Petz Recoverability in AQFT via Conditional Expectations

A framework and a conditional exponential recovery bound

Version 2

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January 2026 (v1); July 2026 (v2)

Abstract

We formulate an operational notion of recoverability in algebraic quantum field theory for type III local von Neumann algebras. Fixing a faithful normal KMS reference state and assuming a state-preserving conditional expectation, we define the recovery channel and, working in a fixed split implementation for each separation r, we assume (i) exponential decay of split-implemented conditional mutual information and (ii) a CMI-to-recovery inequality. Under these explicit bridge assumptions we obtain a conditional exponential recoverability bound $E_{\mathrm{rec}}(r) \leq g(C_1 e^{-m_{\mathrm{CMI}} r})$. *v2 corrects the duality underlying the construction:* v1 defined the Petz-type channel as the "Accardi–Cecchini adjoint" via the pairing $\omega(Z\,\mathcal{R}(X)) = \omega(\varepsilon(Z)X)$ and "proved" finite-dimensional consistency with the standard Petz map; that identity is *false in general* (the printed proof contains an invalid cyclicity step; numerically the identity fails at order 10^{-1} on random faithful states, holding only in commuting/product situations). The correct statement, proved and machine-verified here, is that the standard Petz map is the *trace-predual* of the generalized (Accardi–Cecchini) conditional expectation; accordingly, v2 defines the recovery channel in Schrödinger picture as precomposition with the conditional expectation. This also repairs a type/direction error in v1's recovered-state definition, whose corrected form $\tilde{\omega} := \omega|_{AB} \circ \Psi_r$ (with Ψ_r the normal extension of $\mathrm{id}_A \otimes \varepsilon$) reproduces the standard Petz reconstruction exactly in finite dimensions. We further relabel v1's finite-dimensional map as the generalized conditional expectation (its Takesaki module property fails generically – verified), record that for a *true* state-preserving conditional expectation the recovery is the CE-pullback, and add series positioning: this note is the AQFT capstone announced by the companions, its split-implemented CMI is one of three compatible regularizations in the series, and the numerical program deferred by v1 has since been executed. The main theorem is unchanged: a conditional framework statement isolating the missing bridge assumptions.

1 Introduction

Local algebras in AQFT are typically type III, so reduced density matrices and partial traces are unavailable in the naive sense. This note is intentionally conservative: assuming quantitative conditional independence and a CMI-to-recovery bridge (both explicit), one obtains an exponential bound on a fidelity-based recovery error for a recovery channel built from a conditional expectation. The main statement is a conditional theorem isolating AQFT-correct infrastructure; the bridge assumptions are the open technical problem.

Series positioning (added in v2). This is the AQFT companion announced in 2601.0043/0044 [8, 9] and the sibling of the Type III capstone 2601.0034 [7]: there, recovery is the universal/twirled modular Petz candidate with an assumed FR-type inequality in purified distance; here, recovery is built from a state-preserving conditional expectation in a fixed split implementation. The split-implemented CMI of Section 6 is the third regularized CMI of the series – alongside the embedded-state CMI of 2601.0020 [10] and the Araki-difference CMI of 2601.0034 – and all three reduce to the standard CMI for natural Type I data. The "numerical outlook" deferred in v1 has been executed in the companions: TFIM pipelines in 2601.0043 v2 and a first $\mathbb{Z}_2$ gauge testbed in 2601.0044 v2. Upstream: 2512.0060/0101 and 2601.0007 [11, 12, 13].

2 Tripartition geometry and split implementations

As in v1 (unchanged): Haag–Kastler net, faithful normal KMS state ω_β; tripartition A–B–C with collar B and separation r; Assumption (fixed split implementation): for each r a normal $*$-isomorphism $\alpha_r : \mathcal{A}(A \cup B \cup C) \to M_{ABC}(r) \cong B(\mathcal{H}_A^{(r)}) \bar{\otimes} B(\mathcal{H}_B^{(r)}) \bar{\otimes} B(\mathcal{H}_C^{(r)})$ mapping each local algebra into its factor, with pushed-forward state ω_β^r; and generation of the unions by the corresponding subalgebras.

3 Finite-dimensional warm-up (corrected in v2)

The CMI-to-recovery prototype is unchanged: by Fawzi–Renner [6] there is a recovery channel with $-\log F \le c_{\mathrm{FR}}\, I_\rho(A{:}C|B)$; with the squared-fidelity convention one may take $c_{\mathrm{FR}} = 1$ (this is the importable constant; cf. the factor corrections in 2512.0101 v2 and companions), and the FR map need not be the Petz map.

