(AB) = S(C)$, $S(BC) = S(A)$ and $S(B) = S(AC)$, and $S(ABC) = 0$; substitute into the CMI definition. (Machine-verified on random pure states and on the $\mathbb{Z}_2$ ground state, errors $\sim 10^{-15}$.) □

Remark 2.2 (Consequence for v1's Table 1). v1's wall-geometry table on the 2×3 lattice at $g = 0.5$ reports $I(w{=}0) = 0.4992999$, $I(w{=}1) = 0.8974278$, $I(w{=}2) = 0.4992999$: the exact seven-digit return at $w = 2$ is Lemma 2.1 in action – at $w = 2$ the wall exhausts the complement of $A \cup C$. By v1's own pre-saturation definition that row should be excluded; the informative range of the benchmark instance is $w \in \{0, 1\}$. All v1 numbers stand; what changes is their reading.

Remark 2.3 (Non-monotonicity is geometry-dependent). There is no data-processing theorem for enlarging the conditioning system, so $I(A{:}C|B(w))$ need not decrease in w: v1's wall geometry indeed grows from $w = 0$ to 1 ($0.499 \to 0.897$). On the same class of states, the BFS-patch geometry of the Petz twin decays monotonically across the full coupling sweep (suite: 0/8 growth instances). Collar monotonicity is thus a property of the collaring rule (cf. the collaring-rule framework of 2601.0043), not of CMI itself.

3 ED benchmark and cross-link to the Petz twin (upgraded in v2)

The v1 wall-geometry instance (Table 1, Figures) stands unchanged. v2 adds the coupling-resolved benchmark on the 2×2 lattice with the BFS-patch geometry shared with the Petz twin 2601.0047: over $n = 8$ couplings $g \in [0.5, 2.5]$, $I(A{:}C|B(1))$ falls monotonically from 1.6×10^{-4} to 1.8×10^{-11} while $1/\sigma_{\rm eff}$ falls and the spectral gap grows; the Spearman rank-trend of $I(A{:}C|B(1))$ against $1/\sigma_{\rm eff}$ is $+1.00$ (permutation $p = 10^{-4}$) – *and equally* $+1.00$ against the inverse gap. As in the twin, the CMI benchmark therefore shows a robust monotone rank-trend in the deterministic coupling sweep whose confinement specificity is unresolved at these sizes (the permutation p quantifies ordering strength, not statistical independence). The Petz-error and CMI observables thus tell one consistent story on these lattices.

4 Verification suite (added in v2)

`verification/2601-0050/verify_2601_0050.py` (sparse Lanczos, SVD entropies, chunked JSON cache): the saturation lemma on random pure states and on the $\mathbb{Z}_2$ ground state (machine precision); the geometry-dependence of collar monotonicity; the $n = 8$ CMI benchmark with rank statistics and the gap covariate; Gauss-sector checks; and the root/squared fidelity-convention identity.

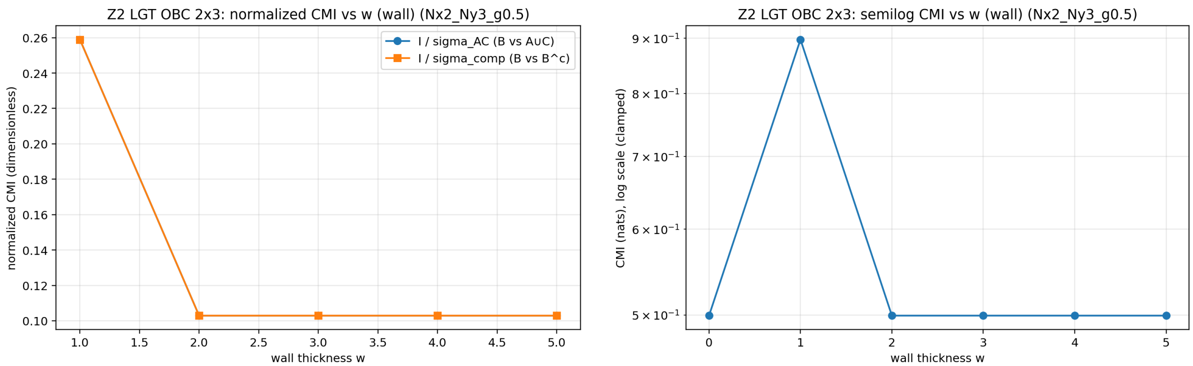

Figure 1: v1 wall-collar benchmark figures (unchanged): I/Σ_B vs w and semilog I vs w, 2×3, $g = 0.5$. Read with Remarks 2.2–2.3.

5 Discussion

The conditional information-horizon proposition and its hypotheses are untouched: (H4$_{\mathrm{QI}}$) remains the structural frontier, and the uniformity-under-refinement remark remains the roadmap link to recovery-based renormalization. What v2 contributes is discipline around the benchmark: saturated rows are identities, collar monotonicity is a collaring-rule property, the CMI and Petz-error twins agree, and the confinement-vs-gap degeneracy identified in 2601.0047 v2 applies verbatim here. Next steps are those of the twin: larger lattices, fixed-coupling geometry sweeps, and sector-wise algebraic prescriptions.

Note added in v2

No v1 number, hypothesis or proposition is changed. v2 adds: (1) the saturation lemma (Lemma 2.1) explaining Table 1's exact $I(w{=}2) = I(w{=}0)$ coincidence and restricting the informative range to $w \in \{0, 1\}$; (2) the geometry-dependence of collar monotonicity (Remark 2.3); (3) the coupling-resolved CMI benchmark cross-linked to 2601.0047 v2, with identical conclusions including the unresolved confinement specificity; (4) the fidelity-convention harmonization note (v1's root-fidelity factor is correct); (5) editorial fixes: internal phase label removed from the Section 1 title, duplicated heading merged; (6) series positioning (v1 cited no companion papers) and the regenerable verification suite.

References

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