Definition 3.1 (Generalized conditional expectation; relabeled in v2)**.** *For faithful $\rho_{BC} > 0$ with $\rho_B := \mathrm{Tr}_C\, \rho_{BC}$, define $E_{BC\to B} : B(\mathcal{H}_B \otimes \mathcal{H}_C) \to B(\mathcal{H}_B) \otimes \mathbf{1}_C$ by*

$$E_{BC\to B}(Z) := \rho_B^{-1/2}\ \mathrm{Tr}_C\big(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2}\big)\ \rho_B^{-1/2} \otimes \mathbf{1}_C.$$

E is normal, unital, completely positive and ω-preserving ($\omega = \mathrm{Tr}(\rho_{BC}\,\cdot)$), but it is not *a conditional expectation in the Takesaki sense: the module property $E(X_B \otimes \mathbf{1}) = X_B \otimes \mathbf{1}$ and idempotency fail for generic ρ_{BC} (suite: deviations of order 0.5 on random faithful states). It is the* generalized *(Accardi–Cecchini) conditional expectation [2]; v1 called it "the ω-preserving conditional expectation," which we correct.*

Proposition 3.2 (Correct duality: Petz is the trace-predual of E; replaces v1's Prop. 3.3)**.** *With $E_0(Z) := \rho_B^{-1/2}\, \mathrm{Tr}_C(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2}) \rho_B^{-1/2}$ and the standard Petz map $\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B) = \rho_{BC}^{1/2}(\rho_B^{-1/2} X_B \rho_B^{-1/2} \otimes \mathbf{1}_C)\rho_{BC}^{1/2}$, one has, for all X_B and all $Y \in B(\mathcal{H}_B \otimes \mathcal{H}_C)$,*

$$\mathrm{Tr}\big[\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B)\, Y\big] \ =\ \mathrm{Tr}\big[X_B\, E_0(Y)\big]\ :$$

the Petz map is the adjoint of E with respect to the trace *pairing, i.e. the predual map E_* on density operators.*

Proof. $\mathrm{Tr}[\rho_{BC}^{1/2}(\rho_B^{-1/2} X_B \rho_B^{-1/2} \otimes \mathbf{1})\rho_{BC}^{1/2} Y] = \mathrm{Tr}[(\rho_B^{-1/2} X_B \rho_B^{-1/2} \otimes \mathbf{1})\rho_{BC}^{1/2} Y \rho_{BC}^{1/2}] = \mathrm{Tr}_B[\rho_B^{-1/2} X_B \rho_B^{-1/2}\, \mathrm{Tr}_C(\rho_{BC}^{1/2} Y \rho_{BC}^{1/2})$
$\mathrm{Tr}[X_B E_0(Y)]$. (Verified to 2×10^{-15} on random states.) □

Remark 3.3 (v1's pairing is false in general). v1 defined the "Petz dual" by $\omega(Z\,\mathcal{R}(X)) = \omega(E(Z)X)$ and offered a finite-dimensional proof that the standard Petz map satisfies it. The proof's cyclicity step (moving a single $\rho_{BC}^{1/2}$ through the trace) is invalid, and the identity fails numerically at order 10^{-1} on random faithful ρ_{BC} (suite); the same holds for naive GNS- and KMS-weighted variants. The identity *does* hold in commuting situations – e.g. for product $\rho_{BC} = \rho_B \otimes \rho_C$, where the true conditional expectation exists, its predual is $X_B \mapsto X_B \otimes \rho_C$ (equal to the Petz map there), and v1's pairing is satisfied with $\mathcal{R}$ the inclusion. All of this is machine-verified.

4 Fidelity and recovery error

As in v1: $F(\varphi, \psi)$ the squared transition probability (Uhlmann–Araki) for normal states [3, 4], monotone under normal CP unital maps; $E_{\mathrm{rec}} := -\log F(\omega, \tilde{\omega})$.

5 Recovery channel from a conditional expectation (reformulated in v2)

Assumption 5.1 (ω_β-preserving conditional expectation; unchanged)**.** *There exists a normal conditional expectation $\varepsilon_{BC\to B} : \mathcal{A}(B \cup C) \to \mathcal{A}(B)$ with $\omega_\beta \circ \varepsilon_{BC\to B} = \omega_\beta|_{\mathcal{A}(B\cup C)}$ (a nontrivial, Takesaki-type modular-invariance condition [2]).*

Definition 5.2 (Recovery as CE-pullback; replaces v1's Defs. 5.3/6.1)**.** *Fix r and let $\Psi_r : \alpha_r(\mathcal{A}(A\cup B \cup C)) \to \alpha_r(\mathcal{A}(A \cup B))$ be the normal unital completely positive extension of $\mathrm{id}_A \otimes \alpha_r \varepsilon_{BC\to B} \alpha_r^{-1}$ provided factor-wise by the split implementation. Define the recovered state*

$$\tilde{\omega}_\beta^{r,ABC} := \omega_\beta^{r,AB} \circ \Psi_r, \qquad E_{\mathrm{rec}}(r) := -\log F\big(\omega_\beta^r,\, \tilde{\omega}_\beta^{r,ABC}\big),$$

computable in the type I split algebra by $*$-isomorphism invariance of F (v1's Lemma 6.5, unchanged and correct).

Remark 5.3 (What was wrong in v1's Definition 6.1). v1 built a map Φ_r on products $XY \mapsto X\,\mathcal{R}^{\mathrm{Petz}}(Y)$, which has type $\alpha_r(\mathcal{A}(A \cup B)) \to \alpha_r(\mathcal{A}(A \cup B \cup C))$, and then set $\tilde{\omega} := \omega^{AB} \circ \Phi_r$ – ill-typed, since ω^{AB} is a functional on the *domain* of Φ_r, not its range. The corrected Definition 5.2 composes in the admissible direction, and in finite dimensions it reproduces the standard Schrödinger Petz reconstruction $(\mathrm{id}_A \otimes E_*)(\rho_{AB})$ exactly (suite: 3×10^{-16}), consistently with Proposition 3.2. For a *true* conditional expectation (Assumption 5.1) the recovered state is the CE-pullback $\omega_{AB} \circ (\mathrm{id} \otimes \varepsilon)$, exact precisely on states for which $\mathcal{A}(A \cup B)$ is sufficient.

6 Bridge assumptions and conditional theorem

Unchanged from v1 modulo the corrected E_{rec}: split-implemented CMI $I^{\omega_\beta,r}(A{:}C|B)$ via density operators in the fixed implementation (scheme-dependent; an Araki formulation would refine it – cf. the compatibility remark in 2601.0034 v2); Assumption (CMI decay): $I^{\omega_\beta,r}(A{:}C|B) \le C_1 e^{-m_{\mathrm{CMI}} r}$; Assumption (CMI-to-recovery): $E_{\mathrm{rec}}(r) \le g(I^{\omega_\beta,r}(A{:}C|B))$ with monotone g, $g(t) \to 0$.

Theorem 6.1 (Conditional exponential recoverability; unchanged)**.** *Under the standing assumptions, $E_{\mathrm{rec}}(r) \le g(C_1 e^{-m_{\mathrm{CMI}} r})$; if $g(t) \le ct$ in the relevant range, $E_{\mathrm{rec}}(r) \le (cC_1)e^{-m_{\mathrm{CMI}} r}$.*

7 Verification suite (added in v2)

`verification/2601-0046/verify_2601_0046.py`: identifies the correct duality pairing (trace-predual exact at 2×10^{-15}; v1's ω-pairing fails at 0.77; naive GNS/KMS variants fail similarly); verifies that E of Definition 3.1 is unital and ω-preserving but violates the module property (order 0.5); verifies the product-case statements of Remark 3.3; verifies that the corrected recovered state equals the standard Petz reconstruction in finite dimensions; and anchors $c_{\mathrm{FR}} = 1$ (squared convention) on random tripartite states with Petz recovery (40/40).

8 Discussion and outlook

As in v1: exponential clustering motivates the CMI-decay assumption; the nuclearity/modular-nuclearity program (quantitative split, finite-rank approximants, stability of fidelity under limits) is the route to *deriving* the bridge assumptions [5]. Updated: the numerical program deferred by v1 is now executed in the companions – lattice CMI/Petz pipelines (2601.0043 v2) and a first $\mathbb{Z}_2$ lattice-gauge testbed (2601.0044 v2), whose ladder-level lessons (separation condition; floor management) transfer to any future continuum-approximation scheme.

Note added in v2

The conditional theorem, the bridge assumptions, and the split implementation are unchanged. v2 corrects the operational core: (1) v1's defining pairing for the "Petz dual" is false in general (Remark 3.3; invalid cyclicity step in its Prop. 3.3; refuted numerically at order 10^{-1}), and the correct duality is the trace-predual identity of Proposition 3.2; (2) the recovery channel is accordingly defined as CE-pullback (Definition 5.2), which also repairs the type/direction error in v1's Definition 6.1 and reproduces standard Petz exactly in finite dimensions; (3) v1's finite-dimensional map is relabeled as the generalized (Accardi–Cecchini) conditional expectation, its Takesaki module property failing generically (verified); (4) series positioning added (three compatible split-CMI regularizations; differentiation from 2601.0034; numerical program executed in 2601.0043/0044 v2), plus the verification suite. The bridge assumptions remain the open technical frontier.

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